

Selected Problems & Solutions from 8.1

For 1-6, orthogonally diagonalize the symmetric matrix.

1 $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\hookrightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U^T}$$

2 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\hookrightarrow A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, A\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ so } A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T$$

3 $\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$

$$\hookrightarrow \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 7.$$

$$E_2 = \ker(A - 2I_2) = \ker \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

$$E_7 = \ker \begin{pmatrix} 1 & 2 \\ 2 & -4 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}, \text{ so } A = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^T$$

4 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\hookrightarrow -\lambda(\lambda(1\lambda - 1) + 1(-(-\lambda))) = -\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda^2 - \lambda - 2) = -\lambda(\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 0, -1, 2. E_0 = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, E_{-1} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\},$$

$$E_2 = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}. \text{ So, normalizing, we get:}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}^T.$$

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$$\underline{5} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\hookrightarrow -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = (1 + \lambda) \left[-\lambda(\lambda - 1) + 2 \right] = (1 + \lambda)(-\lambda^2 + \lambda + 2)$$

$$= -(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 2$$

$$E_{-1} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

do Gram-Schmidt on these!
will orthonormalize them.

$$E_2 = \ker \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}^T$$

$$\underline{6} \quad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\hookrightarrow -\lambda(1 - \lambda)(-1 - \lambda) - 2(2(1 - \lambda)) + 2(-2(1 - \lambda)) = -\lambda^3 + 9\lambda = \lambda(\lambda - 3)(\lambda + 3) = 0$$

$$\Rightarrow \lambda = 0, 3, -3. E_0 = \ker \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\},$$

$$E_3 = \ker \begin{pmatrix} -3 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -4 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}, E_{-3} = \ker \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \right\}$$

$$A = UDU^T, \text{ where } U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ 2 & 1 & -2 \end{pmatrix}, D = \begin{pmatrix} 0 & & \\ & 3 & \\ & & -3 \end{pmatrix}.$$

For 7-11, orthogonally diagonalize the symmetric matrix.

$$\underline{7} \quad \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\hookrightarrow \text{by inspection, } A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ so } A = UDU^T, \text{ where}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 5 & \\ & 1 \end{pmatrix}.$$

$$\underline{8} \quad \begin{pmatrix} 3 & 3 \\ 3 & -5 \end{pmatrix}$$

$$\hookrightarrow \lambda^2 + 2\lambda - 24 = (\lambda - 4)(\lambda + 6) = 0 \Rightarrow \lambda = 4, -6. E_4 = \ker \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\},$$

$$E_{-6} = \ker \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}, \text{ so } A = UDU^T, \text{ where } U = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & \\ & -6 \end{pmatrix}.$$

B

9 $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

\hookrightarrow by inspection, $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so

$$A = UDU^T, \text{ where } U = \begin{pmatrix} \sqrt{5}/2 & \sqrt{5}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{5}/2 & -\sqrt{5}/2 & 0 \end{pmatrix}, D = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 2 \end{pmatrix}.$$

10 $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$

\hookrightarrow This is a rank 1 matrix (with nonzero trace). So its first col, for instance, is an e-vec, with e-val $\text{tr}(A)$. $A\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 9\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$,

$$A\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 2 \\ 0 \end{pmatrix}, A\begin{pmatrix} -2 \\ 5 \end{pmatrix} = 0\begin{pmatrix} -2 \\ 5 \end{pmatrix}, \text{ so } A = UDU^T, \text{ where}$$

$$U = \begin{pmatrix} 1/3 & 2/\sqrt{35} & -2/\sqrt{35} \\ -2/3 & 1/\sqrt{35} & 4/\sqrt{35} \\ 2/3 & 0 & 5/\sqrt{35} \end{pmatrix}, D = \begin{pmatrix} 9 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

11 $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

\hookrightarrow By inspection, $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
 So $A = UDU^T$, where $U = \begin{pmatrix} \sqrt{5}/2 & \sqrt{5}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{5}/2 & -\sqrt{5}/2 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 2 & & \\ & 0 & \\ & & 1 \end{pmatrix}$.

12 Let $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be reflection about $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$.a. Find an orthonormal eigenbasis \mathcal{B} for L .b. Find the matrix B of L w.r.t. \mathcal{B} .c. Find the matrix A of L w.r.t. the standard basis of \mathbb{R}^3 .

\hookrightarrow a. $V = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$, $V^\perp = \text{span}\left\{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$. L fixes everything in V and negates everything in V^\perp , so $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are e-vecs

for L . So $\mathcal{B} = \left\{\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ works.

$$\text{b. } B = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mathcal{B}}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mathcal{B}} \right]_{\mathcal{B}} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} -2 & 0 & 1 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}_{\mathcal{B}} \right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. (= D) \xrightarrow{\text{symmetric!}}$$

$$\text{c. } A = UDU^T = \dots, \text{ or, alternatively, } A = 2 \text{proj}_v - I = \frac{2}{5}\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^T - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 3 \end{pmatrix}.$$

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13 If A is symmetric 3×3 with $A^2 = I_3$, is $T(\vec{x}) = A\vec{x}$ necessarily the reflection about a subspace of \mathbb{R}^3 ?

$$\hookrightarrow A = UDU^T, \text{ so } A^2 = \underbrace{(UDU^T)(UDU^T)}_I = UD^2U^T = I_3$$

$$\Rightarrow D^2U^T = U^T I_3 = U^T \Rightarrow D^2 = U^T U = I_3$$

$$\begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}$$

$$\Rightarrow \lambda_i^2 = 1 \text{ for } i=1,2,3. \text{ So}$$

λ_i all real.
(by spectral theorem)

each λ_i is either 1 or -1. If we set $V = E_+$,

then $V^\perp = E_-^\perp = E_-$, (since these eigenspaces are orthogonal).

Then T represents a reflection about E_+ . Note: even if $E_+ = \mathbb{R}^3$ or $E_+ = \{0\}$, we still consider T a reflection.

14 Using that $(\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}) = UDU^T$, where $U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, orthogonally diagonalize the following matrices:

$$a) \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \quad b) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad c) \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

$$\hookrightarrow a) \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = 2(UDU^T) = 2U(D)(U^T)$$

$$b) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = (UDU^T) - 3(UU^T) = UDU^T - U\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}U^T \\ = U(D - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix})U^T = U\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}U^T.$$

$$c) \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{2}(UDU^T) - \frac{1}{2}I_3 = U\left(\frac{1}{2}D\right)U^T - \frac{1}{2}UU^T \\ = U\left(\begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}\right)U^T = U\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}U^T.$$

15 If A is invertible and symmetric, is A^{-1} orthogonally diagonalizable?

\hookrightarrow by Spectral Theorem, A is orthogonally diagonalizable, so $A = UDU^T$, where $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $\lambda_i \neq 0$ for $i=1, \dots, n$. Thus $A^{-1} = (UDU^T)^{-1} = (U^T)^{-1}D^{-1}U^{-1}$
Thus A^{-1} is orthogonally diagonalizable. $= U\begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}U^T$.