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Math 33a  
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## Teaching Plan, week 8

- Plan for today : 1. hand back & go over midterms ] if midterms aren't ready yet we'll skip this item entirely  
 2. mystery enriching activity #1 ] not for a grade  
 3. mystery enriching activity #2 ]

### 1. Midterm solutions.

#### Problem #1

a)  $V = \text{span}(\cos(x), \sin(x))$ , and

$T: V \rightarrow V$  is given by  $T(f) = f'' + af' + bf$ , where  $a, b \in \mathbb{R}$ .

Show that  $T$  is linear and compute its  $B$ -matrix, where the basis  $B$  for  $V$  is given by  $B = \{\cos(x) - \sin(x), \cos(x) + \sin(x)\}$ .

b) For which  $a, b$  is  $T$  an isomorphism?

Solution: a)  $\forall \lambda, \mu \in \mathbb{R}, \forall f, g \in V$ , we have

$$T(\lambda f + \mu g) = (\lambda f + \mu g)'' + a(\lambda f + \mu g)' + b(\lambda f + \mu g)$$

$$= \lambda [f'' + af' + bf] + \mu [g'' + ag' + bg] = \lambda T(f) + \mu T(g),$$

so  $T$  is linear. Define  $A = \{\cos x, \sin x\}$ , another basis of  $V$ .

$$[T]_A = \begin{bmatrix} [T(\cos x)]_A & [T(\sin x)]_A \end{bmatrix} = \begin{bmatrix} [(b-1)\cos x - a\sin x] & [a\cos x + (b-1)\sin x] \end{bmatrix}_A$$

$$= \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}. S_{B \rightarrow A} = \begin{bmatrix} [\cos x - \sin x]_A & [\cos x + \sin x]_A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$\text{so } [T]_B = S_{A \rightarrow B} [T]_A S_{B \rightarrow A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b-a-1 & a+b-1 \\ -a+b+1 & b-a-1 \end{bmatrix}$$

b)  $T$  is an isomorphism  $\Leftrightarrow [T]_B$  is invertible  $\Leftrightarrow (b-1)^2 + a^2 \neq 0$  and  $(a \neq 0 \text{ or } b \neq 1)$ .  $= \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$

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Problem 2 a) Construct an orthonormal basis of  $\mathbb{R}^3$  from the vectors  $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

b) Compute the QR factorization of  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ .

Solution: a) Use Gram-Schmidt.  $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,

$$\vec{u}_2 = \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\|\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and}$$

$$\vec{u}_3 = \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\|\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

so our desired basis is  $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

b) Set  $Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , the orthonormal columns obtained from Gram-Schmidt above. Then  $A = QR$  for some upper-triangular  $R$  with positive diagonal entries.  $R = Q^T Q R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,

so our desired QR factorization is  $\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_R$ .

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Problem 3 a)  $V = \left\{ \left( x_n \right)_{n=1}^{\infty} : x_n \in \mathbb{R} \text{ for each } n \in \mathbb{N} \right\}$ . Show

that  $\ell_1 = \left\{ \left( x_n \right)_{n=1}^{\infty} \in V : \sum_{n=1}^{\infty} |x_n| < \infty \right\}$  is a linear subspace of  $V$ .

b) If  $T_1 : \ell_1 \rightarrow \ell_1$ ,  $T_1 \left( \left( x_n \right)_{n=1}^{\infty} \right) = (0, x_1, x_2, \dots)$  and

$T_2 : \ell_1 \rightarrow \ell_1$ ,  $T_2 \left( \left( x_n \right)_{n=1}^{\infty} \right) = (1, x_1, x_2, \dots)$ , verify if  $T_1, T_2$  are linear or not, and find the kernel of any that is linear.

Solution: a)  $(0)_{n=1}^{\infty} \in \ell_1$ , since  $\sum_{n=1}^{\infty} |0| = 0 < \infty$ .  
the "0" element of  $V$

Is  $\lambda, \mu \in \mathbb{R}$  and  $\left( x_n \right)_{n=1}^{\infty}, \left( y_n \right)_{n=1}^{\infty} \in \ell_1$ , then

$$\sum_{n=1}^{\infty} |\lambda x_n + \mu y_n| = \lim_{N \rightarrow \infty} \sum_{n=1}^N |\lambda x_n + \mu y_n| \leq \lim_{N \rightarrow \infty} \sum_{n=1}^N |\lambda x_n| + \lim_{N \rightarrow \infty} \sum_{n=1}^N |\mu y_n|$$

definition of an infinite series

triangle inequality &  
monotonicity of limits

$$= \lim_{N \rightarrow \infty} \left( |\lambda| \sum_{n=1}^N |x_n| + |\mu| \sum_{n=1}^N |y_n| \right) = |\lambda| \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N |x_n| \right) + |\mu| \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N |y_n| \right)$$

linearity of convergent limits

$$= |\lambda| \sum_{n=1}^{\infty} |x_n| + |\mu| \sum_{n=1}^{\infty} |y_n| < \infty, \text{ so } \left( \lambda \left( x_n \right)_{n=1}^{\infty} + \mu \left( y_n \right)_{n=1}^{\infty} \right) = \left( \lambda x_n + \mu y_n \right)_{n=1}^{\infty} \in \ell_1,$$

and thus  $\ell_1$  is a subspace of  $V$ . Note that the problem did not ask us to verify that  $V$  was a subspace.

b)  $\forall \lambda, \mu \in \mathbb{R}, \forall \left( x_n \right)_{n=1}^{\infty}, \left( y_n \right)_{n=1}^{\infty} \in \ell_1$ , we have:  $T_1 \left( \lambda \left( x_n \right)_{n=1}^{\infty} + \mu \left( y_n \right)_{n=1}^{\infty} \right) =$

$$(0, \lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \dots) = \lambda (0, x_1, x_2, \dots) + \mu (0, y_1, y_2, \dots) = \lambda T_1 \left( \left( x_n \right)_{n=1}^{\infty} \right) + \mu T_1 \left( \left( y_n \right)_{n=1}^{\infty} \right),$$

so  $T_1$  is linear. Also,  $T_1 \left( \left( x_n \right)_{n=1}^{\infty} \right) = (0, 0, 0, \dots)$  implies  $x_1 = x_2 = x_3 = \dots = 0$ , so  $\ker(T_1) = \left\{ (0)_{n=1}^{\infty} \right\}$ .

$$T_2 \left( 0 \cdot \left( 0 \right)_{n=1}^{\infty} \right) = T_2 \left( (0)_{n=1}^{\infty} \right) = (1, 0, 0, \dots) \neq 0 \cdot (1, 0, 0, \dots) = 0 \cdot T_2 \left( (0)_{n=1}^{\infty} \right) \quad (\text{trivial kernel})$$

So  $T_2$  does not respect scalar multiplication, so it is not linear.

Problem 4 a) Let  $\vec{u} \in \mathbb{R}^n$  be a unit vector (treat  $\vec{u}$  as an  $n \times 1$  column matrix). Show that  $Q = I_n - 2\vec{u}\vec{u}^T$  is orthogonal.

b) Show that for any  $n \times n$  matrix  $A$  we have  $\text{im}(A) = \text{im}(AA^T)$ .

Solution: a)  $\|\vec{u}\| = 1$ , so  $\vec{u}^T \vec{u} = \vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1$ . Thus,

$$\begin{aligned} Q^T Q &= (I_n - 2\vec{u}\vec{u}^T)^T (I_n - 2\vec{u}\vec{u}^T) = (I_n^T - 2(\vec{u}\vec{u}^T)^T)(I_n - 2\vec{u}\vec{u}^T) \\ &= (I_n - 2(\vec{u}^T\vec{u})) (I_n - 2\vec{u}\vec{u}^T) = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T) \\ &= I_n I_n - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T = I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T \\ &= I_n, \end{aligned}$$

which implies that  $Q$  is orthogonal. Alternatively:

If  $V = \text{span}(\vec{u})$ , then  $\text{proj}_V(\vec{x}) = \vec{u}\vec{u}^T\vec{x}$ , and  $\text{refl}_V(\vec{x}) = 2\text{proj}_V(\vec{x}) - \vec{x}$

orthonormal  
set

$= (2\vec{u}\vec{u}^T - I_n)\vec{x}$ . Since reflections preserve length, and  $Q$  is the matrix of a (negative) reflection,  $Q$  preserves

length (treating  $Q$  as a linear map), so  $Q$  is orthogonal.

$$\text{b) } \text{im}(A) = \ker(A^T)^\perp = \ker(A^T A^T)^\perp = \ker(A A^T)^\perp = \text{im}((A A^T)^T) = \text{im}(A^T A) = \text{im}(A A^T)$$

where we used the facts that:

$\forall$  subspaces  $V$ ,  $(V^\perp)^\perp = V$ ,

$\forall p \times q$  matrices  $B$ ,  $(B^T)^T = B$ ,

$\forall p \times q$  matrices  $B$ ,  $\ker(B) = \ker(B^T B)$ ,

$\forall p \times q$  matrices  $B$ ,  $(\text{im}(B))^\perp = \ker(B^T)$ , and

$\forall p \times q$  matrices  $B$  and  $q \times r$  matrices  $C$ ,  $(BC)^T = C^T B^T$ .