

Teaching Plan, week 2

- Plan for today:
1. Gaussian elimination review
 2. Quiz (15 minutes)
 3. Matrix algebra
 4. Linear Transforms
 5. Inverting matrices
 6. Problems

1. Gaussian elimination review: to solve an $n \times k$ linear system, put it in augmented matrix form, perform row operations until you get it in reduced row-echelon form (rref), then translate the system back into variables

$$a) \left| \begin{array}{cccc|c} & & x_3 + x_4 & = & 0 \\ x_1 + x_2 & & x_2 + x_3 & = & 0 \\ x_1 & & & + x_4 & = & 0 \end{array} \right| \rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{(IV)} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{(II)} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow$$

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

so let $x_4 = t$ be free, and have x_1, x_2, x_3 depend on it:

$$x_4 = t, \quad x_1 = -t, \quad x_2 = t, \quad x_3 = -t.$$

$$b) \begin{cases} x_1 + 2x_2 \\ x_1 + 2x_2 + 2x_3 \end{cases} \quad \begin{cases} x_4 + 2x_5 - x_6 = 2 \\ + x_5 - x_6 = 0 \\ - x_5 + x_6 = 2 \end{cases} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & 5 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & 5 & 1 & 2 \end{bmatrix} \xrightarrow{-(I)} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -6 & 2 & 2 \end{bmatrix} \cdot (1/2) \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{bmatrix}$$

this is in rref, and there are 3 "pivots" (leading ones), so these will be determined by the other (free) variables:

$$\begin{cases} x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_3 - 3x_5 + x_6 = 1 \\ x_4 + 2x_5 - x_6 = 2 \end{cases}$$

so let $\begin{cases} x_2 = r \\ x_5 = s \\ x_6 = t \end{cases}$ be free, and

$$\begin{cases} x_1 = -2r - s + t \\ x_3 = 1 + 3s - t \\ x_4 = 2 - 2s + t \end{cases}$$

c) $n \times k$ linear systems always have either: $\begin{cases} \text{exactly one (unique) solution} \\ \text{infinitely many solutions, or} \\ \text{no solutions at all.} \end{cases}$

For a system with $\text{rref} = \begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 9 \end{bmatrix}$, we get $\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$ as our solution.

If $\text{rref} = \begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$, then $x_3 = t = \text{free}$, and $\begin{cases} x_1 = 5 - t \\ x_2 = 7 \end{cases}$.

If $\text{rref} = \begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 0 & | & 9 \end{bmatrix}$, then there are no solutions, because $0 \neq 9$.

2. quiz (15 minutes)

3. Matrix algebra

We want more sophisticated language for discussing linear systems.
 Matrix: rectangular array of numbers.

Two matrices of the same size can be added:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 7 & 4 & 4 \end{bmatrix},$$

and a matrix can be multiplied by a scalar:

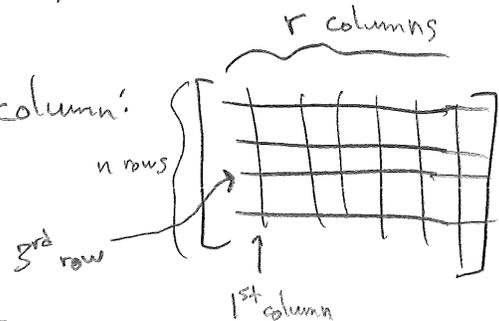
$$6 \begin{bmatrix} 2 & 9 \\ 1 & 4 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 54 \\ 6 & 24 \\ -18 & 0 \end{bmatrix}. \text{ Matrix multiplication is tough,}$$

just accept it. If A is $n \times k$ and B is $k \times r$, then

AB is defined as an $n \times r$ matrix whose $(i,j)^{\text{th}}$ entry

is: $\sum_{l=1}^k a_{il} b_{lj}$, where $a_{il} = (i,l)^{\text{th}}$ entry of A and $b_{lj} = (l,j)^{\text{th}}$ entry of B .

Note: $(i,j)^{\text{th}}$ entry means i^{th} row, j^{th} column:



So $\begin{bmatrix} 1 & 2 & 0 \\ -4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & -5 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & -6 \\ 7 & 9 & -31 \end{bmatrix}.$

Three special interpretations:

If A is $n \times k$ and \vec{x} is $k \times 1$, then think of $A\vec{x}$ as a linear combination of A 's columns, using \vec{x} 's entries as weights:

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \stackrel{\text{new interpretation}}{=} 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\stackrel{\text{definition of matrix multiplication}}{=} \begin{bmatrix} 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 3 \\ 0 \cdot 1 + (-1) \cdot 2 + 4 \cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 10 \end{bmatrix}$$

If \vec{x} is $1 \times k$ and A is $k \times n$, then think of $\vec{x}A$ as a linear combination of A 's rows, using \vec{x} 's entries as weights:

$$\begin{bmatrix} 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 9 & 1 \\ 4 & 5 \end{bmatrix} \stackrel{\text{new interpretation}}{=} 2 \cdot \begin{bmatrix} 2 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 9 & 1 \end{bmatrix} + 7 \cdot \begin{bmatrix} 4 & 5 \end{bmatrix}$$

$$\stackrel{\text{definition of matrix multiplication}}{=} \begin{bmatrix} 2 \cdot 2 + 1 \cdot 9 + 7 \cdot 4 & 2 \cdot 0 + 1 \cdot 1 + 7 \cdot 5 \end{bmatrix} = \begin{bmatrix} 41 & 36 \end{bmatrix}$$

If A is $n \times k$ and B is $k \times r$, then you can think of the (i, j) th entry of AB as being the dot product of the i th row of A and the j th column of B :

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -9 \\ 5 & 8 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ 8 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ 8 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 35 & 7 \\ 13 & -8 \end{bmatrix}$$

Throughout the course, all three of these interpretations, as well as the definition, will be vital!

4. Linear Transforms

T is a linear transform from \mathbb{R}^k to \mathbb{R}^n



$$T: \mathbb{R}^k \rightarrow \mathbb{R}^n, \quad T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \text{ for all } \vec{v}, \vec{w} \in \mathbb{R}^k,$$

and $T(a\vec{v}) = aT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^k, a \in \mathbb{R}$.

Let $\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ \leftarrow 1 in j^{th} slot, 0's elsewhere, if $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^k$,
 k total elements.

then we can write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_k \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_k \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_k \vec{e}_k,$$

so $T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_k \vec{e}_k) = T(x_1 \vec{e}_1) + T(x_2 \vec{e}_2) + \dots + T(x_k \vec{e}_k)$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_k T(\vec{e}_k)$$

$$= [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \dots \quad T(\vec{e}_k)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = A \vec{x}, \text{ where}$$

$A = [T(\vec{e}_1) \quad \dots \quad T(\vec{e}_k)]$ has columns given by T 's image of the \vec{e}_j 's! This tells us that every linear transform from \mathbb{R}^k to \mathbb{R}^n is represented as $T(\vec{x}) = A\vec{x}$ for some matrix A (of size $n \times k$). Also, for each A , we get such a linear transform.

5. Inverting matrices

If $T: \mathbb{R}^k \rightarrow \mathbb{R}^n$ is a linear map, then $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^k$, where A is some $n \times k$ matrix.

We think of inverting T and inverting A as the same thing.

T is invertible $\iff T$ is one-to-one & onto.
(injective) (surjective)

When is T invertible? Think back to the rref!!!

Whenever we had free variables, $A\vec{x} = \vec{b}$ always either had no solutions or infinitely many. So we need T not onto T not 1-to-1

rref(A) to have no free variables. So rref(A) needs to have the same number of pivots as columns.

But also there can't be any rows of zeros on the bottom, since then $A\vec{x} = \vec{b}$ can have no solutions. So therefore:

$$\text{rank}(A) = \# \text{ of pivots of } A \geq \# \text{ columns of } A \geq \# \text{ rows of } A$$

But $\text{rank}(A) \leq \# \text{ columns of } A \leq \# \text{ rows of } A$, by the form of rref(A)!

Thus — T is invertible $\iff A$ is invertible $\iff \text{rank}(A) = \# \text{ columns of } A = \# \text{ rows of } A$.

So only $n \times n$ matrices can be invertible!

($A\vec{x} = \vec{b}$ will have a ^{unique} solution \vec{x} for any \vec{b} , as long as $\text{rref}(A) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$, the $n \times n$ identity matrix).

But - how to find matrix inverses? To answer this we must take a closer look at elementary matrices. Elementary matrices are matrices E we multiply on the left to matrices A to perform elementary row operations.

Examples: $\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{elementary matrix}} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 4 & 9 \end{bmatrix}}_A = \begin{bmatrix} 4 & 6 \\ 1 & 2 \\ 4 & 9 \end{bmatrix}$, so this elementary matrix had the effect of adding $3 \cdot (\text{II})$ to (I) for A .

$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible: $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{elementary matrix}} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 4 & 9 \end{bmatrix}$, so this elementary matrix caused the 1st & 2nd rows to switch.

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Finally, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 8 & 18 \end{bmatrix}$, so this elementary matrix had the effect of multiplying the 3rd row by 2.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

So each row operation is equivalent to left-multiplying by an (invertible) elementary matrix. Since $\text{rref}(A)$ is obtained by elementary row operations,

$E_m E_{m-1} \dots E_2 E_1 A = I_n$ for some elementary matrices $E_i, 1 \leq i \leq m$.

Then $E_m E_{m-1} \dots E_2 E_1 = E_m \dots E_1 I_n$ is the inverse of A , A^{-1} !

So get A^{-1} by performing the same row operations on I_n as you do to A . to get it into rref.

6. Problems

a) Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$?

Solution: set up $a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ and try to solve

for a & b : $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & | & 7 \\ 2 & 5 & | & 8 \\ 3 & 6 & | & 9 \end{bmatrix} \xrightarrow{\substack{-2(I) \\ -3(I)}} \begin{bmatrix} 1 & 4 & | & 7 \\ 0 & -3 & | & -6 \\ 0 & -6 & | & -12 \end{bmatrix} \xrightarrow{\cdot(-1/3)}$

$\begin{bmatrix} 1 & 4 & | & 7 \\ 0 & 1 & | & 2 \\ 0 & 1 & | & 2 \end{bmatrix} \xrightarrow{-(II)} \begin{bmatrix} 1 & 4 & | & 7 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{-4(II)} \begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$, so $\begin{cases} a = -1 \\ b = 2 \end{cases}$

Solves our puzzle.

b) For which values c & d is $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$ a linear combination of

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$?

Solution: $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 1 & | & 5 \\ 1 & 2 & | & 7 \\ 1 & 3 & | & c \\ 1 & 4 & | & d \end{bmatrix} \xrightarrow{\substack{-(I) \\ -(I) \\ -(I)}} \begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 2 \\ 0 & 2 & | & c-5 \\ 0 & 3 & | & d-5 \end{bmatrix} \xrightarrow{\substack{-2(II) \\ -3(II)}} \begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & c-9 \\ 0 & 0 & | & d-11 \end{bmatrix}$, so $\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$,

and we must have $c=9$, $d=11$, or else our system is inconsistent.

c) For which values of c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$? ($a \neq b$ are arbitrary constants)

Solution: $x_1 \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix} = \begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{array} \right] \xrightarrow{\substack{-a(I) \\ -a^2(I)}} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{array} \right] \xrightarrow{(b+a)(II)} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{array} \right]$$

In order for this system to have solutions, it must be the case that $c^2 - a^2 - (c-a)(b+a) = 0$

$$\iff c^2 - a^2 - cb - ca + ab + a^2 = 0$$

$$\iff c^2 - c(a+b) + ab = 0$$

$$\iff (c-a)(c-b) = 0, \text{ which is true}$$

if and only if $c=a$ or $c=b$. If $b-a=0$, then $a=b$, and so the condition $c=a$ or b implies $c-a=0$ (in the case $a=b$), so the system is consistent as long as c equals either a or b , regardless of what $a \neq b$ are (even if $a=b$).

d) if $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is linear, $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$, $T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$, and $T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 7 \end{bmatrix}$, find T .

Solution: $T(\vec{x}) = A\vec{x}$ for some 2×3 matrix A .

$$A = A I_3 = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \left[A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$= \left[T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 7 \end{bmatrix},$$

and this determines $T: T(\vec{x}) = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 7 \end{bmatrix} \vec{x}$.

e) if $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, is this transformation linear? If so, find its matrix.

Solution: $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

where $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. So T is linear, and is represented by this A .

f) if $\vec{v}_1, \dots, \vec{v}_k$ are vectors in \mathbb{R}^n , and $T \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = x_1 \vec{v}_1 + \dots + x_k \vec{v}_k$, is T a linear transform? Also, if it is, then find the matrix representation of T .

Solution: Put $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^k$. $T(\vec{x}) = x_1 \vec{v}_1 + \dots + x_k \vec{v}_k = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = A \vec{x}$, where $A = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$.

So T is a linear transform whose matrix representation is A .

g) Find all matrices X such that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution: write $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ or equivalently,}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\text{So } \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right] \xrightarrow{-2(I)} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{l} x_{11} = -2t \\ x_{21} = t = \text{free} \end{array}$$

$$x_{12} = -2s$$

$$x_{22} = s = \text{free,}$$

$$\text{So } X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -2t & -2s \\ t & s \end{bmatrix}, \text{ where } s, t \in \mathbb{R},$$

captures the full set of solutions to this matrix equation.

h) For which values of k is the matrix $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{k} \\ 1 & \frac{1}{4} & \frac{1}{k^2} \end{bmatrix}$ invertible?

Solution: we know by an earlier problem that $\begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$ is a linear

combination of $\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 2^2 \end{bmatrix}$ if and only if $k = 1$ or 2 ,

in which case there are many solutions to $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{k} \\ 1 & \frac{1}{4} & \frac{1}{k^2} \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

so $\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{k} \\ 1 & \frac{1}{4} & \frac{1}{k^2} \end{bmatrix}}_A$ is not invertible if $k = 1$ or 2 . Conversely, if A is

not invertible, then $\text{ref}(A)$ has a free variable, so $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(which is always consistent) has many solutions, so $\begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$ is a linear

combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 4 \end{bmatrix}$ (since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 4 \end{bmatrix}$ are not multiples of each other), so $k = 1$ or 2 , again by the earlier exercise.

So A is invertible $\iff k$ is not 1 or 2. But this is a "high level" solution. Let's see if we can solve this through manual labor.

Let's try to find the inverse of A directly:

$$[A | I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & k & 0 & 1 & 0 \\ 1 & 4 & k^2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{-(I) \\ -(I)}}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & k-1 & -1 & 1 & 0 \\ 0 & 3 & k^2-1 & -1 & 0 & 1 \end{array} \right] \xrightarrow{-3(I)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2-k & 2 & -1 & 0 \\ 0 & 1 & k-1 & -1 & 1 & 0 \\ 0 & 0 & k^2-3k+2 & 2 & -3 & 1 \end{array} \right]$$

note $k^2-3k+2 = (k-2)(k-1)$

In order for $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, it is necessary & sufficient for $(k-2)(k-1) \neq 0$. Thus A is invertible \iff

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \iff (k-2)(k-1) \neq 0 \iff k \text{ is not } 1 \text{ or } 2.$$

So we have our answer. Note: we could have simply found $\text{rref}(A)$ instead of trying to find A^{-1} .

i) For which constants a, b, & c is the matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ invertible?

Solution: $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{bmatrix} c \\ -b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $A\vec{x} = \vec{0}$ has a nontrivial

solution \vec{x} (if $a=b=c=0$, then $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$)

so $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ works as the nontrivial solution)

j) Find all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$ad-bc=1$ and $A^{-1}=A$

Solution: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(check this! It's a general fact.)

so if $ad-bc=1$, then

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ so } a=d, -c=c, -b=b, \text{ and } a=d.$$

So $b=c=0$, which

forces $ad-bc=a^2=1$, so $a=\pm 1$. Thus the only solutions are $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
 $d=a \dots$

so $\text{rref}(A) \neq I_2$ (otherwise the only solution \vec{x} to $A\vec{x} = \vec{0}$ would be the trivial solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$), so A is not invertible for any values of a, b, & c.