

Jean-Michel
Maldague
Math 3c
4-17-17

Teaching Plan, week 3

Warm-up: Find the CDF associated with the

$$\text{PDF } f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-x & \text{for } 0 \leq x \leq 1 \\ 1/2 & \text{for } 1 < x < 2 \\ 0 & \text{for } 2 \leq x. \end{cases}$$

- Plan for today:
1. Solution to warm-up
 2. More on CDFs
 3. Improper integrals/comparison test
 4. Homework questions
 5. Quiz (5 minutes)

1. Solution to Warm-up

Note: f is p.c. and $f \geq 0$, and after we find the CDF $F(x)$, it'll be clear that $\lim_{x \rightarrow \infty} F(x) = 1$, which will imply that $\int_{-\infty}^{\infty} f(t) dt = 1$,

so f really is a PDF.

Find F in stages, according to the "parts" on which f is defined:

$$\text{For } x \leq 0, \quad F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^x 0 dt = 0 \quad (\text{even if } x=0, \text{ the}$$

integral $\int_{-\infty}^0 f(t) dt$ is independent of what $f(0)$ is!).

$$\text{For } x \in [0, 1], \quad F(x) - F(0) = \int_0^x f(t) dt = \int_0^x (1-t) dt = \left[t - \frac{t^2}{2} \right]_0^x = x - \frac{x^2}{2},$$

$$\text{So } F(x) = x - \frac{x^2}{2} + \underbrace{F(0)}_{=0, \text{ by the previous part}} = x - \frac{x^2}{2}.$$

For $x \in [1, 2]$, $F(x) - F(1) = \int_1^x f(t) dt = \int_1^x \frac{1}{2} dt = \left[\frac{t}{2} \right]_1^x = \frac{x}{2} - \frac{1}{2}$,

so $F(x) = \frac{x}{2} - \frac{1}{2} + F(1) = \frac{x}{2}$.
 $= 1 - \frac{1}{2} = \frac{1}{2}$, by the previous part

For $x \in [2, \infty)$, $F(x) - F(2) = \int_2^x f(t) dt = \int_2^x 0 dt = 0$, so $F(x) = F(2) = \frac{2}{2} = 1$

by the previous part. Putting it all together, we get

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x - \frac{x^2}{2} & 0 \leq x \leq 1 \\ x/2 & 1 \leq x \leq 2 \\ 1 & 2 \leq x \end{cases}$$

It's ok to use " \leq " everywhere since the "parts" of F agree @ the endpoints.

2. More on CDFs

a) If $F(x) = \begin{cases} \frac{x^4}{1+x^4} & \text{if } x \geq 0 \\ 0 & \text{else,} \end{cases}$ verify that F is

a CDF, and find $P(-50 \leq X < 1)$, $P(1 < X < 10)$, where X is a random variable with CDF F .

Solution: $\frac{0^4}{1+0^4} = 0$, so F 's parts agree @ the endpoint 0.

Thus F is continuous. $F'(x) = \begin{cases} \frac{4x^3(1+x^4) - x^4(4x^3)}{(1+x^4)^2} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

$= \begin{cases} \frac{4x^3}{(1+x^4)^2} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$, so F is nondecreasing (F is nondecreasing on $(-\infty, 0]$ and $[0, \infty)$).

$\lim_{x \rightarrow -\infty} F(x) = \lim_{x \rightarrow -\infty} 0 = 0$ and $\lim_{x \rightarrow \infty} F(x) = \lim_{x \rightarrow \infty} \frac{x^4}{1+x^4} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x^4} + 1} = 1$,

so F is a CDF.

$$P(-50 \leq X < 1) = P(-50 < X \leq 1) = P(-\infty < X \leq 1) - P(-\infty < X \leq -50)$$

since point probabilities are 0 for continuous R.V.s

$$= F(1) - F(-50) = \frac{1^4}{1+1^4} - 0 = \frac{1}{2}$$

$$P(1 < X < 10) = P(-\infty < X \leq 10) - P(-\infty < X \leq 1)$$

$$= F(10) - F(1) = \frac{10^4}{1+10^4} - \frac{1^4}{1+1^4} = \frac{10,000}{10,001} - \frac{1}{2} = \frac{20,000 - 10,001}{20,002}$$

$$= \frac{9,999}{20,002}$$

b) If $\frac{dy}{dt} = -.12y$, $y(0) = y_0 > 0$, solve for $y(t)$ ($t \geq 0$).

If $y(t)$ represents the amount of drug in the body at time t , find the proportion of the drug that has left the body by time t , and call it $F(t)$. Define $F(t) = 0$ for $t < 0$, and verify that F is a CDF. Find the probability that a randomly chosen drug molecule leaves the body between $t=1$ and $t=5$.

Solution: use separation of variables to get $y(t)$:

$$\frac{dy}{dt} = -.12y, \text{ so } \int \frac{dy}{y} = \int -.12 dt, \text{ so } \ln|y| = -.12t + C, \text{ so}$$

$$|y| = e^{-.12t + C} = e^{-.12t} e^C, \text{ so } y(t) = A e^{-.12t} \text{ for } t \geq 0. y(0) = y_0 = A e^0 = A$$

so $y(t) = y_0 e^{-.12t}$ for $t \geq 0$. $\frac{y(t)}{y_0}$ is the fraction of drug

still in the body, so $F(t) = 1 - \frac{y(t)}{y_0}$ is the fraction of drug

that has left the body by time $t \geq 0$. So $F(t) = \begin{cases} 1 - e^{-.12t} & t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$

$1 - e^{-.12(0)} = 1 - e^0 = 1 - 1 = 0$, so F 's parts agree @ the endpoint 0, so F is continuous.

$$F'(t) = \begin{cases} .12e^{-.12t} & t > 0 \\ 0 & \text{if } t < 0 \end{cases}, \text{ so } F \text{ is nondecreasing.}$$

$$\lim_{t \rightarrow -\infty} F(t) = \lim_{t \rightarrow -\infty} 0 = 0 \text{ and } \lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} (1 - e^{-.12t}) = 1 - 0 = 1,$$

so F is a CDF. If X represents the time a drug particle leaves the body, then $P(1 \leq X \leq 5) = F(5) - F(1) = (1 - e^{-.12(5)}) - (1 - e^{-.12(1)}) = e^{-.12} - e^{-.6}$. (This quantity is positive).

c) If $f(x) = \begin{cases} 3ax^2 & \text{for } 1 \leq x \leq 4 \\ 0 & \text{else,} \end{cases}$ find "a" so that f is

a PDF. Then find the CDF, $F(x)$. Then find the 1st quartile, median, and 3rd quartile.

Solution: f is p.c. and ≥ 0 as long as $a \geq 0$. $\int_{-\infty}^{\infty} f(t) dt = \int_1^4 3at^2 dt$

$$= [at^3]_1^4 = a \cdot 64 - a \cdot 1 = 63a = 1, \text{ so } a = \frac{1}{63}. \text{ This makes } f$$

a PDF. For $x \in (-\infty, 1]$, $F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^x 0 dt = 0$. For $x \in [1, 4]$,

$$F(x) - F(1) = \int_1^x f(t) dt = \int_1^x 3at^2 dt = [at^3]_1^x = a(x^3 - 1) = \frac{x^3 - 1}{63}, \text{ so}$$

$$F(x) = \frac{x^3 - 1}{63} + F(1) = \frac{x^3 - 1}{63} + 0 = \frac{x^3 - 1}{63}, \text{ For } x \in [4, \infty),$$

$$F(x) - F(4) = \int_4^x f(t) dt = \int_4^x 0 dt = 0, \text{ so } F(x) = F(4) = \frac{64 - 1}{63} = 1.$$

Putting all this together yields $F(x) = \begin{cases} 0 & x \leq 1 \\ \frac{x^3-1}{63} & \text{for } 1 \leq x \leq 4 \\ 1 & 4 \leq x \end{cases}$

Let q_1 & q_3 be the 1st & 3rd quartiles, respectively, and let m be the median. q_1 is the x -value below which 25% of the data lies, or equivalently, the x -value such that $P(-\infty < X \leq x) = \frac{1}{4}$.

So $\int_{-\infty}^x f(t) dt = \frac{1}{4} = F(x)$. We can solve for x by isolating it

from either $\int_{-\infty}^x f(t) dt = \frac{1}{4}$, or $F(x) = \frac{1}{4}$. Let's do the latter since

it's easier. Then clearly $q_1 \in [1, 4]$, so $\frac{1}{4} = F(q_1) = \frac{q_1^3-1}{63}$, so

$q_1 = (1 + \frac{63}{4})^{1/3}$. Similarly, $\frac{1}{2} = F(m) = \frac{m^3-1}{63}$, so $m = (1 + \frac{63}{2})^{1/3}$,

and $\frac{3}{4} = F(q_3) = \frac{q_3^3-1}{63}$, so $q_3 = (1 + \frac{189}{4})^{1/3}$.

3. Improper integrals/comparison test (refer to week 2 teaching plan)

a) $\int_2^{\infty} \frac{dx}{x^2} = ?$ Solution: $\int_2^{\infty} \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_2^t = \lim_{t \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{t} \right) = \frac{1}{2}$,

so this integral turned out to be convergent.

b) $\int_2^{\infty} \frac{dx}{x} = ?$ Solution: $\int_2^{\infty} \frac{dx}{x} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x} = \lim_{t \rightarrow \infty} \left[\ln x \right]_2^t = \lim_{t \rightarrow \infty} (\ln t - \ln 2) = \infty$,

so this integral turned out to be divergent.

c) $\int_{-\infty}^0 x e^x dx = ?$ Solution: $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$, so $\int_{-\infty}^0 x e^x dx =$
 $u=x \quad v=e^x \quad (IBP)$
 $du=dx \quad dv=e^x dx$

$\lim_{t \rightarrow -\infty} \int_t^0 x e^x dx = \lim_{t \rightarrow -\infty} \left[x e^x - e^x \right]_t^0 = \lim_{t \rightarrow -\infty} (-1 - t e^t + e^t) = \lim_{t \rightarrow -\infty} \left(-1 - \frac{t-1}{e^t} \right)$
 $\stackrel{H}{=} \lim_{t \rightarrow -\infty} \left(\frac{1}{e^t} \right) - 1 = -1$, so the integral converges ($t \rightarrow -1$).

d) $\int_{-\infty}^{\infty} x e^x dx = ?$ Solution: split into two parts:
 $\int_{-\infty}^0 x e^x dx = -1$ from (c), and $\int_0^{\infty} x e^x dx = \lim_{t \rightarrow \infty} \int_0^t x e^x dx = \lim_{t \rightarrow \infty} [x e^x - e^x]_0^t$
 $= \lim_{t \rightarrow \infty} (1 + t e^t - e^t) = 1 + \lim_{t \rightarrow \infty} (t-1)e^t = \infty$, so this part of the whole integral diverges. Thus, automatically, $\int_{-\infty}^{\infty} x e^x dx$ diverges.

If any of the integral's parts diverge, then the whole integral diverges. This is always the case, even if $\int_{-\infty}^0 f(x) dx = -\infty$ and $\int_0^{\infty} f(x) dx = \infty$. (can't just cancel out the infinities).

e) Does $\int_1^{\infty} \frac{dx}{1+x^2}$ converge or diverge? Solution: $\frac{1}{1+x^2} \leq \frac{1}{x^2}$ for $1 \leq x$,
 so $\int_1^{\infty} \frac{dx}{1+x^2} \leq \int_1^{\infty} \frac{dx}{x^2} < \infty$, by (a). So $\int_1^{\infty} \frac{dx}{1+x^2}$ converges, (bottom bound by comparison test. 1 or 2, doesn't change anything)

f) Does $\int_2^{\infty} \frac{5 + \sin^9(x^3)}{x} dx$ converge or diverge?
 Solution: $|\sin^9(x^3)| \leq 1$, so $\frac{5 + \sin^9(x^3)}{x} \geq \frac{4}{x}$ for $x \geq 2$, so $\int_2^{\infty} \frac{5 + \sin^9(x^3)}{x} dx \geq \int_2^{\infty} \frac{4}{x} dx = 4 \int_2^{\infty} \frac{dx}{x} = \infty$, by (b). So $\int_2^{\infty} \frac{5 + \sin^9(x^3)}{x} dx$ diverges, by the comparison test.

g) Find the CDF associated with PDF $f(x) = 3e^{-6|x|}$. Solution:
 For $x \leq 0$, $F(x) = \int_{-\infty}^x f(t) dt = \lim_{a \rightarrow -\infty} \int_a^x 3e^{-6|t|} dt = \lim_{a \rightarrow -\infty} 3 \int_a^x e^{6t} dt$
 $= \lim_{a \rightarrow -\infty} 3 \left[\frac{e^{6t}}{6} \right]_a^x = \lim_{a \rightarrow -\infty} \frac{1}{2} (e^{6x} - e^{6a}) = \frac{e^{6x}}{2}$. For $x \geq 0$, $F(x) - F(0) = \int_0^x 3e^{-6|t|} dt$
 $= \int_0^x 3e^{-6t} dt = \left[-\frac{3}{6} e^{-6t} \right]_0^x = \frac{1}{2} - \frac{e^{-6x}}{2}$, so $F(x) = \frac{1}{2} - \frac{e^{-6x}}{2} + F(0) = 1 - \frac{e^{-6x}}{2}$.
 Since $H = t$ for $t \geq 0$
 4. HW a's 5 Quiz (5 mins)