

Teaching Plan, week 5

- Plan for today:
1. Warm-up
 2. Lognormal distributions
 3. Problems
 4. Quiz (memorize the PDFs of the normal & lognormal distributions, given parameters μ and σ), (10 mins)

1. Warm-up

using the following properties of logs:

$$\log_a(bc) = \log_a(b) + \log_a(c) \quad (\text{log changes } \cdot \text{ to } +)$$

$$\log_a(b^c) = c \cdot \log_a(b) \quad (\text{log changes } ^c \text{ to } \cdot)$$

$$\log_a(a^b) = b, a^{\log_a(b)} = b \quad (\text{log and exponentiation are inverses})$$

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)} \quad (\text{change of base formula})$$

solve the following questions.

a) If $x, y > 0$, simplify $\log_{xy}(x^y)(1 + \log_x(y))$.

b) If $a, x > 0$ and $a^2 = A$, simplify $a^{(\log_a(x) + \log_A(x))}$.

c) If $x, y > 0$ and $x^2 + y^2 = 14xy$, find k so that
 $\ln(k(x+y)) = \frac{1}{2}(\ln x + \ln y)$.

Solutions: a) $\log_{xy}(x^y)(1 + \log_x(y)) = \frac{\log_x(x^y)}{\log_x(xy)} (1 + \log_x(y))$

$$= \frac{y}{\log_x(x) + \log_x(y)} (1 + \log_x(y)) = y \cdot \frac{1 + \log_x(y)}{1 + \log_x(y)} = y$$

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b) $\log_a(x) + \log_a(x) = a(\log_a(x) + \log_{a^2}(x)) = a(\log_a(x) + \frac{\log_a(x)}{\log_a(a^2)})$

 $= a(\log_a(x) + \frac{\log_a(x)}{2}) = a \frac{3}{2} \log_a(x) = a \log_a(x^{3/2}) = x^{3/2}$

c) $\ln(k(x+y)) = \frac{1}{2}(\ln x + \ln y) \iff \ln(k(x+y)) = \frac{1}{2}\ln(xy)$

 $\iff \ln(k(x+y)) = \ln((xy)^{1/2}) \iff e^{\ln(k(x+y))} = e^{\ln(\sqrt{xy})}$
 $\iff k(x+y) = \sqrt{xy} \iff k > 0 \text{ and } k^2(x^2 + 2xy + y^2) = xy$
 $\iff k^2(14xy + 2xy) = xy, k > 0 \iff 16xyk^2 = xy, k > 0$
 $\iff k^2 = \frac{1}{16}, k > 0 \iff k = \frac{1}{4}$

2. Lognormal distributions

Suppose we have some normal distribution T with mean μ and standard deviation $\sigma > 0$. Define $X = e^T$.

Then $P(a \leq T \leq b) = P(e^a \leq e^T \leq e^b) = P(e^a \leq X \leq e^b)$

$$= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \stackrel{t=\ln x, x=e^t}{=} \int_{e^a}^{e^b} \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx, \text{ so}$$

$\left\{ \begin{array}{l} t = \ln x, \text{ so } x = e^t \\ dt = \frac{1}{x} dx \end{array} \right.$

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad \text{is the PDF of } X.$$

Note that $\ln X = T$. Distributions X such that $\ln X$ is normally distributed are called lognormal distributions.

B

Just like normal distributions, lognormal distributions don't admit a nice CDF for us to use, because the PDF is too hard to integrate (without infinite series or numerical approximation methods). We can express probabilities involving lognormal distributions as integrals of their PDFs, but to actually compute these probabilities just translate the probability involving X into a probability involving $\ln X$, which is normally distributed.

If $\ln X$ is normal with mean μ and standard deviation σ , we say X has "log mean" μ and "log standard deviation" σ . This is very unfortunate terminology, because if X has log mean μ , that does not mean that the mean of X is e^μ : even though $\ln X$ has mean μ , X does not have mean e^μ . (Same goes for log standard deviation). Then how can we find the mean and standard deviation of X with log mean μ and log standard deviation σ ? By direct calculation, using the definitions, and the "completing the square" trick:

Say $m = \text{mean of } X$ and $s^2 = \text{standard deviation of } X$.

$$\text{Then } m = \int_{-\infty}^{\infty} x f(x) dx \text{ and } s^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - (m)^2, \text{ where}$$

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} & \text{for } x > 0 \end{cases} \quad \text{is the PDF for } X.$$

Warning! A nightmare lies beyond. Feel free to skip this section.

$$S_o m = \int_0^\infty \frac{x}{\sqrt{2\pi}\sigma x} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx = \lim_{a \rightarrow 0^+} \int_a^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx$$

this is necessary to deal with the integrand's "singularity" @ $x=0$, and with ∞ as the upper bound

$$+ \lim_{b \rightarrow \infty} \int_0^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx = \lim_{a \rightarrow 0^+} \int_a^{\ln b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt + \lim_{b \rightarrow \infty} \int_0^{\ln b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

$\begin{cases} \ln x = t, \text{ so } x = e^t \\ \frac{1}{x} dx = dt, \text{ so } dx = x dt = e^t dt \end{cases}$

aside: $e^t e^{-\frac{(t-\mu)^2}{2\sigma^2}} = e^{t - \frac{(t-\mu)^2}{2\sigma^2}}$. $t - \frac{(t-\mu)^2}{2\sigma^2} = \frac{2\sigma^2 t - t^2 + 2t\mu - \mu^2}{2\sigma^2}$

$$= -(t^2 - (2\mu + 2\sigma^2)t + \mu^2) \cdot \frac{1}{2\sigma^2} = \left(\frac{-1}{2\sigma^2}\right) \cdot ((t - \mu - \sigma^2)^2 - (\mu + \sigma^2)^2 + \mu^2)$$

$$= \frac{-(t - (\mu + \sigma^2))^2}{2\sigma^2} + \frac{\mu^2 + 2\mu\sigma^2 + \sigma^4 - \mu^2}{2\sigma^2} = \frac{-(t - (\mu + \sigma^2))^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2},$$

$$\text{so } e^t e^{-\frac{(t-\mu)^2}{2\sigma^2}} = e^{\frac{-(t - (\mu + \sigma^2))^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2}} = e^{\mu + \frac{\sigma^2}{2}} e^{\frac{-(t - (\mu + \sigma^2))^2}{2\sigma^2}},$$

and so $m = e^{\mu + \frac{\sigma^2}{2}} \cdot \lim_{a \rightarrow 0^+} \int_a^{\ln b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - (\mu + \sigma^2))^2}{2\sigma^2}} dt + e^{\mu + \frac{\sigma^2}{2}} \cdot \lim_{b \rightarrow \infty} \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - (\mu + \sigma^2))^2}{2\sigma^2}} dt$

$$= e^{\mu + \frac{\sigma^2}{2}} \cdot \int_{-\infty}^{\ln a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - (\mu + \sigma^2))^2}{2\sigma^2}} dt = e^{\mu + \frac{\sigma^2}{2}} \cdot \int_0^0 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - (\mu + \sigma^2))^2}{2\sigma^2}} dt. \text{ So } m = e^{\mu + \frac{\sigma^2}{2}}.$$

$\overbrace{= 1}$, since this is the integral of the PDF of the normal distribution with mean $(\mu + \sigma^2)$ and standard deviation σ

$$\text{Similarly, } m^2 + s^2 = \int_0^\infty \frac{x^2}{\sqrt{2\pi}\sigma x} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{2t} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-[t^2 - (2\mu + 4\sigma^2)t + \mu^2]/2\sigma^2} dt = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-[(t - (\mu + 2\sigma^2))^2 - (\mu + 2\sigma^2)^2 + \mu^2]/2\sigma^2} dt$$

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$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-[(t-(\mu+\sigma^2))^2]/2\sigma^2 + 2\sigma^2 + 2\mu} dt \\
 &= e^{2\sigma^2 + 2\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-[(t-(\mu+\sigma^2))^2]/2\sigma^2} dt = e^{2\sigma^2 + 2\mu},
 \end{aligned}$$

= 1, since this is the integral of the PDF of the normal distribution with mean $(\mu+\sigma^2)$ and standard deviation σ .

$$\begin{aligned}
 \text{so } S^2 &= e^{2\sigma^2 + 2\mu} - m^2 = e^{2\sigma^2 + 2\mu} - (e^{\mu + \frac{\sigma^2}{2}})^2 \\
 &= e^{2\sigma^2 + 2\mu} - e^{2\mu + \sigma^2} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}
 \end{aligned}$$

— — — (Ok, the nightmare is over.)

Thus if X is lognormally distributed with log mean μ and log standard deviation σ , then X has mean $m = e^{\mu + \frac{\sigma^2}{2}}$

and variance $S^2 = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$.

3. Problems

- a) For X lognormally distributed with log mean $\mu = 44$ and log standard deviation $\sigma = 7$, calculate $P(e^{30} < X \leq e^{51})$.

$$P(e^{30} < X \leq e^{51}) = P(\ln(e^{30}) < \ln X \leq \ln(e^{51}))$$

$$= P(30 < \ln X \leq 51) = P\left(\frac{30-44}{7} < Z \leq \frac{51-44}{7}\right) = P(-2 < Z \leq 1),$$

normally distributed,
with mean $\mu = 44$ and
standard deviation $\sigma = 7$

z-scores of 30 & 51,
respectively

$$\begin{aligned}
 &\hookrightarrow P(0 \leq Z \leq 2) + P(0 \leq Z \leq 1) \quad (\text{by symmetry}) \\
 &\approx .4772 + .3413 = .8185
 \end{aligned}$$

b) If X is lognormally distributed with log mean $\mu=0$ and log standard deviation $\sigma=1$, find $P(0 \leq X < 10)$.

Solution: $X > 0$, so $P(0 \leq X < 10) = P(X < 10) = P(\ln X < \ln 10)$

$$= P(Z \leq \frac{\ln(10) - 0}{\sqrt{1}}) = P(Z \leq \ln(10)) \approx P(Z \leq 2.30)$$

$$= \underbrace{P(-\infty < Z \leq 0)}_{=\frac{1}{2}, \text{ by symmetry}} + P(0 \leq Z \leq 2.30) \approx \frac{1}{2} + .4893 = .9893$$

Note: we would get the same answer had the problem asked us to find $P(-5 \leq X < 10)$ instead, again since $X > 0$.

$X > 0$ means $P(X > 0) = 1$, or equivalently, $P(X \leq 0) = 0$, which we can easily see since X 's PDF is 0 for negative inputs, or by definition of X as e^T for some (normal) distribution T .

c) Solve the differential equation $\frac{dy}{dt} = ry(1-y)$, $y(0) = y_0$, where $0 < y_0 < 1$.

Solution: Use partial fractions. $\int \frac{dy}{y(1-y)} = \int r dt$, so decompose $\frac{1}{y(1-y)}$:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}, \quad 1 = A(1-y) - By, \quad y=1: B=-1 \\ y=0: A=1, \quad \text{so } \int \frac{dy}{y(1-y)} = \int \frac{1}{y} - \frac{1}{1-y} dy$$

$$= \ln|y| - \ln|y-1| = \ln\left|\frac{y}{y-1}\right| = \ln\left|\frac{y-1+1}{y-1}\right| = \ln\left|1 + \frac{1}{y-1}\right| = rt + C, \quad \text{so } \left|1 + \frac{1}{y-1}\right| = e^{rt+C}$$

$$= e^{ert}, \quad \text{so } 1 + \frac{1}{y-1} = De^{rt}, \quad \text{so } \frac{1}{y-1} = De^{rt}-1, \quad \text{and thus } y = 1 + \frac{1}{De^{rt}-1}$$

$$= \frac{De^{rt}}{De^{rt}-1} = \frac{1}{1 - \frac{1}{D}e^{-rt}} = \frac{1}{1 + Ee^{-rt}}, \quad y(0) = y_0 = \frac{1}{1 + Ee^0} = \frac{1}{1+E}, \quad \text{so } E = \frac{1}{y_0} - 1 = \frac{1-y_0}{y_0}$$

$$\text{so } y(t) = \frac{1}{1 + \frac{1-y_0}{y_0}e^{-rt}} = \frac{y_0}{y_0 + (1-y_0)e^{-rt}}$$



, and you can check that y is a CDF using the criteria to be a CDF!