

Teaching Plan, week 6

- Plan for today:
1. Quiz (15 minutes)
 2. Matrix of a linear transform
 3. Orthogonal projections & bases
 4. Gram-Schmidt & QR factorization
 5. Problems

1. Quiz!

2. Matrix of a linear transform

If V is an n -dimensional linear space, with basis, say,

$\mathcal{B} = \{f_1, \dots, f_n\}$, then for any $f \in V$ we can find its

\mathcal{B} -coordinates: $[f]_{\mathcal{B}}$ is the vector in \mathbb{R}^n of weights

of f_1, \dots, f_n used to produce f . If $T: V \rightarrow V$ is a linear map,

then $B = \begin{bmatrix} [T(f_1)]_{\mathcal{B}} & \cdots & [T(f_n)]_{\mathcal{B}} \end{bmatrix}$ is the \mathcal{B} -matrix

of T , since $[T(f)]_{\mathcal{B}} = B[f]_{\mathcal{B}}$ for all $f \in V$.

B basically lets us do computations in \mathbb{R}^n as opposed to V

[2]

If $B = \{b_1, \dots, b_n\}$ and $\mathcal{U} = \{u_1, \dots, u_n\}$ are two bases for the same n -dimensional vector space V , then the $n \times n$ matrix $S = \begin{bmatrix} [b_1]_{\mathcal{U}} & \cdots & [b_n]_{\mathcal{U}} \\ | & \cdots & | \\ 1 & \cdots & 1 \end{bmatrix}$, also denoted $S_{B \rightarrow \mathcal{U}}$, is the change-of-basis matrix from B to \mathcal{U} , and it has the property that $[f]_{\mathcal{U}} = S[f]_B$ for all $f \in V$, so S is one-to-one & onto, hence invertible.

Now if V is a subspace of \mathbb{R}^n with two bases $\mathcal{U} = \{\vec{a}_1, \dots, \vec{a}_m\}$ and $B = \{\vec{b}_1, \dots, \vec{b}_m\}$, then

$$\begin{aligned} [\vec{a}_1 \ \cdots \ \vec{a}_m] S &= [\vec{a}_1 \ \cdots \ \vec{a}_m] \begin{bmatrix} [b_1]_{\mathcal{U}} & \cdots & [b_m]_{\mathcal{U}} \end{bmatrix} \\ &= \begin{bmatrix} [\vec{a}_1 \ \cdots \ \vec{a}_m] [b_1]_{\mathcal{U}} & \cdots & [\vec{a}_1 \ \cdots \ \vec{a}_m] [b_m]_{\mathcal{U}} \end{bmatrix} \\ &= \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}. \end{aligned}$$

Finally, if V is an n -dimensional vector space with bases \mathcal{U} & B , and A, B are the \mathcal{U}, B -matrices of T , respectively, and S is the change of basis matrix from B to \mathcal{U} , then

3

$\forall f \in V$, $[T(f)]_A = A[f]_A = AS[f]_B$, and

$[T(f)]_A = S[T(f)]_B = SB[f]_B$. Then

$$AS = AS I_n = AS \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = AS \begin{bmatrix} 1 & & \\ [b_1]_B & \cdots & [b_n]_B \\ 1 & & 1 \end{bmatrix}$$

$$= [AS[b_1]_B \cdots AS[b_n]_B] = [SB[b_1]_B \cdots SB[b_n]_B]$$

$$= SB[[b_1]_B \cdots [b_n]_B] = SB \begin{bmatrix} 1 & & \\ \vdots & \ddots & \vdots \\ 1 & & 1 \end{bmatrix} = SB, \text{ so}$$

A & B are similar.

3. Orthogonal projections & bases

We are back in \mathbb{R}^n ! Woohoo!!

If $\vec{v}, \vec{w} \in \mathbb{R}^n$, we say \vec{v} & \vec{w} are perpendicular, or orthogonal, if $\vec{v} \cdot \vec{w} = 0$. The length of \vec{v} is

$\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$, by multiple applications of the Pythagorean theorem. If $\|\vec{v}\|=1$ then \vec{v} is a unit vector.

If you have a bunch of vectors $\vec{u}_1, \dots, \vec{u}_m$ in \mathbb{R}^n then we say they are orthonormal if they are orthogonal unit vectors, so $\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

These groups of vectors are always linearly independent:

if $c_1 \vec{u}_1 + \dots + c_m \vec{u}_m = \vec{0}$, then dot both sides with \vec{u}_i to get

$$0 = \vec{0} \cdot \vec{u}_i = (c_1 \vec{u}_1 + \dots + c_m \vec{u}_m) \cdot \vec{u}_i = c_1 \vec{u}_1 \cdot \vec{u}_i + \dots + c_m \vec{u}_m \cdot \vec{u}_i = c_1 \vec{u}_1 \cdot \vec{u}_i = c_1,$$

4

Then each c_i must be 0, which establishes linear independence. So any set of n orthonormal vectors in \mathbb{R}^n automatically is a basis of \mathbb{R}^n , since any n linearly independent vectors in \mathbb{R}^n must be a basis of \mathbb{R}^n . This works for (nonzero) orthogonal vectors as well!

If $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a set of linearly independent vectors in \mathbb{R}^n , then we can always find an orthonormal set of vectors $\{\vec{u}_1, \dots, \vec{u}_m\}$ with the same span: set

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \text{ and } \vec{w}_k = \vec{v}_k - (\vec{v}_k \cdot \vec{u}_1)\vec{u}_1 - \dots - (\vec{v}_k \cdot \vec{u}_{k-1})\vec{u}_{k-1}$$

$$\vec{u}_k = \frac{\vec{w}_k}{\|\vec{w}_k\|}. \text{ Then } \{\vec{u}_i\} \text{ is a linearly independent set (of course!)}$$

and since $\|\vec{u}_1\|=1$, $\{\vec{u}_i\}$ is an orthonormal set with the same span as $\{\vec{v}_i\}$. $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$ is nonzero, since $\vec{v}_2 \notin \text{span}(\vec{u}_1) = \text{span}(\vec{v}_1)$, by linear independence of the \vec{v}_i 's. Also, $\vec{w}_2 \cdot \vec{u}_1 = (\vec{v}_2 \cdot \vec{u}_1) - (\vec{v}_2 \cdot \vec{u}_1)\underbrace{(\vec{u}_1 \cdot \vec{u}_1)}_{=1} = 0$, so $\vec{u}_2 \cdot \vec{u}_1 = 0$, and since $\|\vec{u}_2\|=1$, we get that $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal set.

Since $\vec{v}_2 \in \text{span}(\vec{u}_1, \vec{w}_2) = \text{span}(\vec{u}_1, \vec{u}_2)$ and $\vec{u}_2 \in \text{span}(\vec{u}_1, \vec{v}_2) = \text{span}(\vec{v}_1, \vec{v}_2)$, we get that $\text{span}(\vec{u}_1, \vec{u}_2) = \text{span}(\vec{v}_1, \vec{v}_2)$.

$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$ is nonzero, since $\vec{v}_3 \notin \text{span}(\vec{u}_1, \vec{u}_2) = \text{span}(\vec{v}_1, \vec{v}_2)$. $\vec{w}_3 \cdot \vec{u}_1 = \vec{v}_3 \cdot \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_1)\underbrace{(\vec{u}_1 \cdot \vec{u}_1)}_1 - (\vec{v}_3 \cdot \vec{u}_2)\underbrace{(\vec{u}_2 \cdot \vec{u}_1)}_0 = 0$ and

$$\vec{w}_3 \cdot \vec{u}_2 = \vec{v}_3 \cdot \vec{u}_2 - (\underbrace{\vec{v}_3 \cdot \vec{u}_1}_{0})(\underbrace{\vec{u}_1 \cdot \vec{u}_2}_{0}) - (\vec{v}_3 \cdot \vec{u}_2)(\underbrace{\vec{u}_2 \cdot \vec{u}_2}_{1}) = 0, \text{ so}$$

\vec{u}_3 is orthogonal to \vec{u}_1 & \vec{u}_2 , and since $\|\vec{u}_3\|=1$, we get that

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set. Since $\vec{v}_3 \in \text{span}(\vec{u}_1, \vec{u}_2, \vec{w}_3) = \text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $\vec{u}_3 \in \text{span}(\vec{u}_1, \vec{u}_2, \vec{v}_3) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, this shows that $\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

Continue in this manner! End up with the orthonormal set

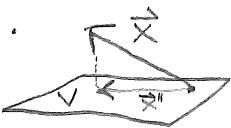
$\{\vec{u}_1, \dots, \vec{u}_m\}$ which has the same span as $\{\vec{v}_1, \dots, \vec{v}_m\}$.

This process is called Gram-Schmidt.

Now let V be a subspace of \mathbb{R}^n , and say $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for V . We're going to try to project onto this subspace. Previously we only projected onto lines. Do Gram-Schmidt on those vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ to get an orthonormal set of vectors $\{\vec{u}_1, \dots, \vec{u}_m\}$ with the same span as the \vec{v}_i 's. Since the \vec{v}_i 's formed a basis for V , the \vec{u}_i 's also form a basis for V (since their span is V and they are LI).

For any $\vec{x} \in \mathbb{R}^n$, define $\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m$ the orthogonal projection of \vec{x} onto the subspace V .

We also use $\vec{x}'' = \underbrace{\text{proj}_V(\vec{x})}_{\text{"}\vec{x}\text{ parallel"}}, \vec{x}^\perp = \vec{x} - \vec{x}''.$



Then $\vec{x} = \vec{x}'' + \vec{x}^\perp$, and for any $i \in \{1, \dots, m\}$ we have:

$$\begin{aligned}\vec{x}^\perp \cdot \vec{u}_j &= (\vec{x} - \vec{x}^{\parallel}) \cdot \vec{u}_j = (\vec{x} \cdot \vec{u}_j) - (\vec{x}^{\parallel} \cdot \vec{u}_j) \\ &= (\vec{x} \cdot \vec{u}_j) - ((\vec{x} \cdot \vec{u}_j)(\vec{u}_j \cdot \vec{u}_j) + \dots + (\vec{x} \cdot \vec{u}_m)(\vec{u}_m \cdot \vec{u}_j)) \\ &\quad \text{by linearity of the dot product} \\ &= (\vec{x} \cdot \vec{u}_j) - (\vec{x} \cdot \vec{u}_j) = 0,\end{aligned}$$

only one that isn't killed off, since $\vec{u}_j \cdot \vec{u}_i = \delta_{ij}$

so \vec{x}^\perp is, true to its name, perpendicular to all the \vec{u}_i 's, and thus to everything in V , since V is spanned by the \vec{u}_i 's.

The representation $\vec{x} = \vec{x}^{\parallel} + \vec{x}^\perp$ is unique: if $\vec{x} = \vec{y} + \vec{z}$, where $\vec{y} \in V$ and \vec{z} is perpendicular to everything in V ,

then $\vec{y} = c_1 \vec{u}_1 + \dots + c_m \vec{u}_m$, since $\{\vec{u}_1, \dots, \vec{u}_m\}$ spans V ,

$$\begin{aligned}\text{so } (\vec{y} \cdot \vec{u}_i) &= (c_1 \vec{u}_1 + \dots + c_m \vec{u}_m) \cdot \vec{u}_i = c_i (\vec{u}_i \cdot \vec{u}_i) + \dots + c_m (\vec{u}_m \cdot \vec{u}_i) \\ &= c_i \text{ (all other terms get killed off)},\end{aligned}$$

which are the coefficients of the \vec{u}_i 's in the formula for $\text{proj}_V(\vec{x})$. The transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$T(\vec{x}) = \text{proj}_V(\vec{x})$, is linear: if $\lambda, \mu \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\text{then } T(\lambda \vec{x} + \mu \vec{y}) = \text{proj}_V(\lambda \vec{x} + \mu \vec{y}) = ((\lambda \vec{x} + \mu \vec{y}) \cdot \vec{u}_1) \vec{u}_1 + \dots + ((\lambda \vec{x} + \mu \vec{y}) \cdot \vec{u}_m) \vec{u}_m$$

by linearity of the dot product

$$\begin{aligned}&\stackrel{\text{by linearity of the dot product}}{=} (\lambda (\vec{x} \cdot \vec{u}_1) + \mu (\vec{y} \cdot \vec{u}_1)) \vec{u}_1 + \dots + (\lambda (\vec{x} \cdot \vec{u}_m) + \mu (\vec{y} \cdot \vec{u}_m)) \vec{u}_m \\ &= \lambda [\vec{x} \cdot \vec{u}_1] \vec{u}_1 + \dots + [\vec{x} \cdot \vec{u}_m] \vec{u}_m + \mu [\vec{y} \cdot \vec{u}_1] \vec{u}_1 + \dots + [\vec{y} \cdot \vec{u}_m] \vec{u}_m\end{aligned}$$

$= \lambda T(\vec{x}) + \mu T(\vec{y})$, so T is linear. Thus the orthogonal projection onto any subspace of \mathbb{R}^n is always linear.

The projection $\text{proj}_{\mathbb{R}^n}(\vec{x})$ onto \mathbb{R}^n is silly since it just maps every vector to itself, but our projection formula says that if $\{\vec{u}_1, \dots, \vec{u}_n\}$ is any orthonormal basis of \mathbb{R}^n (and we know, by gram-schmidtting any basis, that at least one orthonormal basis of \mathbb{R}^n exists), then

$$\vec{x} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{x} \cdot \vec{u}_n)\vec{u}_n.$$
 We call $V^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}$ the orthogonal complement of V . Since $\vec{x} \in V^\perp$ implies $\text{proj}_V(\vec{x}) = 0$ conversely and $\text{proj}_V(\vec{x}) = 0$ implies $\vec{x} \cdot \vec{u}_i = 0$ for every \vec{u}_i in some orthonormal basis of V , which in turn implies that $\vec{x} \cdot \vec{v}$ for every $\vec{v} \in V$ (since $\vec{v} \in V = \text{span}\{\vec{u}_1, \dots, \vec{u}_n\}$ and by linearity of the dot product), which means $\vec{x} \in V^\perp$, we get that V^\perp is exactly the kernel of the orthogonal projection map onto V . So V^\perp is a subspace of \mathbb{R}^n , and since $V = \text{im}(\underbrace{\text{proj}_V(\cdot)}_{\substack{\text{the projection map onto } V \\ (\text{orthogonal})}})$, by rank-nullity, we get $\dim(V) + \dim(V^\perp) = n$. If $\exists \vec{w} \in V \cap V^\perp$ then $\underbrace{\vec{w} \cdot \vec{w}}_{\in V^\perp \in V} = 0$, so $\|\vec{w}\|^2 = 0$, which means $\vec{w} = \vec{0}$. So $V \cap V^\perp = \{\vec{0}\}$.

18

By definition, $(V^\perp)^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in V^\perp\}$.

Surely $V \subset (V^\perp)^\perp$, since every $\vec{v} \in V$ is also in $(V^\perp)^\perp$.

Also, if $\vec{z} \in \mathbb{R}^n \setminus V$, in other words if \vec{z} is in \mathbb{R}^n but not V , then $\vec{z} = \underbrace{\vec{z}''}_{\in V} + \underbrace{\vec{z}^\perp}_{\in V^\perp}$, where $\vec{z}^\perp \neq \vec{0}$ (else \vec{z} would be in V), so $\vec{z} \cdot \vec{z}^\perp = (\vec{z}'' + \vec{z}^\perp) \cdot \vec{z}^\perp$

$$= \underbrace{\vec{z}'' \cdot \vec{z}^\perp}_{=0} + \underbrace{\vec{z}^\perp \cdot \vec{z}^\perp}_{= \|\vec{z}\|^2 \neq 0} \neq 0, \text{ so } \vec{z} \notin (V^\perp)^\perp. \text{ This means}$$

$(V^\perp)^\perp \subset V$. Thus $V = (V^\perp)^\perp$. Graphically, if $\vec{x}, \vec{y} \in \mathbb{R}^n$

are any perpendicular vectors in \mathbb{R}^n , then the Pythagorean theorem tells us that $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ (if $n=2$ or 3).



This is true in general (for any $n \in \mathbb{N}$), since:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &= \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2(\vec{x} \cdot \vec{y}) = \|\vec{x}\|^2 + \|\vec{y}\|^2 \end{aligned} \quad \begin{matrix} \text{"is perpendicular to"} \\ \downarrow \\ \text{if and only if } \vec{x} \perp \vec{y}! \end{matrix}$$

If V is a subspace of \mathbb{R}^n and $\vec{x} \in \mathbb{R}^n$, then $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$, and equality holds iff $\vec{x} \in V$: $\vec{x} = \underbrace{\vec{x}''}_{=\text{proj}_V(\vec{x})} + \underbrace{\vec{x}^\perp}_{\in V^\perp}$, so by Pythagorean

theorem, since $\vec{x}'' \perp \vec{x}^\perp$, we get: $\|\vec{x}\|^2 = \|\vec{x}''\|^2 + \|\vec{x}^\perp\|^2 \geq \|\vec{x}''\|^2 = \|\text{proj}_V(\vec{x})\|^2$, so $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$, and equality holds iff $\vec{x}^\perp = \vec{0}$, which happens iff $\vec{x} \in V$.

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. If $\vec{y} = \vec{0}$, then $|\underbrace{\vec{x} \cdot \vec{y}}_{\vec{0}}| = \|\vec{x}\| \|\vec{y}\|$. Otherwise, let $V = \text{span}(\vec{y})$. Then $\vec{x} = \vec{x}'' + \vec{x}^\perp$ (where $\vec{x}'' = \text{proj}_V(\vec{x})$), and by the Pythagorean theorem, $\|\vec{x}\|^2 = \|\vec{x}''\|^2 + \|\vec{x}^\perp\|^2$, since $\vec{x}'' \perp \vec{x}^\perp$. Let $\vec{u} = \frac{\vec{y}}{\|\vec{y}\|}$, so that $\|\vec{u}\| = 1$. Then

$$\begin{aligned} \vec{x}'' &= \text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}, \text{ so } \|(\vec{x} \cdot \vec{u})\vec{u}\| = |(\vec{x} \cdot \vec{u})| \underbrace{\|\vec{u}\|}_{=1} = |(\vec{x} \cdot \vec{u})| \\ &= \|\vec{x}''\|, \text{ so since } \vec{u} = \frac{\vec{y}}{\|\vec{y}\|}, \text{ this means } |(\vec{x} \cdot \frac{\vec{y}}{\|\vec{y}\|})| = \|\vec{x}''\|, \\ \text{so that } |(\vec{x} \cdot \vec{y})| &= \underbrace{\|\vec{x}''\|}_{\|\vec{x}\|} \|\vec{y}\| \leq \|\vec{x}\| \|\vec{y}\|, \text{ where equality holds} \\ \text{iff } \vec{x}^\perp &= \vec{0}, \text{ which happens iff } \vec{x} \in \text{span}(\vec{y}) = V. \text{ This} \\ \text{gives us the Cauchy-Schwarz inequality: for any } \vec{x}, \vec{y} \in \mathbb{R}^n, \\ |(\vec{x} \cdot \vec{y})| &\leq \|\vec{x}\| \|\vec{y}\|, \text{ where equality holds iff } \vec{x} \text{ & } \vec{y} \text{ are collinear} \\ \text{(aka. parallel).} \end{aligned}$$

4. Gram-Schmidt & QR factorization

nearly independent

We already talked about Gram-Schmidt. Given LI vectors $\vec{v}_1, \dots, \vec{v}_m$ lying in \mathbb{R}^n , we obtained an orthonormal set of vectors $\vec{u}_1, \dots, \vec{u}_m$ in \mathbb{R}^n such that $\forall k \in \{1, \dots, m\}$, $\text{span}(\vec{u}_1, \dots, \vec{u}_k) = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$, by the construction. Furthermore since we had $\vec{w}_k = \vec{v}_k - (\vec{v}_k \cdot \vec{u}_1)\vec{u}_1 - \dots - (\vec{v}_k \cdot \vec{u}_{k-1})\vec{u}_{k-1}$ and $\vec{u}_k = \frac{\vec{w}_k}{\|\vec{w}_k\|}$ for each $k = 1, \dots, m$, this means that each \vec{v}_k is a linear combination of $\vec{u}_1, \dots, \vec{u}_k$, where the weight of \vec{u}_k is positive.

10

So for each $k \in \{1, \dots, m\}$, we have:

$$\vec{v}_k = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_k & \cdots & \vec{u}_m \end{bmatrix}}_{n \times m \text{ matrix}} \begin{bmatrix} \approx \\ \vdots \\ \star \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{\text{k^{th} entry}} \text{m} \times 1 \text{ column vector}$$

by the interpretation of a matrix times a vector being a weighting of the matrix's columns using the corresponding elements of the vector as the weights. Here, " \approx " represents some real number, and " \star " represents a positive number, per the discussion at the end of the previous page. Now, stacking the \vec{v}_i 's together, we get:

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}}_{n \times m \text{ matrix}} = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}}_{n \times m \text{ matrix}} \underbrace{\begin{bmatrix} \star & \approx & \approx & \approx & \approx \\ 0 & \star & \approx & \approx & \approx \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}}_{m \times m \text{ matrix, with positive diagonal elements}}$$

This is the QR factorization of matrix M with

LI columns $\vec{v}_1, \dots, \vec{v}_m$ in \mathbb{R}^n (of course, this means $m \leq n$).

The representation is unique: $\vec{v}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$ is forced, since no other

unit vector is a positive multiple of \vec{v}_1 . Then $\vec{v}_2 = c_1 \vec{u}_1 + c_2 \vec{u}_2$,

$\{\vec{u}_1, \vec{u}_2\}$ orthonormal, forces $(\vec{v}_2 \cdot \vec{u}_1) = (c_1 \vec{u}_1 + c_2 \vec{u}_2) \cdot \vec{u}_1 = c_1 (\vec{u}_1 \cdot \vec{u}_1) + c_2 (\vec{u}_2 \cdot \vec{u}_1) = c_1$

so if $\vec{w}_2 = \vec{v}_2 - (c_1 \vec{u}_1) \vec{u}_1$, then \vec{u}_2 must be a positive multiple of \vec{w}_2 , which forces $\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$, etc. So we just "recover" the Gram-Schmidt algorithm, and so this means that the QR factorization always exists & is unique.

In practice, what's the best way to obtain R ?
 (we already know how to get Q - just gram-schmidt the matrix M 's columns). One (painful) way is to keep track of all the coefficients in each step of gram-schmidt. Another (not as bad, but still awful) way is to write out $\vec{v}_i = c_1 \vec{u}_1 + \dots + c_i \vec{u}_i$ and dot both sides with $\uparrow i^{\text{th}} \text{ column of } M$

\vec{u}_k for $k=1, \dots, i$ to get each c_k and painstakingly record each c_k in the $(k, i)^{\text{th}}$ entry of R . But thankfully, there is another way... Do gram-schmidt on M to get Q .

$Q = [\vec{u}_1 \dots \vec{u}_m]$, the \vec{u}_i 's orthonormal. What happens when we left-multiply Q by Q^T ? (Q is $n \times m$, Q^T is $m \times n$, $m \leq n$)

$$Q^T Q = \underbrace{\begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} | & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & | \end{bmatrix}}_{n \times m} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \dots & \vec{u}_1 \cdot \vec{u}_m \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & & \\ \vdots & & \ddots & \\ \vec{u}_m \cdot \vec{u}_1 & \dots & \dots & \vec{u}_m \cdot \vec{u}_m \end{bmatrix}$$

$$= \begin{bmatrix} I_m \\ 0 \end{bmatrix} = I_m,$$

since $\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$. Thus we know $M = QR$

for some $m \times m$ matrix R , so left-multiplying by Q^T gives:

$Q^T M = Q^T Q R = I_m R = R$. So to find R , you can just use this trick. Now let's solve some problems.

5. Problems

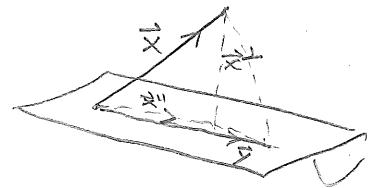
a) Let V be a subspace of \mathbb{R}^n , and let $\vec{x} \in \mathbb{R}^n$. Prove that $\text{proj}_V(\vec{x})$ is the closest vector in V to \vec{x} .

Soln: suppose $\vec{y} \in V$. Draw a picture:

$$(\vec{x} - \vec{y}) = (\vec{x}'' - \vec{y}) + (\vec{x}^\perp),$$

$\in V$, since V
is a subspace

\perp to $(\vec{x}'' - \vec{y})$, a vector in V



So the Pythagorean theorem tells us that $\|\vec{x} - \vec{y}\|^2 = \|\vec{x}'' - \vec{y}\|^2 + \|\vec{x}^\perp\|^2$.

Since $\|\vec{x} - \text{proj}_V(\vec{x})\| = \|\vec{x} - \vec{x}''\| = \|\vec{x}^\perp\|$, we get that

$$\begin{aligned} \|\vec{x} - \vec{y}\|^2 &= \|\vec{x}'' - \vec{y}\|^2 + \|\vec{x}^\perp\|^2 = \|\vec{x}'' - \vec{y}\|^2 + \|\vec{x} - \text{proj}_V(\vec{x})\|^2 \\ &\geq \|\vec{x} - \text{proj}_V(\vec{x})\|^2 \text{ with equality iff } \vec{y} = \vec{x}'' = \text{proj}_V(\vec{x}). \end{aligned}$$

Thus, $\text{proj}_V(\vec{x})$ is the closest vector in V to \vec{x} . ^{"if & only if"}

b) $B = \{1, t, t^2\}$ is the standard basis for P_2 . If $T: P_2 \rightarrow P_2$, $T(f(t)) = f(-t)$, find if T is linear, and if so, find the B -matrix for T , and bases for $\ker(T)$ & $\text{im}(T)$.

Soln: $\forall \lambda, \mu \in \mathbb{R}$, $f, g \in P_2$, we have $T((\lambda f + \mu g)(t)) = (\lambda f + \mu g)(-t)$
 $= \lambda f(-t) + \mu g(-t) = \lambda T(f(t)) + \mu T(g(t))$, so T is linear.

$$[T]_B = B = [T(1)]_B \quad [T(t)]_B \quad [T(t^2)]_B = [[1]]_B \quad [t]_B \quad [t^2]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

what I use to represent
the B -matrix for T

which is invertible. So T is also invertible, hence an isomorphism (= linear bijection), so $\ker(T) = \{0\}$, $\text{im}(T) = \text{span}(1, t, t^2)$

\emptyset = the empty set = basis for $\ker(T)$.

invertible, a.k.a. one-to-one & onto basis for $\text{im}(T)$

c) Same as the previous problem, but $T(f) = f'' + 4f'$.

Soln: $\forall \lambda, \mu \in \mathbb{R}, f, g \in P_2, T(\lambda f + \mu g) = (\lambda f + \mu g)'' + 4(\lambda f + \mu g)' = \lambda f'' + \mu g'' + 4\lambda f' + 4\mu g' = \lambda T(f) + \mu T(g)$, so T is linear.

$$[T]_B = B = \begin{bmatrix} [T(1)]_B & [T(t)]_B & [T(t^2)]_B \end{bmatrix} = \begin{bmatrix} [0]_B & [4]_B & [2+8t]_B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} . \text{ rref}(B) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ after a few elementary row ops,}$$

$$\text{so this shows } \ker(B) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} x_1 = \text{free} \\ x_2 = 0 \\ x_3 = 0 \end{array}\right), \text{ and}$$

since the pivots of $\text{rref}(B)$ are in the 2nd & 3rd columns, it must be that the 2nd & 3rd columns of B form a basis for $\text{im}(B)$:

$$\text{im}(B) = \text{span}\left(\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \text{ so switching to}$$

"vectors" in P_2 (which are in fact polynomials) from their B -coordinates,

$$\text{we get: } \ker(T) = \text{span}(\underbrace{1}_{\text{basis for } \ker(T)}), \quad \text{im}(T) = \text{span}(\underbrace{1+t}_{\text{basis for } \text{im}(T)}).$$

Since T has nontrivial kernel, it is not one-to-one, therefore not invertible, therefore not an isomorphism. You could also say that since B isn't invertible, neither is T .

d) Let V be the plane $x_1 + 2x_2 + 3x_3 = 0$ in \mathbb{R}^3 . If V has basis

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \right\}$, find the \mathcal{B} -matrix for $T: V \rightarrow V$, where

T is the orthogonal projection onto the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$\text{Soln: } [T]_{\mathcal{B}} = B = \begin{bmatrix} [T(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix})]_{\mathcal{B}} & [T(\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix})]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{bmatrix}_{\mathcal{B}}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2/9 & -14/9 \\ -1/9 & 7/9 \end{bmatrix} \end{bmatrix}_{\mathcal{B}}, \text{ since } \begin{array}{c|ccc} 1 & 5 & 1 & 1 \\ -1 & 4 & -2 & -1 \\ 1 & 1 & 1 & 2 \end{array} \xrightarrow{\text{II} \leftrightarrow \text{III}} \begin{array}{c|ccc} 1 & 5 & 1 & 1 \\ -1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 2 \end{array} \xrightarrow{\text{III} \cdot 2} \begin{array}{c|ccc} 1 & 5 & 1 & 1 \\ -1 & 4 & -2 & -1 \\ 0 & 0 & 1 & 4 \end{array} \xrightarrow{\text{III} \leftrightarrow \text{II}} \begin{array}{c|ccc} 1 & 5 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 4 \end{array} \xrightarrow{\text{II} \cdot (-2)} \begin{array}{c|ccc} 1 & 5 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array} \end{array}, \text{ so } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -2/13 \\ 1/13 \\ 7/13 \end{bmatrix}_{\mathcal{B}}$$

- e) There exists a 3×3 matrix P such that the linear transformation $T(M) = MP - PM$ from $\mathbb{R}^{3 \times 3}$ to $\mathbb{R}^{3 \times 3}$ is an isomorphism.
True or false?

Soln: False. Suppose such a P exists. Then $T(P) = PP - PP = 0$, so if $P \neq 0$, then T has nontrivial kernel, and is thus not an isomorphism. If $P = 0$, then $T(I_3) = I_3 0 - 0 I_3 = 0$, so again T has nontrivial kernel, and is hence not an isomorphism.

- f) If the image of linear map T is infinite dimensional, then the domain of T must be infinite dimensional. True or false?

Soln: True. Suppose the domain of T is finite dimensional, with basis, say, $\{\vec{v}_1, \dots, \vec{v}_n\}$. Then $\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is a spanning set for $\text{im}(T)$, by linearity of T , so it can be reduced to a basis for $\text{im}(T)$ with finitely many vectors, contradicting that it's infinite dimensional.

- g) There exists a linear map T from P_6 to P_6 such that the kernel of T is isomorphic to the image of T . True or false?

Soln: False. $\dim(P_6) = 7$, so use rank-nullity to say $\dim(\text{im}(T)) + \dim(\ker(T)) = \dim(P_6) = 7$, but since $\ker(T)$ is isomorphic to $\text{im}(T)$, they have the same dimension, $\frac{7}{2}$, which cannot be.

h) Prove the triangle inequality: if $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

$$\text{Solu: } \|\vec{x} + \vec{y}\|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \leq \vec{x} \cdot \vec{x} + 2|\langle \vec{x}, \vec{y} \rangle| + \vec{y} \cdot \vec{y}$$

$$\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2, \text{ so taking square roots yields}$$

Cauchy-Schwarz $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

i) Find all vectors perpendicular to $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

$$\text{Solu: } \vec{x} \cdot \vec{v} = 0 \iff \vec{v}^T \vec{x} = 0 \iff \vec{x} \in \ker(\vec{v}^T)$$

So this is equivalent to finding the kernel of \vec{v}^T .
 treat as a matrix for the moment

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \iff x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$\begin{array}{l} \uparrow \\ x_1 = -2r - 3s - 4t \\ \downarrow \end{array}$$

$$x_2 = r$$

$$x_3 = s$$

$$x_4 = t$$

$$\iff \vec{x} \in \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

basis for the ker
of \vec{v}^T

j) Find a basis for W^\perp , where $W = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right)$.

Solu: as above, $W^\perp = \ker \left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \right)$. Do Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{-5(I)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \xrightarrow{\cdot(-\frac{1}{4})} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-2(II)}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \text{ so } \ker \left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \right) = \text{span} \left(\underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\text{basis}} \right) = W^\perp.$$

k) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ onto $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$

Soln: $V = \text{span}\left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}\right)$

orthonormal set, so no Gram-Schmidt necessary

$$\begin{aligned} \text{proj}_V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\right) \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}\right) \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}\right) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

l) If $\vec{u}_1, \dots, \vec{u}_5$ are orthonormal and live in \mathbb{R}^10 , find the length of $\vec{x} = 7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5$.

Soln: $\|\vec{x}\|^2 = \|7\vec{u}_1\|^2 + \underbrace{\|-3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5\|_2^2}_{\perp \vec{u}_i \text{ by assumption}} = 7^2 + 3^2 + 2^2 + 1^2 + 1^2 = 64$, so $\|\vec{x}\| = 8$.
(use Pythagorean theorem)

m) Let V be a subspace of \mathbb{R}^n and let $\vec{x} \in \mathbb{R}^n$. Define $\vec{y} = \text{proj}_V(\vec{x})$. What is the relationship between $\|\vec{y}\|^2$ and $\vec{y} \cdot \vec{x}$?

Soln: $\vec{x} = \vec{x}'' + \vec{x}^\perp$, so $\vec{y} \cdot \vec{x} = \vec{y} \cdot (\vec{x}'' + \vec{x}^\perp) = \vec{y} \cdot \vec{y} + \underbrace{\vec{y} \cdot \vec{x}^\perp}_0 = \|\vec{y}\|^2$,
so the two quantities are equal.

n) A plane V in \mathbb{R}^3 has an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$. Let $\vec{x} \in \mathbb{R}^3$. Find a formula for the reflection $R(\vec{x})$ of \vec{x} about the plane V .

Soln: $R(\vec{x}) = \vec{x}'' - \vec{x}^\perp = \vec{x}'' - (\vec{x} - \vec{x}'') = 2\vec{x}'' - \vec{x} = 2\text{proj}_V(\vec{x}) - \vec{x}$
 $= 2[(\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2] - \vec{x}$

o) Can you find a line L in \mathbb{R}^n and a vector $\vec{x} \in \mathbb{R}^n$ st. $\vec{x} \cdot \text{proj}_L(\vec{x}) < 0$?

Soln: No, because $\underbrace{\vec{x} \cdot \text{proj}_L(\vec{x})}_{=\vec{x}''} = (\vec{x}'' + \vec{x}^\perp) \cdot \vec{x}'' = \vec{x}'' \cdot \vec{x}'' + \underbrace{\vec{x}^\perp \cdot \vec{x}''}_0 = \|\vec{x}''\|^2 \geq 0$.

p) Gram-Schmidt the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

$$\text{Soln: } \vec{u}_1 = \frac{1}{\sqrt{1+1+1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$\text{so } \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{\frac{1}{2}}} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{bmatrix} \quad (\vec{w}_2 \text{ already had unit length}).$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right) - \left(\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \left(\frac{1}{2}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \text{so } \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \text{so our orthonormal set obtained via Gram-Schmidt is: } \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

q) Get the QR factorization of the matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Soln: use the previous question to get Q :

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \text{so } M = QR \Leftrightarrow Q^T M = Q^T Q R = R. \quad R = \underbrace{Q^T Q}_{= I_3} R$$

$$R = Q^T M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 2 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $M = QR$, where Q and R have been specified.

(Q has orthonormal columns & R is upper triangular with positive diagonal elements).

r) Find an orthonormal basis of $\ker(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Soln: by inspection, $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ works as our orthonormal basis for $\ker(A)$, which has dimension 2 since $\text{rref}(A)$ has 2 pivots & by rank-nullity. We could also find any basis for $\ker(A)$ and then do Gram-Schmidt on those basis vectors.

