

Teaching Plan, week 4

Plan for today: 1. go over midterm  
 2. group problems

1. go over midterm:

Problem 1 a)  $A$  is  $n \times n$  invertible. Prove  $(A^T)^{-1} = (A^{-1})^T$ .  
 b) Find inverse of  $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ .

Solution: a) We don't know that  $(A^T)^{-1}$  exists yet, so we can't write it down.

$$A^T(A^{-1})^T = (A^{-1}A)^T = (I_n)^T = I_n, \text{ so } A^T \text{ is invertible and } (A^T)^{-1} = (A^{-1})^T.$$

$I_n$  is diagonal

if  $B$  is  $n \times k$  and  $C$  is  $k \times p$ , then for  $1 \leq i \leq n$  and  $1 \leq j \leq p$ , we have:

$$((BC)^T)_{ji} = (BC)_{ij} = \sum_{\ell=1}^k B_{i\ell} C_{\ell j} = \sum_{\ell=1}^k (C^T)_{j\ell} (B^T)_{\ell i} = (C^T B^T)_{ji}, \text{ so } (BC)^T = C^T B^T.$$

$$b) \left( \begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \uparrow \\ \downarrow \end{array} \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array} \right) \xrightarrow{-4(I)+3(II)} \left( \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array} \right) \cdot \left( \frac{1}{2} \right)$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array} \right) \begin{array}{l} -3(III) \\ -2(III) \end{array} \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array} \right)$$

Problem 2 a) Find the matrix of the linear transform that first projects onto  $y=x$  (in  $\mathbb{R}^2$ ), then rotates ccw by  $\frac{\pi}{4}$  radians, then scales by a factor  $k > 0$ .

b) Let  $\vec{w}$  be a vector on a line  $L$  in  $\mathbb{R}^2$  passing through the origin. Let  $\vec{v} \in \mathbb{R}^2$ . Let  $A$  be the  $2 \times 2$  matrix whose columns are the vectors  $\vec{w}$  and  $\text{proj}_L(\vec{v})$ . Is  $A$  invertible? Prove your answer.

Solution: a) For  $\vec{x} \in \mathbb{R}^2$  this transform  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  outputs:

$$T(\vec{x}) = \underbrace{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}_{\text{scaling by } k} \underbrace{\begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}}_{\text{rotating ccw } \pi/4 \text{ rad}} \underbrace{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}}_{\substack{\uparrow \text{projecting onto span}([1]) \\ (u_1, u_2) = \frac{1}{\sqrt{2}}(1, 1)}} \vec{x} = \frac{k}{2\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x}$$

$$= \frac{k}{2^{3/2}} \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 & 0 \\ \frac{k}{\sqrt{2}} & \frac{k}{\sqrt{2}} \end{pmatrix} \vec{x}, \text{ so the matrix representation}$$

for  $T$  is  $\begin{pmatrix} 0 & 0 \\ \frac{k}{\sqrt{2}} & \frac{k}{\sqrt{2}} \end{pmatrix}$ .

b) Intuitively: no,  $A$  cannot be invertible, because it has nontrivial kernel since its columns are linearly dependent, which is due to the fact that  $\text{proj}_L(\vec{v})$  and  $\vec{w}$  must be collinear.

explicitly:  $A = (\vec{w} \quad \text{proj}_L(\vec{v})) = \left( \vec{w} \quad \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} \right)$ , so

$$A \begin{pmatrix} \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \\ -1 \end{pmatrix} = \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} - \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = \vec{0}, \text{ so } A \text{ has a nontrivial}$$

kernel, and thus is not injective, and thus is not invertible.

Problem 3 a) Find the matrix of the linear transform  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

b) Let  $A$  be a given  $3 \times 3$  matrix, and  $\vec{v}$  be a given vector in  $\mathbb{R}^3$ .

Is the transformation  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $F(\vec{y}) = \vec{v} \times \vec{y} + A\vec{y}$  for all  $\vec{y} \in \mathbb{R}^3$ , where  $\vec{v} \times \vec{y} = \begin{bmatrix} v_2 y_3 - v_3 y_2 \\ v_3 y_1 - v_1 y_3 \\ v_1 y_2 - v_2 y_1 \end{bmatrix}$  for  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ ,  $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ ,

a linear transformation?

Solution: a)  $T$  is linear, so it has a matrix representation  $A$ .

$$\text{Then } A\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = A\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \left(A\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad A\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \left(T\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 9 & 2 \\ 3 & 4 \end{pmatrix}, \text{ so } A = \begin{pmatrix} 9 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 9 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \cdot \frac{1}{6-1}$$

$$= \frac{1}{5} \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 16/5 & -3/5 \\ 2/5 & 9/5 \end{pmatrix}.$$

$$b) \forall \vec{y} \in \mathbb{R}^3, F(\vec{y}) = \vec{v} \times \vec{y} + A\vec{y} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + A\vec{y}$$

$$= B\vec{y}, \text{ where } B = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} + A. \text{ Then by laws of matrix}$$

algebra,  $\forall \lambda, \mu \in \mathbb{R}$  and  $\forall \vec{x}, \vec{y} \in \mathbb{R}^3$ , we get  $F(\lambda\vec{x} + \mu\vec{y})$

$$= B(\lambda\vec{x} + \mu\vec{y}) = \lambda B\vec{x} + \mu B\vec{y} = \lambda F(\vec{x}) + \mu F(\vec{y}), \text{ so}$$

$F$  is indeed a linear transformation.

Problem 4 a) Is the set  $V = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$

a subspace of  $\mathbb{R}^2$ ?

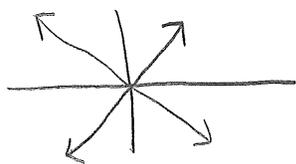
b) Let  $V, W$  be two subspaces of  $\mathbb{R}^n$ . Show that

$V \cap W = \{x \in \mathbb{R}^n : x \in V \text{ and } x \in W\}$  is also a subspace of  $\mathbb{R}^n$ .

Solution: a)  $(1) \in V$ , since  $1^2 = 1^2$ .  $(-1) \in V$ , since  $(-1)^2 = 1^2$ .

But  $(1) + (-1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin V$ , since  $0^2 \neq 2^2$ . Therefore,

$V$  is not a subspace of  $\mathbb{R}^2$ , since it is not closed under vector addition.



(Note:  $V = \text{span}([1]) \cup \text{span}([-1])$ )

b)  $\vec{0} \in V$  and  $\vec{0} \in W$ , so  $\vec{0} \in V \cap W$ .

If  $\lambda, \mu \in \mathbb{R}$  and  $\vec{x}, \vec{y} \in V \cap W$ , then  $\vec{x}, \vec{y} \in V$  and

$\vec{x}, \vec{y} \in W$ , so  $\lambda\vec{x} + \mu\vec{y} \in V$  and  $\lambda\vec{x} + \mu\vec{y} \in W$ , since  $V$

and  $W$  are subspaces. So  $\lambda\vec{x} + \mu\vec{y} \in V \cap W$ .

Thus, since  $\vec{0} \in V \cap W$ , and  $V \cap W$  is closed under taking

linear combinations (equivalently,  $V \cap W$  is closed under vector addition and scalar multiplication),  $V \cap W$  is

a subspace of  $\mathbb{R}^n$ .

2. group problems!