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math 33a
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Teaching Plan, week 5

- Plan for today:
1. Midterm 1 stats & pass back
 2. Brief review of dimension & coordinates in \mathbb{R}^n
 3. Quiz (15 minutes)
 4. Brief refresher on vector spaces, linear transformations, and isomorphisms
 5. Group problems (Solve what we don't cover in class at home!)

1. Midterm 1 stats & pass back

Rough statistics: mean \sim mid 50's
 median \sim mid 50's
 standard deviation \sim around 19

Score	grade
≥ 70	at least A-
≥ 55	at least B-
≥ 40	at least C-

How to improve for Midterm 2 & the Final:

SOLVE PROBLEMS!!! Check solutions only after attempting problems yourself. Master homework and quiz problems, as well as problems in my notes, and proofs in lecture. If the professor says "you can try this at home," or something to that effect, actually solve that problem at home! Email me/attend OH if you're stuck on problems.

2. Brief review of dimension & coordinates in \mathbb{R}^n . Let V be a subspace of \mathbb{R}^n .

If $\vec{v}_1, \dots, \vec{v}_p \in V$ are linearly independent and $\vec{w}_1, \dots, \vec{w}_q \in V$ span V , then $\vec{v}_1 = \sum_{i=1}^q c_i \vec{w}_i$, where not all the c_i 's are 0 (since $\vec{v}_1 \neq \vec{0}$). So we can isolate one of the w_i 's, say w_1 (after relabelling them), so $\vec{w}_1 \in \text{span}\{\vec{v}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_q\}$, so $\{\vec{v}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_q\}$ still spans V .

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Then $\vec{v}_2 = \sum_{i=2}^q c_i^2 \vec{w}_i + c_1^2 \vec{v}_1$ for some constants $c_i^2, i=1, \dots, q$. If all c_i^2 are 0 for $i>1$, then \vec{v}_2 would be a multiple of \vec{v}_1 , which is impossible, since the \vec{v}_i 's are linearly independent. So one of the \vec{w}_i 's for $i\geq 2$ can be isolated. Relabel the \vec{w}_i 's so that this \vec{w}_i is \vec{w}_2 .

Then $\vec{w}_2 \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4, \dots, \vec{w}_q\}$, so $\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \dots, \vec{w}_q\}$ still spans V . Continue in this manner, ending with the set

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_{p+1}, \vec{w}_{p+2}, \dots, \vec{w}_q\}$ that still spans V .

This is the "replacement argument," and it shows that $p \leq q$.

Note that if $q < p$, then $\{\vec{v}_1, \dots, \vec{v}_q\}$ would span V by replacement, so \vec{v}_{q+1} would be in $\text{span}\{\vec{v}_1, \dots, \vec{v}_q\}$, contradicting that the \vec{v}_i 's are linearly independent. Thus, spanning sets always have at least as many vectors as linearly independent sets do. Then if

$\{\vec{v}_1, \dots, \vec{v}_p\}$ and $\{\vec{w}_1, \dots, \vec{w}_q\}$ are both bases for V , we know each of these bases is both a linearly independent and a spanning set.

So $p \leq q$ and $p \geq q$, so $p = q$. If $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an LI set in $V \subset \mathbb{R}^n$, then we can always extend this to a basis for V :

If $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$, then we're done. Otherwise, pick $\vec{v}_{p+1} \in V$ that isn't in $\text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$. $\{\vec{v}_1, \dots, \vec{v}_{p+1}\}$ must then be LI, since $\sum_{i=1}^{p+1} c_i \vec{v}_i = \vec{0}$ forces $c_{p+1} = 0$, which then forces $c_i = 0$ for $i=1, \dots, p$, by linear independence of $\{\vec{v}_1, \dots, \vec{v}_p\}$. Continue adding more vectors in V

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Eventually we have to span V with these increasing LI sets, since any n LI vectors in \mathbb{R}^n have to span \mathbb{R}^n (the $n \times n$ matrix whose columns are those LI vectors must have trivial kernel, therefore rref equal to I_n , therefore image equal to \mathbb{R}^n , which means those columns span \mathbb{R}^n).

Since $V \subset \mathbb{R}^n$, eventually V will be spanned. So any LI set of vectors in V can be extended to a basis for V . Similarly, by removing redundant vectors one-by-one from any spanning set of vectors in V , we can shrink any spanning set in V to a basis for V . So any subspace V of \mathbb{R}^n always admits a basis, and any two bases for V must have the same number of vectors in it. So the dimension of V , denoted $\dim(V)$, which equals the number of vectors in any basis for V , always exists and is well-defined! Now given $n \times k$ matrix A , how can we find bases for $\ker(A)$, which is a subspace of \mathbb{R}^k , and $\text{im}(A)$, which is a subspace of \mathbb{R}^n ? rref(A) will give us these.

$\text{rref}(A) = E_p \cdots E_1 A \vec{x} = \vec{0}$ for some elementary $n \times n$ matrices E_i , so $E_p \cdots E_1 A \vec{x} = \vec{0}$ implies $A \vec{x} = E_1^{-1} \cdots E_p^{-1} \vec{0} = \vec{0}$ (since elementary matrices are invertible), so $\vec{x} \in \ker(\text{rref}(A)) \Rightarrow \vec{x} \in \ker(A)$. Also, $A \vec{x} = \vec{0}$ implies $E_p \cdots E_1 A \vec{x} = E_p \cdots E_1 \vec{0} = \vec{0}$, so $\vec{x} \in \ker(A) \Rightarrow \vec{x} \in \ker(\text{rref}(A))$. So $\ker(A) = \ker(\text{rref}(A))$! This is a very important fact.

This means if some collection of $\text{rref}(A)$'s columns is linearly dependent, then the corresponding columns of A obey the same linear dependency (with the same column weights, these weights forming vectors in the kernel common to both A and $\text{rref}(A)$), and vice-versa. Since pivot columns of $\text{rref}(A)$ are clearly LI (the leading one's can't be formed from any pivot column appearing to the left), and the remaining columns are in the span of the pivot columns, we get that the columns of A corresponding to the pivot columns of $\text{rref}(A)$ are LI, and every other column of A is in the span of these columns. So a basis for $\text{im}(A)$ is the columns of A corresponding to the pivot columns of $\text{rref}(A)$. A basis for $\ker(A)$ is easily found by finding a basis for $\text{rref}(A)$ (the same basis will work!). Every non-pivot column of $\text{rref}(A)$ corresponds to a free variable, which corresponds to a vector in a basis for $\ker(\text{rref}(A))$. Every column of $\text{rref}(A)$ is either a pivot column or a non-pivot column. The number of pivot columns = $\underbrace{\dim(\text{im}(A))}_{\begin{array}{l}=\text{rank}(A) \\ =\text{rank of } A\end{array}}$ and the number of non-pivot columns = $\underbrace{\dim(\ker(A))}_{\begin{array}{l}=nul(A) \\ =\text{nullity of } A\end{array}}$.

Since $\text{rref}(A)$ and A have the same number of columns, this means:

$\text{rank}(A) + \text{nul}(A) = \# \text{ of columns of } A$. This known as the rank-nullity theorem.

If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of \mathbb{R}^n , then every $\vec{x} \in \mathbb{R}^n$ has a unique representation as $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$, and every weight vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ corresponds to a vector $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n \in \mathbb{R}^n$.

But how do we go from $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ = the B -coordinate vector of \vec{x} to \vec{x} , and vice-versa? Simply by putting $S = [\vec{v}_1 \dots \vec{v}_n]$. Then S is the $n \times n$ (invertible) matrix which takes in B -coordinates (weights) and splits out the true vector in \mathbb{R}^n that those weights correspond to:

$$S \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{v}_1 \dots \vec{v}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n. S^{-1}$$
 must then take

in actual vectors in \mathbb{R}^n and splits out the B -coordinates of those true vectors, since inputs & outputs are switched for inverses. Thus

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1} \vec{x}. \text{ If } T: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is a linear transformation with}$$

matrix representation A , then $T(\vec{x}) = A\vec{x}$, so $[T(\vec{x})]_B = S^{-1}A\vec{x}$,

$$\text{and since } \vec{x} = S[\vec{x}]_B, \text{ we get } [T(\vec{x})]_B = S^{-1}AS[\vec{x}]_B$$

$$= S^{-1}A[\vec{v}_1 \dots \vec{v}_n][\vec{x}]_B = S^{-1}[A\vec{v}_1 \dots A\vec{v}_n][\vec{x}]_B$$

$$= [S^{-1}A\vec{v}_1 \dots S^{-1}A\vec{v}_n][\vec{x}]_B = [[A\vec{v}_1]_B \dots [A\vec{v}_n]_B][\vec{x}]_B$$

$$= [[T(\vec{v}_1)]_B \dots [T(\vec{v}_n)]_B][\vec{x}]_B \text{ gives } T \text{ in } B\text{-coordinates}$$

when its input is also given in B -coordinates. $B = S^{-1}AS$ is the "B-matrix of T "; it takes in the B -coordinates of \vec{x} , and outputs the B -coordinates of $T(\vec{x})$.

For any $n \times n$ matrices A & B , if \exists invertible $n \times n$ matrix S such that $B = S^{-1}AS$, then we say A & B are similar.

In such a scenario, B always represents the B -matrix of T , where $T(\vec{x}) = A\vec{x}$, and \mathcal{B} is the basis of \mathbb{R}^n consisting of S 's columns.

If $B = S^{-1}AS$ and $A = U^{-1}CU$, then $B = S^{-1}U^{-1}CUS = (US)^{-1}CUS$, so similarity is transitive (it's also clearly reflexive & symmetric).

A & B are similar iff they represent the same linear transformation with respect to different bases. For now, check if two matrices are similar by the brute force approach, i.e. trying to solve for every element of S s.t. $B = S^{-1}AS$ (or, equivalently, $SB = AS$ and S is invertible), or by reasoning that A & B represent the same linear transform in different coordinates. More sophisticated tools which we'll develop later will enable us to tackle these problems with more grace in the future.

3. Quiz (15 mins)

4. Brief refresher on vector spaces; linear transformations, and isomorphisms

Vector spaces are spaces where we can add vectors & multiply vectors by constants, and have 8 nice properties that we are used to having in \mathbb{R}^n (see pg. 167 of textbook for them).

"Vectors" can be very general! For instance, they can be sequences of real numbers, functions, or matrices, just to name a few possibilities.

Subspaces, span, linear independence, basis, and coordinates all have an analogous meaning for general vector spaces as they do for \mathbb{R}^n . If V is finite-dimensional, it's often easier to use coordinates to prove things about V .

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by using things that we know about \mathbb{R}^n . For finite-dimensional vector spaces, the proof of the existence & uniqueness of dimension given at the start can be directly applied here too, since there was no mention of matrices, rref, etc.

Linear transforms, image, and kernel also extend to general vector spaces in the obvious manner, as do rank and nullity for finite-dimensional vector spaces, where the rank-nullity theorem holds (can be proven using isomorphisms). Isomorphisms are just invertible linear transforms between vector spaces. One nice property of isomorphisms is they take bases in one vector space to bases in another vector space. Two finite-dimensional vector spaces are isomorphic iff they have the same dimension. (There's an isomorphism that maps the basis vectors in one of the vector spaces to basis vectors in the other vector space).

All right — enough talk! Time to solve some problems.

5. Group problems

a) Find bases for $m(A)$ and $\ker(A)$, where $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

$$\text{Sln: } \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{-2(I)} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-(II)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \text{rref}(A),$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $m(A)$, and $A\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} x_1 = -t \\ x_2 = -2t \\ x_3 = t \end{cases} \Leftrightarrow \vec{x} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \text{ so } \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \ker(A).$$

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b) Find bases for $\text{im}(A)$ and $\ker(A)$, where $A = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Soln: A is already in mref. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\text{im}(A)$.

$$A\vec{x} = \vec{0} \Leftrightarrow \begin{aligned} x_1 &= 2s + t \\ x_2 &= s \\ x_3 &= -5t \\ x_4 &= t \\ x_5 &= 0 \end{aligned} \Leftrightarrow \vec{x} = s \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \text{ so } \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ forms}$$

a basis for $\ker(A)$.

c) Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -4 \\ -8 \end{bmatrix} \right\}$ a basis for \mathbb{R}^4 ?

Soln: Put $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 4 \\ 1 & 1 & 8 & -8 \end{bmatrix}$ and find if $\text{rref}(A) = I_4$ or not.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 4 \\ 1 & 1 & 8 & -8 \end{bmatrix} \xrightarrow{(I)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 1 & 2 & 4 & 4 \\ 1 & 1 & 8 & -8 \end{bmatrix} \xrightarrow{(1/3)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 8 & -8 \end{bmatrix} \xrightarrow{(II)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 6 & -6 \end{bmatrix} \xrightarrow{(-6)(III)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -12 \end{bmatrix}, \text{ which we can}$$

already tell will have a rref of I_4 , so A 's columns are LI and span \mathbb{R}^4 , so these vectors do form a basis for \mathbb{R}^4 .

d) Find a basis of the subspace of \mathbb{R}^3 defined by the equation

$$2x_1 + 3x_2 + x_3 = 0.$$

Soln: $x_1 = -\frac{3}{2}s - \frac{1}{2}t$, so $\vec{x} = s \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ captures all the solutions to the given equation, so a basis for these solutions is $\left\{ \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

e) If $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, what is the dimension of the space W of all vectors in \mathbb{R}^n that are perpendicular to \vec{v} ?

Soln: $\vec{v} \cdot \vec{x} = 0 \Leftrightarrow \underbrace{\vec{v}^\top \vec{x}}_{\text{rank } 1 \text{ } 1 \times n \text{ matrix}} = 0 \Leftrightarrow \vec{x} \in \ker(\vec{v}^\top)$, so by rank-nullity, $1 + \underbrace{\ker(\vec{v}^\top)}_{= W} = n$, so our answer is $\dim W = n-1$.

f) Find a 2×2 matrix A with $\text{im}(A) = \ker(A)$.

Soln: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\ker(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \text{im}(A)$.

g) Can you find a 3×3 matrix A with $\text{im}(A) = \ker(A)$?

Soln: nope: $\dim(\text{im}(A)) = \dim(\ker(A))$ would be forced,

$$\text{so } \dim(\text{im}(A)) + \dim(\ker(A)) = 2 \cdot \dim(\text{im}(A)) = 3$$

by rank-nullity. But $\dim(\text{im}(A))$ is an integer, a contradiction.

So no such matrix A can exist.

h) If A is $n \times n$ and $\text{im}(A) = \ker(A)$, what can you say about n ?

Soln: $\dim(\text{im}(A)) = \dim(\ker(A))$, so $n = \dim(\text{im}(A)) + \dim(\ker(A))$

$= 2 \dim(\text{im}(A))$, so n is even. Note: if n is even,

$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ works as our A (blank spaces are 0's).

i) A is 5×5 , B is 5×4 , C is 4×5 , and $A = BC$.

Is A invertible?

Soln: no, since $\dim(\text{im}(C)) \leq 4$ (since C outputs to \mathbb{R}^4), and therefore

$5 = \underbrace{\dim(\text{im}(C))}_{\leq 4} + \dim(\ker(C))$, so $\dim(\ker(C)) \geq 1$, so C

has nontrivial kernel, and thus so does A ($\ker(C) \subset \ker(A)$),

because $C\vec{x} = \vec{0}$ implies $A\vec{x} = BC\vec{x} = B\vec{0} = \vec{0}$, so A can't be

invertible.
j) If $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$ is a basis for \mathbb{R}^3 , find

the B -matrix for T , B . ($T(\vec{x}) = A\vec{x}$)

Soln: Put $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 6 & 6 \end{bmatrix}$. Then $B = S^{-1}AS$. S^{-1} is hard to compute, so write

$$SB = AS = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 12 \end{bmatrix}$$

and solve for B using elementary row ops:

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Performing Gaussian elimination on $[S | \begin{smallmatrix} 0 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & 3 & 12 \end{smallmatrix}]$ will let us

Find B: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 & 2 & 6 \\ 1 & 3 & 6 & 0 & 3 & 12 \end{array} \right] \xrightarrow{-\text{(I)}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 4 \\ 0 & 2 & 5 & 0 & 2 & 10 \end{array} \right] \xrightarrow{-\text{(II)}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array} \right] \xrightarrow{-2\text{(III)}}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array} \right], \text{ so } B = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right].$$

k) If \vec{v}_1 and $\vec{v}_2 \in \mathbb{R}^3$ are perpendicular unit vectors,
 $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ ($\text{so } \vec{v}_3 \in \mathbb{R}^3$ is a unit vector perpendicular to both
 \vec{v}_1 & \vec{v}_2), $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of \mathbb{R}^3 , and $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$,
 $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_2$, find B, the \mathcal{B} -matrix for T,
and interpret T geometrically.

Soh: $B = [T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}} = [\vec{v}_1 - 2\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_3]_{\mathcal{B}}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } T \text{ shears inputs in the } \vec{v}_1 \text{-coordinate according to the } \vec{v}_2 \text{-coordinate}$$

l) Find a basis \mathcal{B} of the plane $2x_1 - 3x_2 + 4x_3 = 0$ such that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
for $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

Soh: $2x_1 - 3x_2 + 4x_3 = 0 \iff \begin{cases} x_1 = 3s - 2t \\ x_2 = 2s \\ x_3 = t \end{cases} \iff \vec{x} = s \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

If \vec{v}_1, \vec{v}_2 are the basis vectors in \mathcal{B} , then $\vec{v}_1 = s_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ for some s_1, t_1 ,
and $\vec{v}_2 = s_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Also, $2\vec{v}_1 + 3\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, so $(2s_1 + 3s_2) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + (2t_1 + 3t_2) \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

For some s_2, t_2 .
 $\begin{cases} 2s_1 + 3s_2 = 0 \\ 2t_1 + 3t_2 = -1 \end{cases} \quad \text{choose } s_1 = 3, s_2 = 2 \quad (s_1, s_2 \neq 0 \text{ would make } \vec{v}_1, \vec{v}_2 \text{ LD})$

$\begin{cases} 2s_1 + 3s_2 = 0 \\ 2t_1 + 3t_2 = -1 \end{cases} \quad t_1 = 1, t_2 = -1. \text{ Then } \vec{v}_1 = \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \text{ works,}$

so $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \right\}$. There are many possible answers here, depending on our choice of s_1 & t_1 , (since then s_2 & t_2 are determined).