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Math 33a  
4-17-17

## Teaching Plan, week 3

- Plan for today:
1. geometric properties of linear transforms
  2. quiz (15 mins)
  3. image & kernel
  4. Subspaces, bases, & Linear Independence
  5. problems

### 1. geometric properties of linear transforms

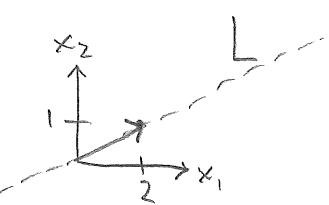
We'll just consider linear transforms from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , but all of the transforms discussed in this section can be generalized to transforms from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  (we'll get to these later in the course).

Scaling The matrix  $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$  has the effect of multiplying both elements of a vector  $\vec{x} \in \mathbb{R}^2$  by 3:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}, \text{ so } \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ was scaled by a factor of 3.}$$

In general,  $T(\vec{x}) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$  is the linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that scales  $\vec{x} \in \mathbb{R}^2$  by a factor of  $k$ .

Orthogonal Projections Consider the vector  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ :



and extend this vector to a line. For any  $\vec{x} \in \mathbb{R}^2$

we can decompose  $\vec{x}$  into  $\vec{x}''$  and  $\vec{x}'$ , where  $\vec{x}''$  is the closest vector in the line  $L$  to  $\vec{x}$ , and  $\vec{x}'$  is the remaining part of  $\vec{x}$ ,

which will be perpendicular to the L:

Say  $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Since  $\vec{x}''$  lies on the L,

$\vec{x}'' = k\vec{w}$  for some  $k \in \mathbb{R}$ . Once we've found  $k$ , we've found our projection  $\vec{x}''$ .

Since  $\vec{x}^\perp$  is orthogonal to  $\vec{w}$ , we know that

$\vec{x}^\perp \cdot \vec{w} = 0$  (from math 32a). So  $\vec{x} = \vec{x}'' + \vec{x}^\perp$ , and dotting

both sides by  $\vec{w}$  yields  $\vec{x} \cdot \vec{w} = (\vec{x}'' + \vec{x}^\perp) \cdot \vec{w} = \vec{x}'' \cdot \vec{w} + \underbrace{\vec{x}^\perp \cdot \vec{w}}_{=0}$

$= \vec{x}'' \cdot \vec{w} = (k\vec{w}) \cdot \vec{w} = k\|\vec{w}\|^2$ , so  $k = \frac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2}$ . Thus the projection

$\text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$ . (This is just the  $\vec{x}''$  vector). Since

for all  $a_1, a_2 \in \mathbb{R}$  and  $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^2$  we get  $\text{proj}_L(a_1\vec{x}_1 + a_2\vec{x}_2) = \left( \frac{(a_1\vec{x}_1 + a_2\vec{x}_2) \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w}$

$= a_1 \frac{\vec{x}_1 \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} + a_2 \frac{\vec{x}_2 \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = a_1 \text{proj}_L(\vec{x}_1) + a_2 \text{proj}_L(\vec{x}_2)$

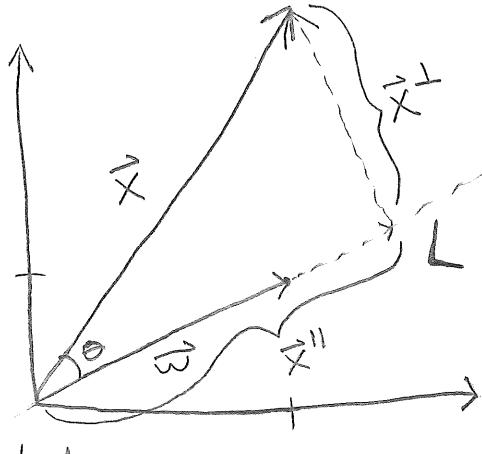
So the orthogonal projection is a linear map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

Since all linear maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  are represented by matrices (this was proved in last week's discussion notes),  $\text{proj}_L$  has a matrix representation:  $\text{proj}_L(\vec{x}) = A\vec{x}$  for some  $2 \times 2$  matrix A.

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{proj}_L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\vec{1} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{proj}_L \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\vec{0} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix}, \text{ so}$$

$$A = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}. \text{ (This is a great tactic to get } A \dots)$$



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This works for any  $\vec{w} \in \mathbb{R}^2$ , not just  $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  (well...  $\vec{w}$  shouldn't be  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , or else there's no line  $L$ ). So for any  $\vec{w} \in \mathbb{R}^2$ ,  $\vec{w} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , if  $L$  is the line spanned by  $\vec{w}$ , then the orthogonal projection along  $L$  is given by:

$$\text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}. \text{ "Orthogonal" because we decompose } \vec{x}$$

into orthogonal parts  $\vec{x}''$  and  $\vec{x}^\perp$ . We'll study other projections later. This formula makes intuitive sense considering the interpretation of the dot product from math 32a:  $\vec{x} \cdot \vec{w} = \|\vec{x}\| \|\vec{w}\| \cos \theta$ ,

where  $\theta$  is the angle between  $\vec{x}$  &  $\vec{w}$ . By geometry, we know

$$\vec{x}'' = (\|\vec{x}\| \cos \theta) \underbrace{\frac{\vec{w}}{\|\vec{w}\|}}_{\text{unit vector in direction of line } L} = \frac{\|\vec{x}\| \|\vec{w}\| \cos \theta}{\|\vec{w}\|^2} \vec{w} = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}.$$

Reflections Now instead of projecting  $\vec{x}$  orthogonally onto  $L$ , we'll reflect it about  $L$ . But this is really easy!

We can reuse all the work we just did. The reflection

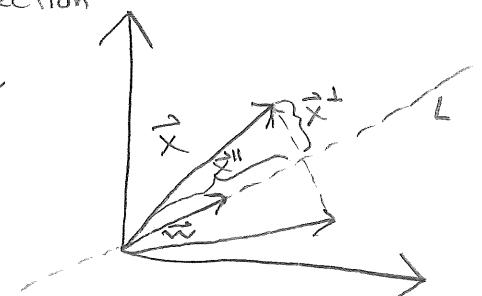
$$\text{ref}_L(\vec{x}) = \vec{x}'' - \vec{x}^\perp. \text{ Since } \vec{x} = \vec{x}'' + \vec{x}^\perp,$$

we get  $\vec{x}^\perp = \vec{x} - \vec{x}''$ , so

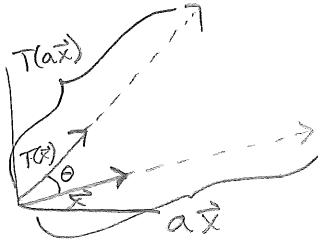
$$\text{ref}_L(\vec{x}) = \vec{x}'' - (\vec{x} - \vec{x}'') = 2\vec{x}'' - \vec{x}$$

$$= 2 \text{proj}_L(\vec{x}) - \vec{x} = 2 \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} - \vec{x}.$$

Since  $\text{ref}_L$  is a linear combination of two linear functions ( $\text{proj}_L$  and  $I_2$ ), it is itself linear. If  $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ , then  $\text{ref}_L(\vec{x}) = 2 \text{proj}_L(\vec{x}) - I_2 \vec{x} = 2 \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \vec{x} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \vec{x}$ .

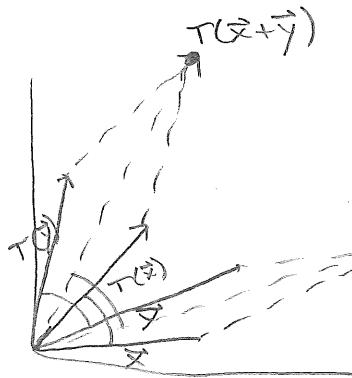


Rotations Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the map that rotates vectors in  $\mathbb{R}^2$  by an angle of  $\theta$  (think of it either radians or degrees), my preference



If  $\vec{x} \in \mathbb{R}^2$  and  $a \in \mathbb{R}$ , then  $T(a\vec{x}) = aT(\vec{x})$ , since scaling the input by a will result in scaling the (rotated) output by a too. If  $\vec{x}, \vec{y} \in \mathbb{R}^2$ ,

then  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ , since the rotation of the sum is the sum of the rotated vectors. (Imagine just rotating the whole parallelogram).



Thus  $T$  is a linear transform, so it has a matrix representation. To find that representation, see where  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  output to.

$$T(\vec{x}) = A\vec{x} \text{ for some } 2 \times 2 \text{ matrix } A.$$

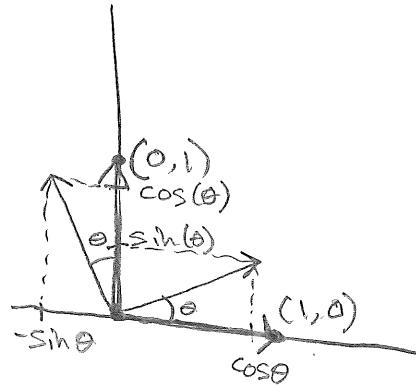
$$\text{By the picture, } T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}, \text{ so}$$

$$A = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} 1 \\ 0 \end{bmatrix} & A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

So a rotation  $\frac{\pi}{3}$  radians counter-clockwise is given by  $T(\vec{x}) = \begin{bmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{bmatrix} \vec{x} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \vec{x}$ .



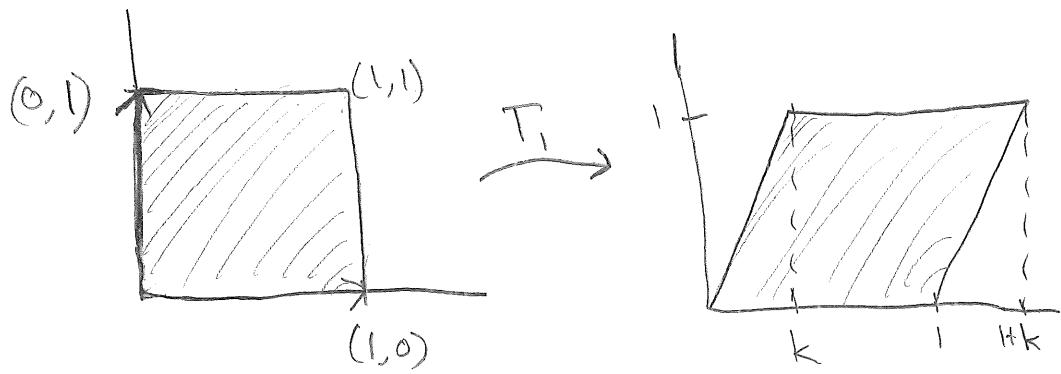
The linear maps

$$\text{Shears} \quad T_1(\vec{x}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \vec{x} \quad \text{and} \quad T_2(\vec{x}) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \vec{x}$$

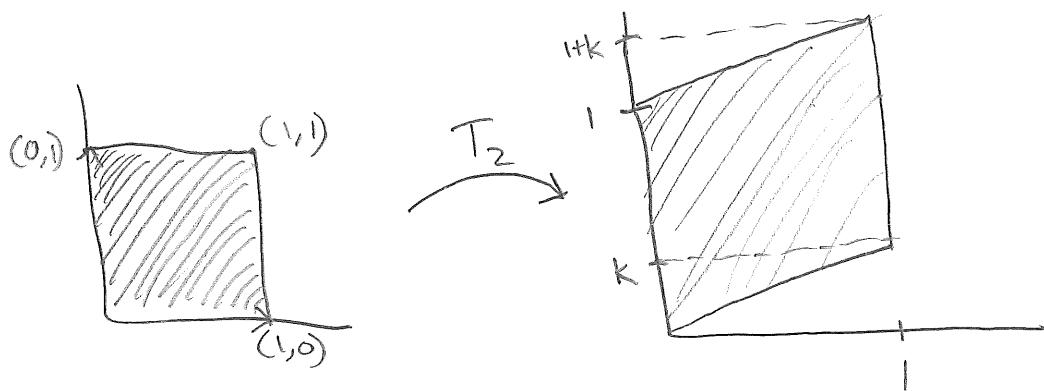
are called horizontal and vertical shears, respectively.

Why? → because, see what they do to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ :

$$T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$



$$T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



You can combine any of these easily. For example, if your task is to find some  $T$  that first shears horizontally by a factor of  $\frac{1}{2}$

and then rotates clockwise by  $\frac{\pi}{2}$  radians, then

$$T(\vec{x}) = \begin{bmatrix} \cos(-\pi/2) & -\sin(-\pi/2) \\ \sin(-\pi/2) & \cos(-\pi/2) \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \vec{x}}_{\text{first, shear the input}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \vec{x}.$$

then rotate it

2. quiz (15 minutes)

3. Image & kernel

The image of a function is the set of all possible outputs. If  $L: \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,  $L(\vec{x}) = A\vec{x}$  for

some  $n \times k$  matrix  $A$ , then  $\text{im}(L) = \text{im}(A) =$

$\{\vec{y} \in \mathbb{R}^n : \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^k\}$ . If we put  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$ ,

$A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_k]$ , where each  $\vec{v}_i \in \mathbb{R}^n$  is a column of  $A$ , then  $A\vec{x} = [\vec{v}_1 \ \dots \ \vec{v}_k] \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = x_1\vec{v}_1 + \dots + x_k\vec{v}_k$ ,

so the set of all possible outputs of  $L$  is just all linear combinations of  $A$ 's columns. We call the set of

all linear combinations of vectors  $\vec{v}_1, \dots, \vec{v}_k$  the span of these vectors. So  $\text{im}(L) = \text{im}(A) = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ .

So if  $A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$ , then  $\text{im}(A) = \text{span}([\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}], [\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}]) = \left\{ a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 5 \\ 7 \end{bmatrix} : a_1, a_2 \in \mathbb{R} \right\}$ .

The kernel of a function is the set of all inputs that make the output 0. If  $L: \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,  $L(\vec{x}) = A\vec{x}$ , then  $\text{ker}(L) = \text{ker}(A) = \{\vec{x} \in \mathbb{R}^k : A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n\}$ . So if  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ,

then  $A\vec{x} = \vec{0} \iff \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ \hline 0 & 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ \hline 0 & 0 \end{bmatrix} \iff x_1 + 2x_2 = 0 \iff x_1 = -2t$

So  $\text{ker}(A) = \left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} \in \mathbb{R}^2 : t \in \mathbb{R} \right\} = \{a \begin{bmatrix} -2 \\ 1 \end{bmatrix} : a \in \mathbb{R}\}$ .  $x_2 = t$  (free).

If  $A$  is  $n \times k$ , then  $\ker(A) = \{\vec{0}\}$  if and only if  $\text{rref}(A)$  has a pivot in each column (otherwise we get free variables, and thus many solutions to  $A\vec{x} = \vec{0}$ ), which happens if and only if  $\text{rank}(A) = k$ , which means we must have  $n \geq k$ .  $\text{im}(A) = \mathbb{R}^n$  if and only if  $\text{rref}(A)$  has a pivot in each row (otherwise we get some rows of zeros on the bottom, which means there are some  $\vec{b} \in \mathbb{R}^n$  such that  $A\vec{x} = \vec{b}$  has no solutions, so  $\vec{b} \notin \text{im}(A)$ ), which happens if and only if  $\text{rank}(A) = n$ , which means we must have  $k \geq n$ . So if you know  $\ker(A) = \{\vec{0}\}$  and  $\text{im}(A) = \mathbb{R}^n$ , then  $k = n$  and  $\text{rank}(A) = n = k$ , so  $\text{rref}(A) = I_n = I_k$ , so  $A$  is invertible.

If we know  $A$  is  $n \times n$  from the start, and we are given that  $\ker(A) = \{\vec{0}\}$ , then  $\text{rank}(A) = n$ , so again  $A$  is invertible.

If we know  $A$  is  $n \times n$  and we know  $\text{im}(A) = \mathbb{R}^n$ , then  $\text{rank}(A) = n$ , so again  $A$  is invertible.

If all we are given is that  $A$  is invertible, then  $A$  must be  $n \times n$ , and it's one-to-one & onto. One-to-one means  $A\vec{x} = \vec{0}$  has no more than one solution (so  $\ker(A) = \{\vec{0}\}$ ) and onto means  $A\vec{x} = \vec{b}$  always has a solution  $\vec{x}$  (so  $\text{im}(A) = \mathbb{R}^n$ ).

#### 4. Subspaces, bases, and linear independence

If  $W \subset \mathbb{R}^n$  is a subset of  $\mathbb{R}^n$ , then  $W$  is a subspace of  $\mathbb{R}^n$  if and only if:

$\vec{0} \in W$ ,  $\vec{x}, \vec{y} \in W$  implies  $\vec{x} + \vec{y} \in W$  ( $W$  is closed under addition),

and  $\vec{x} \in W$  implies  $k\vec{x} \in W$  for each  $k \in \mathbb{R}$  ( $W$  is closed under scalar multiplication). Let  $A$  be some  $n \times k$  matrix.

$\vec{x} \in \ker(A)$  implies  $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$ , so  $k\vec{x}$ , for each  $k \in \mathbb{R}$ . If  $\vec{x}, \vec{y} \in \ker(A)$ , then  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$ . Also,  $A\vec{0} = \vec{0}$ , so  $\vec{0} \in \ker(A)$ . Thus  $\ker(A)$  is a subspace (of  $\mathbb{R}^k$ ).

$\equiv$

$\vec{x} \in \text{im}(A)$  means there is some  $\vec{v} \in \mathbb{R}^k$  with  $A\vec{v} = \vec{x}$ , so  $k\vec{x} = kA\vec{v} = A(k\vec{v})$ , and thus  $k\vec{x} \in \text{im}(A)$ . If  $\vec{x}, \vec{y} \in \text{im}(A)$ , then  $A\vec{v}_1 = \vec{x}$  and  $A\vec{v}_2 = \vec{y}$  for some  $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^k$ , so  $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{x} + \vec{y}$ , so  $\vec{x} + \vec{y} \in \text{im}(A)$ .  $A\vec{0} = \vec{0}$ , so  $\vec{0} \in \text{im}(A)$ . Thus  $\text{im}(A)$  is a subspace (of  $\mathbb{R}^n$ !).

Given vectors  $\{\vec{v}_1, \dots, \vec{v}_k\}$ , we say

that they are linearly independent if  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$

implies  $c_1 = c_2 = \dots = c_k = 0$ . Otherwise, we say they are linearly dependent. If each  $\vec{v}_i \in \mathbb{R}^n$  and  $V$  is a subspace of  $\mathbb{R}^n$ , then we say  $\vec{v}_1, \dots, \vec{v}_k$  form a basis of  $V$  if  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$  and the  $\vec{v}_1, \dots, \vec{v}_k$  are linearly independent.

## 5. problems

- a) Prove that  $\vec{v}_1, \dots, \vec{v}_k$  form a basis for  $V$ , some subspace of  $\mathbb{R}^n$ , if and only if every  $\vec{v} \in V$  can be expressed uniquely as a linear combination of the  $\vec{v}_i$ 's. (assume each  $\vec{v}_i \in V$ ).

Solution:  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is a basis for  $V \iff V = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$   
and  $\vec{v}_1, \dots, \vec{v}_k$  are linearly independent.

If  $V$  is a basis for the  $\vec{v}_i$ 's, then each  $\vec{v} \in V$  is a linear combination of the  $\vec{v}_i$ 's, since  $\text{span}(\vec{v}_1, \dots, \vec{v}_k) = V$ . Also, if

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = d_1 \vec{v}_1 + \dots + d_k \vec{v}_k, \text{ then}$$

$$\vec{0} = \vec{v} - \vec{v} = (c_1 - d_1) \vec{v}_1 + \dots + (c_k - d_k) \vec{v}_k, \text{ which implies}$$

$$c_1 - d_1 = 0, c_2 - d_2 = 0, \dots, c_k - d_k = 0, \text{ by linear independence}$$

of the  $\vec{v}_i$ 's. Thus  $c_1 = d_1, \dots, c_k = d_k$ , and so the representation of each  $\vec{v} \in V$  as a linear combination of the

$\vec{v}_i$ 's is unique. Conversely, if each  $\vec{v} \in V$  has a unique representation as a linear combination of the  $\vec{v}_i$ 's, then  $V \subseteq \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ , and since  $V$  is a subspace, hence closed under linear combinations,  $\text{span}(\vec{v}_1, \dots, \vec{v}_k) \subseteq V$ . Thus  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ . If  $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$ , then by the uniqueness of the representation,  $c_1 = c_2 = \dots = c_k = 0$  is the only possibility (since  $0 \vec{v}_1 + \dots + 0 \vec{v}_k = \vec{0} \in V$ ), so the  $\vec{v}_i$ 's are linearly independent. So  $\vec{v}_1, \dots, \vec{v}_k$  form a basis for  $V$ .

b) Is  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+y+z=1 \right\}$  a subspace of  $\mathbb{R}^3$ ?

Solution: no, because  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$  ( $0+0+0 \neq 1$ ).

c) Is  $W = \left\{ \begin{bmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^3$ ?

Solution: yes, because  $W = \text{im} \left( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right)$ , which is a subspace for any matrix.

d) Give a geometrical description of all subspaces of  $\mathbb{R}^3$ .

Justify your answer.

Solution: no subspace of  $\mathbb{R}^3$  can have more than 3 vectors

in a basis, since then we can take four of those vectors,

$\vec{v}_1, \dots, \vec{v}_4$ , and say  $A = [\vec{v}_1 \dots \vec{v}_4]$ , which is  $3 \times 4$ . So  $\text{rank}(A) \leq 3$ , so  $\text{rref}(A)$  has free variables, so there are many solutions to  $A\vec{x} = \vec{0}$ , contradicting linear dependence of  $A$ 's columns, the  $\vec{v}_i$ 's.

If  $\vec{x}_1, \vec{x}_2, \vec{x}_3$  form a basis for

$V \subset \mathbb{R}^3$ , then  $A = [\vec{x}_1 \vec{x}_2 \vec{x}_3]$  is  $3 \times 3$  and has  $\ker(A) = \{\vec{0}\}$

(by linear independence of the  $\vec{x}_i$ 's), so  $A$  is invertible, hence onto,

so  $\text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \mathbb{R}^3 = V$ . If  $V = \text{span}(\vec{x}_1, \vec{x}_2)$ , then

$V$  is a plane in  $\mathbb{R}^3$  passing through  $\vec{0}$ . If  $V = \text{span}(\vec{x}_1)$ , then

$V$  is a line in  $\mathbb{R}^3$  through the origin,  $\vec{0}$ .

e) If  $W \subset \mathbb{R}^n$  is closed under addition and scalar multiplication,  
is it necessarily a subspace?

Solution: nope, it could be the empty set,  $\emptyset$ , in which case  
 $\vec{0}$  is not in it.

f) Are the columns of an invertible matrix linearly independent?

Solution: yep, because  $A\vec{x} = \vec{0}$  implies  $\vec{x} = \vec{0}$ .

g) Find a basis for  $\ker\left(\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}\right)$ .

Solution: it's already in rref, so jump straight to the variables (add column

$$\begin{array}{lll} \text{of zeros} & x_1 = -2x_2 - 3x_4 - 5x_5 & x_2 = r \\ \text{to get} & x_3 = -4x_4 - 6x_5 & \left. \begin{array}{l} x_4 = s \\ x_5 = t \end{array} \right\} \text{free, so the solutions } \vec{x} \text{ can} \\ \text{augmented} & & \end{array}$$

$$\text{matrix)} \quad \text{be written as } r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \text{ so } \ker(A) = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right),$$

and this is a basis, because these vectors are clearly L.I.,

since they each have a 1 in a coordinate, in which the others are 0.  
Linearly Independent

h) If  $B = \text{rref}(A)$ , is  $\ker(A) = \ker(B)$ ? Is  $\text{im}(A) = \text{im}(B)$ ?

Solution:  $B = \text{rref}(A) \Rightarrow E_m E_{m-1} \cdots E_2 E_1 A = B$  for some elementary (invertible)  
matrices  $E_i$ .  $\vec{x} \in \ker(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow B\vec{x} = E_m \cdots E_1 A \vec{x} = E_m \cdots E_1 \vec{0} = \vec{0}$ .

Conversely,  $\vec{x} \in \ker(B) \Rightarrow B\vec{x} = \vec{0} \Rightarrow A\vec{x} = E_1^{-1} \cdots E_m^{-1} B\vec{x} = \vec{0}$ , so yes,  
 $\ker(A) = \ker(B)$ . If  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , then  $\text{rref}(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \vec{0}$ , and so

$$\text{im}(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \text{im}(B), \text{ so } \text{im}(A) \neq \text{im}(B).$$

i) Give an example of a matrix  $A$  such that  $\text{im}(A)$  is the plane with normal vector  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$  in  $\mathbb{R}^3$ .

Solution: just find an  $A$  whose columns are in that plane, including two LI columns (lest  $\text{im}(A)$  be a line).

$$x_1 + 3x_2 + 2x_3 = 0 \iff \left[ \begin{array}{ccc|c} 1 & 3 & 2 & 0 \end{array} \right] \iff \begin{cases} x_1 = -3s - 2t \\ x_2 = s \\ x_3 = t \end{cases} \text{ free}$$

so  $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$  will do. Put  $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ , and we are done.

$$A = \begin{bmatrix} -3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} -3 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ also work.}$$

j) Give an example of a linear transform whose image is spanned by  $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$  in  $\mathbb{R}^3$ .

Solution: Set  $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ . Then  $A: \mathbb{R}^1 \rightarrow \mathbb{R}^3$  and clearly  $\text{im}(A) = \text{span}\left(\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}\right)$ .

You could also do something like  $A = \begin{bmatrix} 0 & -7 & 14 \\ 0 & -6 & 12 \\ 0 & -5 & 10 \end{bmatrix}$ . Then  $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  but it has the same span.

k) Give an example of a linear transformation whose kernel is the plane  $x+2y+3z=0$  in  $\mathbb{R}^3$ .

Solution: Set  $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ ,  $A: \mathbb{R}^3 \rightarrow \mathbb{R}^1$ . Then  $A\vec{x} = \vec{0}$

if and only if  $x_1 + 2x_2 + 3x_3 = 0$  if and only if  $\vec{x}$  is in that plane.

You can also say  $A = \begin{bmatrix} 5 & 10 & 15 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix}$ , for example.

l) Suppose  $A$  is  $n \times n$  and  $\text{rank}(A) = n$ . How many  $n \times n$  matrices  $X$  are there such that  $AX = I_n$ ?

Solution:  $A$  is invertible, so there's at least one solution,  $X = A^{-1}$ . If  $AB = AC = I_n$ , then left-multiply by  $A^{-1}$  to get  $A^{-1}AB = A^{-1}AC$ , so  $B = C$ . Thus there's only one  $X$  that solves this. This shows that matrix inverses are unique.

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w) Prove the  $\sin$  &  $\cos$  angle addition formulae.

Solution: use a composition of rotation maps:

$$\begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix} \underbrace{\begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix}}_{\text{first rotate ccw by } a \text{ radians}} \vec{x} = \begin{bmatrix} \cos(b)\cos(a) & -\sin(b)\sin(a) \\ \sin(b)\cos(a) & +\cos(b)\sin(a) \end{bmatrix} \vec{x}$$

then rotate ccw by  $b$  radians

$$\rightarrow = \begin{bmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{bmatrix} \vec{x}, \text{ since rotating by } a \text{ and then by } b \text{ is the same as rotating by } a+b$$

so equating components (which must necessarily be unique, e.g. take  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ),

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Now do you think linear algebra is powerful?

n) Find the scaling matrix  $A$  that transforms  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  into  $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ .

Solution:  $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$  scales by a factor of 4, so this works.

o) Find the orthogonal projection matrix  $B$  that transforms  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$  into  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ .

Solution:  $B$  outputs to a line, and  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$  is on that line. Set  $\vec{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ .

$$\text{Then } B(\vec{x}) = \text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} = \left( \frac{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}} \right) \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2x_1}{4} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}, \text{ so } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

p) Find the rotation matrix  $C$  that transforms  $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$  into  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

Solution:  $C \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ , so  $C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ .  $C = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , so

$$C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \text{ so } \theta = \sin^{-1}(-3/5), \text{ in the IV'th quadrant.}$$

$$\text{So } C = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}.$$

q) Find the shear matrix  $D$  that transforms  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  into  $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ .

Solution: must be a horizontal shear since only the  $x_1$ -coordinate was affected. So  $D = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$  for some  $k \in \mathbb{R}$ .  $D \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$$\text{so } 1+3k=7, \text{ so } k=2. \text{ Thus } D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

r) Find the reflection matrix  $E$  that transforms  $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$  into  $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ .

Solution: the reflection is determined by some line  $L$ .

$\vec{x} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \vec{x}'' + \vec{x}^\perp$ ,  $\vec{x}'' - \vec{x}^\perp = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ , so we can find  $L$  if we find  $\vec{x}''$ , since  $\vec{x}''$  lies on  $L$ .  $\vec{x}'' = \frac{(\vec{x}'' + \vec{x}^\perp) + (\vec{x}'' - \vec{x}^\perp)}{2}$

$$= \frac{\begin{bmatrix} 7 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix}}{2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \text{ so } \text{ref}_L(\vec{x}) = E\vec{x} = 2\text{proj}_L(\vec{x}) - \vec{x}$$

$$= 2 \left( \frac{\vec{x} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \vec{x} = 2 \left( \frac{1+3}{1+9} \right) \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} = \begin{bmatrix} 1/5 & 3/5 \\ 3/5 & 9/5 \end{bmatrix} \vec{x} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \vec{x}, \text{ so } E = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}.$$

s) Suppose a line  $L$  in  $\mathbb{R}^3$  contains a unit vector  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ .

Find the matrix  $A$  of the linear transformation  $T(\vec{x}) = \text{proj}_L(\vec{x})$ .

Solution: just like in the  $2 \times 2$  case,  $\text{proj}_L(\vec{x}) = \frac{(\vec{x} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u} = (\vec{x} \cdot \vec{u}) \vec{u}$

$$= (x_1 u_1 + x_2 u_2 + x_3 u_3) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^2 x_1 + u_1 u_2 x_2 + u_1 u_3 x_3 \\ u_1 u_2 x_1 + u_2^2 x_2 + u_2 u_3 x_3 \\ u_1 u_3 x_1 + u_2 u_3 x_2 + u_3^2 x_3 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & u_2^2 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & u_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\text{so } A = \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & u_2^2 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & u_3^2 \end{bmatrix}.$$

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f) Which of the following linear transformations  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  are invertible? Find the inverse if it exists.

- reflection about a plane (invertible,  $A^2 = I_3$ , get back to where you started)
- orthogonal projection onto a plane (not invertible, since the normal to that plane in  $\mathbb{R}^3$  gets squashed, and thus  $\ker(A) \neq \{\vec{0}\}$ ).
- scaling by a factor of 5 (invertible,  $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$ ).
- rotation about an axis (invertible, for example

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ reverse the rotation to get the inverse.)}$$

u) Find the matrix  $A$  of the linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \text{ and } T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\text{Solution: } T(\vec{x}) = A\vec{x}, \quad A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} 1 \\ 2 \end{bmatrix} & A\begin{bmatrix} 2 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} T\begin{bmatrix} 1 \\ 2 \end{bmatrix} & T\begin{bmatrix} 2 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{5-4} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix},$$

$$\text{so } A = A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}.$$

v) The formula  $AB = BA$  holds for all non matrices  $A$  and  $B$ , true or false?

$$\text{Solution: False! } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so these matrices don't commute.

w) True or false: there exists an invertible  $2 \times 2$  matrix  $A$  such that  $A^{-1} = -A$ .

Solution: true, try a rotation of  $\pi/2$ :

$$A = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Then } A(-A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so  $A^{-1} = -A$ .

x) There exist an invertible  $10 \times 10$  matrix that has 92 ones among its entries, true or false?

Solution: false, because then there are only 8 positions for numbers that are not 1's, so there must be at least two columns that are all 1's, so  $A\vec{x} = \vec{0}$  has many solutions, so  $A$  is not one-to-one, so  $A$  is not invertible.

y) There exists an invertible matrix  $S$  such that  $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$  is a diagonal matrix, true or false?

Solution: false, since if  $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = D$  = diagonal, then  $D^2 = (S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S)^2 = (S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S)(S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S) = S^{-1} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , so  $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  (since if  $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ,

then  $D^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ ).  
And then  $\underbrace{SS^{-1}}_{I_2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = SD = S \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , so  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , and so  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} SS^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , a contradiction.

z) True or false - if  $A$  is an invertible  $2 \times 2$  matrix and  $B$  is any  $2 \times 2$  matrix, then  $\text{rref}(AB) = \text{rref}(B)$ .

Solution: true, because  $A$  invertible  $\Rightarrow E_m E_{m-1} \cdots E_2 E_1 A = I_2$  for some elementary  $E_i$ , so  $\text{rref}(AB) = F_p \cdots F_1 AB = F_p \cdots F_1 E_m \cdots E_1 B = G_q \cdots G_1 B = \text{rref}(B)$ , by uniqueness of rref.