

## Answers to True/False from ch. 7

- 1 If  $0$  is an e-val of  $A$ , then  $\det(A)=0$ .  
 ↳ True, since  $\det(A-0I_n) = \det(A)=0$ .
- 2 The e-vals of a  $2\times 2$  matrix  $A$  are the solutions of the equation  $\lambda^2 - (\text{tr}(A))\lambda + \det(A) = 0$ .  
 ↳ T; this is an alternative formula for the characteristic polynomial for  $2\times 2$  matrices.
- 3 The e-vals of any triangular matrix are its diagonal entries.  
 ↳ T; the characteristic eqn of  $\begin{pmatrix} a_1 & \text{stuff} \\ 0 & a_n \end{pmatrix}$  is  $(a_1-\lambda)\cdots(a_n-\lambda)=0$ .
- 4 The trace of any square matrix is the sum of its diagonal entries.  
 ↳ T, by definition. Trace is only defined for square matrices (same for the determinant), by the way.
- 5 The algebraic multiplicity of an e-val cannot exceed its geometric multiplicity.  
 ↳ False;  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  has  $\text{almul}(0)=2$ , but  $\text{gemu}(0)=1$ , as a counterexample. The correct formula is  $1 \leq \text{gemu}(\lambda) \leq \text{almul}(\lambda)$  for each e-val  $\lambda$ .
- 6 If an  $n\times n$  matrix  $A$  is diagonalizable (over  $\mathbb{R}$ ), then there must be a basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .  
 ↳ T. Diagonalizable over  $\mathbb{R}$  means  $A$ 's e-vals are all real, and  $\text{gemu}(\lambda) = \text{almul}(\lambda)$  for each e-val  $\lambda$ . Then we can form a basis for  $\mathbb{R}^n$  by concatenating bases for all the e-spaces  $E_\lambda$ .

7 If the standard vectors  $\vec{e}_1, \dots, \vec{e}_n$  are e-vecs of an  $n \times n$  matrix  $A$ , then  $A$  must be diagonal.

↪ T:  $A = I_n D I_n^{-1} = D$  in this case, since  $S$  in the formula  $A = SDS^{-1}$  (or  $AS = SD$ ) houses the e-vecs in the chosen eigenbasis for  $A$ .

8 If  $\vec{v}$  is an e-vec of  $A$ , then it's also an e-vec for  $A^3$ .

↪ T:  $A\vec{v} = \lambda\vec{v}$  implies  $A^3\vec{v} = A^2\underset{\lambda\vec{v}}{A\vec{v}} = \lambda A^2\vec{v} = \dots = \lambda^3\vec{v}$ , so  $\vec{v}$  is an e-vec of  $A^3$ , with corresponding e-val  $\lambda^3$ .

9 There exists a real  $5 \times 5$  matrix with only two distinct e-vals (over  $\mathbb{C}$ ).

↪ T:  $\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}$  works. "over  $\mathbb{C}$ " just means we are allowing for complex e-vals.

10 There exists a real  $5 \times 5$  matrix without any real e-vals.

↪ T: the characteristic polynomial will be a 5<sup>th</sup>-order polynomial, which (by continuity & the intermediate value theorem) must have at least one real root; since that polynomial approaches  $\infty$  on one end and  $-\infty$  on the other, it must cross the x-axis somewhere.



11 If  $A$  &  $B$  have the same e-vals (over  $\mathbb{C}$ , including multiplicities), then  $\text{tr}(A) = \text{tr}(B)$ .

↪ T:  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n = \text{tr}(B)$ .

12 If a real matrix  $A$  has only the eigenvalues  $\pm 1$ , then  $A$  must be orthogonal.

↪ F:  $A = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}$  serves as a counterexample, since it is not orthogonal.

13

13 Any rotation-scaling matrix in  $M_{2 \times 2}(\mathbb{R})$  is diagonalizable over  $\mathbb{C}$ .

↪ T: these are of the form  $A = C \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ , so the char. eqn. is  $\lambda^2 - (2\cos\theta)\lambda + c^2(\cos^2\theta + \sin^2\theta)$

$$\begin{aligned} &= \lambda^2 - (2\cos\theta)\lambda + c^2 = (\lambda - c\cos\theta)^2 + c^2 - c^2\cos^2\theta \\ &= (\lambda - c\cos\theta)^2 - c^2(1 - \cos^2\theta) = (\lambda - c\cos\theta)^2 + c^2\sin^2\theta = 0 \end{aligned}$$

$\iff \lambda = c\cos\theta \pm i\sin\theta = ce^{\pm i\theta}$  (note that  $\cos(\cdot)$  is an even function and  $\sin(\cdot)$  is an odd function).

So we get two distinct e-vals if  $0 < \theta < 2\pi$ . If  $\theta = 0$ , then  $A$  is the identity. In either case,  $A$  is diagonalizable (over  $\mathbb{C}$ ).

14 If  $A$  is a noninvertible  $n \times n$  matrix, then  $\text{genu}(0) = n - \text{rank}(A)$ .

↪ T:  $\det(A) = \det(A - 0I_n) = 0$ , so  $\lambda = 0$  is an e-val of  $A$ . Then  $\text{genu}(0) = \text{nul}(A - 0I_n) = \text{nul}(A)$

rank-nullity theorem  $\hat{=} (\# \text{ of } \underline{\text{cols}}) - \text{rank}(A)$

$$= n - \text{rank}(A).$$

15 If  $A$  is diagonalizable, then so is its transpose.

↪ T:  $A = SDS^{-1} \Rightarrow A^T = (SDS^{-1})^T = (S^{-1})^T D^T S^T = (S^T)^{-1} D^T S^T$ , so  $A^T$  is diagonalizable.

16 If  $A$  &  $B$  are  $3 \times 3$  s.t.  $\text{tr}(A) = \text{tr}(B)$  and  $\det(A) = \det(B)$ , then  $A$  &  $B$  must have the same e-vals.

↪ F:  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  serves as a counterexample.

If  $A$  &  $B$  were  $2 \times 2$  then the statement would hold.

17 If 1 is the only e-val of  $n \times n$  A, then  $A = I_n$ .

↪ F:  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is a counterexample.

18 If  $A \& B$  are  $n \times n$  and  $\alpha$  is an e-val of  $A$  and  $\beta$  is an e-val of  $B$ , then  $\alpha\beta$  is an e-val of  $AB$ .

↪ F:  $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \Rightarrow AB = 0$ . If one of  $\alpha, \beta$  is 0, then the statement is true though, since  $AB$  will fail to be invertible.

19 If 3 is an e-val of  $A$ , then 9 is an e-val of  $A^2$ .

↪ T:  $A\vec{v} = 3\vec{v} \Rightarrow A^2\vec{v} = AA\vec{v} = A3\vec{v} = 3A\vec{v} = 3 \cdot 3\vec{v} = 9\vec{v}$  ( $\vec{v} \neq 0$ )

20 The matrix of any orthogonal projection onto a subspace  $V$  of  $\mathbb{R}^n$  is diagonalizable.

↪ T: Get a basis  $\{\vec{w}_1, \dots, \vec{w}_k\}$  for  $V$ . Gram-Schmidt this basis to make it orthonormal:  $V = \underbrace{\text{span}(\vec{v}_1, \dots, \vec{v}_k)}_{\text{orthonormal set of vectors}}$

Then extend this to a basis  $(\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n)$  for  $\mathbb{R}^n$  (by, for instance, successively throwing in vecs from  $\{\vec{e}_1, \dots, \vec{e}_n\}$  that aren't already in the span, until that set of vecs spans all of  $\mathbb{R}^n$ ). Then do Gram-Schmidt again (first  $k$  vecs won't change...) to get an orthonormal basis  $\{\vec{u}_1, \dots, \vec{u}_k, \vec{u}_{k+1}, \dots, \vec{u}_n\}$  for  $\mathbb{R}^n$ . Then  $\{\vec{v}_1, \dots, \vec{v}_k\}$

is an o.n. basis for  $V$ , and  $\{\vec{u}_{k+1}, \dots, \vec{u}_n\}$  is an o.n. basis for  $V^\perp$ . The orthogonal projection  $T$  will have:  $T(\vec{v}_i) = \vec{v}_i$  for  $i=1, \dots, k$  and  $T(\vec{u}_j) = 0 \vec{u}_j$  for  $j=k+1, \dots, n$ , so  $T$  has an eigenbasis, thus is diagonalizable. Its e-vals will be 0 and 1.

21 All diagonal matrices are invertible.

↪ F:  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  is diagonal, but not invertible.

22 If  $\vec{v}$  is an e-vec for both  $A$  &  $B$ , then  $\vec{v}$  must also be an e-vec for  $A+B$ .

↪ T:  $\begin{cases} A\vec{v} = \lambda\vec{v} \\ B\vec{v} = \mu\vec{v} \end{cases} \Rightarrow (A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda + \mu)\vec{v}.}$

23 If  $A^2$  is diagonalizable, then  $A$  must be diagonalizable as well.

↪ F:  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Then  $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , which is diagonalizable.

But  $A$  is not diagonalizable, since  $\text{gemu}(0) = 1 < 2 = \text{almu}(0)$ .

24 The determinant of a matrix is the product of its eigenvalues (over  $\mathbb{C}$ , counting multiplicities).

↪ T: this is proven by looking at the constant terms of the characteristic eqn  $\det(A - \lambda I_n) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda) = 0$ .

25 All lower triangular matrices are diagonalizable (over  $\mathbb{C}$ ).

↪ F:  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  isn't.

26 If two  $n \times n$  matrices  $A$  &  $B$  are diagonalizable, then  $AB$  must be diagonalizable as well.

↪ F:  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1/2 & 1 \\ 0 & 2 \end{pmatrix}$ . Then  $A$  &  $B$  each have 2 distinct e-vals ( $\frac{1}{2}$  &  $2$ ), hence are diagonalizable.

However,  $AB = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ , which is not diagonalizable, since  $1 = \text{gemu}(1) < \text{almu}(1) = 2$ . Hard problem!

27 If  $A$  is invertible and diagonalizable, then  $A^{-1}$  must be diagonalizable as well.

→ T:  $A = SDS^{-1}$  invertible means  $A$ 's e-vals are all nonzero, so  $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$  has nonzero diagonal entries, so  $D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{pmatrix}$ , so  $A^{-1} = (SDS^{-1})^{-1} = (S^{-1})^{-1} D^{-1} S^{-1} = S D^{-1} S^{-1}$ , so  $A^{-1}$  is diagonalizable.

28 If  $\det(A) = \det(A^T)$ , then  $A$  must be symmetric.

→ F:  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is a counterexample. Note:  $\det(A)$  is always equal to  $\det(A^T)$ , for any square matrix  $A$ .

29 If matrix  $A = \begin{pmatrix} 7 & a & b \\ & 7 & c \\ & & 7 \end{pmatrix}$  is diagonalizable, then  $a, b, \& c$  must all be zero.

→ T: the only eval is 7, with  $\text{almu}(7)=3$ . In order for  $\text{genu}(7)=3$ , it must be that  $\text{nul}(A-7I_3)$  is 3; in other words,  $A-7I_3$  must be the zero matrix.

$$A-7I_3 = \begin{pmatrix} 0 & ab \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \text{ so } a=b=c=0 \text{ is forced.}$$

30 If two  $n \times n$  matrices  $A$  &  $B$  are diagonalizable, then  $A+B$  must be diagonalizable as well.

→ F:  $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ ,  $B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$  are both diagonalizable, since they each have two distinct e-vals. But  $A+B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$  is not diagonalizable, since  $1 = \text{genu}(0) < \text{almu}(0) = 2$ .

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31 If  $\vec{u}, \vec{v}, \& \vec{w}$  are e-vecs of a  $4 \times 4$  matrix  $A$  with e-vals 3, 7, & 11, respectively, then  $\vec{u}, \vec{v}, \& \vec{w}$  must be LI.

→ T: e-vecs corresponding to distinct e-vals are always linearly independent.

32 If a  $4 \times 4$  matrix  $A$  is diagonalizable, then  $A + 4I_4$  must be diagonalizable as well.

→ T: if  $A = SDS^{-1}$ , then  $A + 4I_4 = SDS^{-1} + S(4I_4)S^{-1} = S(D + 4I_4)S^{-1}$ , still diagonal

$$A + 4I_4 = SDS^{-1} + S(4I_4)S^{-1} = S(D + 4I_4)S^{-1},$$

so  $A + 4I_4$  is diagonalizable.

33 If an  $n \times n$  matrix  $A$  is diagonalizable, then it must have  $n$  distinct e-vals.

→ F:  $A = I_n$  is a counterexample, since  $\text{genu}(1) = n = \text{alge}(1)$ .

34 If two  $3 \times 3$  matrices  $A$  &  $B$  have the e-vals 1, 2, & 3, then they must be similar.

→ T:  $A$  &  $B$  are both diagonalizable, so

$$A = S_1 D S_1^{-1}, B = S_2 D S_2^{-1}, \text{ where } D = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}.$$

$$\text{Then } D = S_1^{-1} A S_1 = S_2^{-1} B S_2, \text{ so } A = S_1 S_2^{-1} B S_2 S_1^{-1}.$$

$$\text{Note that } (S_1 S_2^{-1})^{-1} = S_2 S_1^{-1}.$$

35 If  $\vec{v}$  is an e-vec of  $A$ , then  $\vec{v}$  must be an e-vec for  $A^T$ .

→ F:  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Then  $A\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{v}$ , but  $A^T\vec{v} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , and  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$  is not a multiple of  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

36 All invertible matrices are diagonalizable.

↪ F:  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is invertible, but not diagonalizable.

37 If  $\vec{v}$  &  $\vec{w}$  are LI e-vecs of  $A$ , then  $\vec{v} + \vec{w}$  must also be an e-vec of  $A$ .

↪ F:  $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then  $A\vec{v} = 2\vec{v}$ ,  $A\vec{w} = 3\vec{w}$ , and  $\vec{v}$  &  $\vec{w}$  are LI, but  $A(\underbrace{\vec{v} + \vec{w}}_{= \begin{pmatrix} 1 \\ 1 \end{pmatrix}}) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ , which is not in  $\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ .

38 If a  $2 \times 2$  matrix  $R$  represents a reflection about a line  $L$ , then  $R$  must be diagonalizable.

↪ T: say  $L = \text{span}(\vec{v})$ ,  $L^\perp = \text{span}(\vec{w})$ . Then

$R\vec{v} = \vec{v}$  and  $R\vec{w} = -\vec{w}$ , so  $R$  is diagonalizable (has e-vals 1 & -1). Alternatively:  $\text{refl}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$

and  $\text{proj}_L(\vec{x}) = UU^T\vec{x}$ , where  $U$  has orthonormal cols that form a basis for the subspace  $L$ . Then the projection matrix  $UU^T$  is symmetric, so the reflection matrix

$R = \underbrace{2UU^T}_{\text{symmetric}} - \underbrace{I}_{\text{symmetric}}$  is symmetric, so it is diagonalizable.

For this reason, all reflection & projection matrices are symmetric (and thus orthogonally diagonalizable) by the spectral theorem.

39 If  $A$  is a  $2 \times 2$  matrix s.t.  $\text{tr}(A)=1$  and  $\det(A)=-6$ , then  $A$  must be diagonalizable.

$$\hookrightarrow T: \det(A - \lambda I_2) = \lambda^2 - (\text{tr}(A))\lambda + \det(A)$$

$$= \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

$\Rightarrow \lambda = -2, 3$ , so  $A$  has 2 distinct e-vals and thus is diagonalizable.

40 If a matrix is diagonalizable, then  $\text{algeu}(\lambda) = \text{geomu}(\lambda)$  for each of the e-vals.

$\hookrightarrow T$ : this is one of the main characterizations of diagonalizability.

41 All orthogonal matrices are diagonalizable over  $\mathbb{R}$ .

$\hookrightarrow F$ :  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is orthogonal and diagonalizable over  $\mathbb{C}$ , but not over  $\mathbb{R}$  (the e-vals are  $\pm i$ ). It is true in general that orthogonal matrices are diagonalizable over  $\mathbb{C}$ , but this is a difficult theorem that you should see in math 115A if you take that class!

42 If  $A$  is an  $n \times n$  matrix and  $\lambda$  is an e-val of the block matrix  $M = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ , then  $\lambda$  must be an e-val of  $A$ .

$$\hookrightarrow T: \det(M - \lambda I_{2n}) = \det \begin{pmatrix} A - \lambda I_n & A \\ 0 & A - \lambda I_n \end{pmatrix} = \det(A - \lambda I_n) \det(A - \lambda I_n) = (\det(A - \lambda I_n))^2,$$

so every e-val of  $M$  comes from  $A$  and has twice the algebraic multiplicity.

43 If two matrices  $A$  &  $B$  have the same characteristic polynomials, then they must be similar.

↪ T:  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Both have the same char. poly. and hence the same e-vals with the same almu's, but they have different genmu's, and thus cannot be similar. Alternatively, suppose they are similar.

Then  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = S S^{-1} = I$ , a contradiction.

44 If  $A$  is  $4 \times 4$  diagonalizable, and  $A^4 = 0$ , then  $A$  must be the zero matrix.

$$\hookrightarrow T: A = S D S^{-1}, \text{ so } A^4 = (S D S^{-1})^4 = (S D^4 S^{-1})^2 = (S D^2 S^{-1})^2 = S D^2 \underbrace{S^{-1} S D^2 S^{-1}}_I = S D^4 S^{-1} = S \underbrace{D^4}_I S^{-1} = 0,$$

So  $D^4 S^{-1} = S^{-1} 0 = 0$ , and thus  $D^4 = 0 S = 0$ , so

$$D^4 = \begin{pmatrix} d_1^4 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_4^4 \end{pmatrix} = 0 \Rightarrow d_1 = \dots = d_4 = 0, \text{ so } A = S D S^{-1} = 0.$$

45 If an  $n \times n$  matrix  $A$  is diagonalizable (over  $\mathbb{R}$ ), then every vector  $\vec{v} \in \mathbb{R}^n$  can be expressed as a sum of eigenvectors of  $A$ .

↪ T: this is the same as saying that diagonalizable matrices admit an eigenbasis. Note that nonzero multiples of e-vecs are still e-vecs.

46 If  $\vec{v}$  is an e-vec for both  $A$  &  $B$ , then  $\vec{v}$  is an e-vec for  $AB$ .

↪ T:  $(AB)\vec{v} = A(B\vec{v}) = A(\lambda_2 \vec{v}) = \lambda_2 A\vec{v} = \lambda_2 \lambda_1 \vec{v}$ , so  $\vec{v}$  is an e-vec for  $AB$ , with corresponding e-val  $(\lambda_1 \lambda_2)$ .

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47 Similar matrices have the same characteristic polynomials.

$$\hookrightarrow T: \text{If } A = SBS^{-1}, \text{ then } \det(A - \lambda I) = \det(SBS^{-1} - \lambda I)$$

$$= \det(SBS^{-1} - \lambda SIS^{-1}) = \det(S(BS^{-1} - \lambda IS^{-1}))$$

$$= \det(S(B - \lambda I)S^{-1}) = \det(S)\det(B - \lambda I)\det(S^{-1})$$

$$= \det(\underbrace{SS^{-1}}_I)\det(B - \lambda I) = \det(B - \lambda I).$$

48 If  $A$  has  $k$  distinct e-vals, then  $\text{rank}(A) \geq k$ .

$\hookrightarrow F: A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$  has 4 distinct e-vals but  $\text{rank}(A)=3$ .

49 If the rank of a square matrix is 1, then all the nonzero vectors in  $\text{im}(A)$  are e-vecs of  $A$ .

$\hookrightarrow T: \text{Say } \text{im}(A) = \text{span}(\vec{v}), \vec{v} \neq \vec{0}. \text{ Say } A = (c_1\vec{v} \cdots c_n\vec{v}).$

$$\text{Then } A(c\vec{v}) = cA\vec{v} = c(c_1\vec{v}, c_2\vec{v}, \dots, c_n\vec{v})$$

$$= c(c_1\vec{v}, \dots, c_n\vec{v})\vec{v} = (\text{tr}(A))(c\vec{v}) \text{ for each } c \neq 0,$$

so such  $c\vec{v}$  are e-vecs of  $A$ .

50 If  $\text{rank}(A)=1$ , then  $A$  is diagonalizable.

$\hookrightarrow F: A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  has rank 1, but is not diagonalizable.

Note: see solutions to problems from 7.1 for a proof that rank 1 matrices are diagonalizable iff they have nonzero trace!

51 If  $A$  is  $4 \times 4$  and  $A^4 = \vec{0}$ , then  $\vec{0}$  is the only e-val of  $A$ .

$\hookrightarrow T: \text{Suppose } A\vec{v} = \lambda\vec{v}, \text{ where } \vec{v} \neq \vec{0}. \text{ Then } \underbrace{A^4\vec{v}}_{\vec{0}} = \lambda^4\vec{v} = \vec{0},$   
forcing  $\lambda = 0$ .

52 If two  $n \times n$  matrices  $A$  &  $B$  are both diagonalizable, then they must commute.

↪ T:  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then both  $A$  &  $B$  are diagonalizable, but  $AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = BA$ .

53 If  $\vec{v}$  is an e-vec of  $A$ , then  $\vec{v} \in \text{im}(A)$  or  $\vec{v} \in \text{ker}(A)$ .

↪ T:  $\vec{v} \neq \vec{0}, A\vec{v} = \lambda\vec{v} \Rightarrow \begin{cases} \text{if } \lambda = 0, \text{ then } A\vec{v} = \vec{0}, \text{ so } \vec{v} \in \text{ker}(A). \\ \text{if } \lambda \neq 0, \text{ then } A(\frac{1}{\lambda}\vec{v}) = \vec{v}, \text{ so } \vec{v} \in \text{im}(A). \end{cases}$

54 All symmetric  $2 \times 2$  matrices are diagonalizable (over  $\mathbb{R}$ ).

↪ T: this is part of the spectral theorem from ch. 8.

Alternatively, say  $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ . Then  $\det(A - \lambda I_2) = \lambda^2 - (a+c)\lambda + ac - b^2 = 0 \Rightarrow \lambda = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 + 4b^2 - 4ac}}{2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$ , so we get two distinct e-vals if either  $a \neq c$  or  $b \neq 0$ . If  $b=0$  and  $a=c$ , then  $A$  is already diagonal. Thus for any  $a, b, \neq c$ ,  $A$  is diagonalizable.

55 If  $A$  is  $2 \times 2$  with e-vals 3 & 4 and if  $\vec{u}$  is a unit e-vec of  $A$ , then  $\|A\vec{u}\| \leq 4$ .

↪ T:  $A\vec{u} = \lambda\vec{u}$ , where  $\lambda = 3$  or  $4$ . So  $\|A\vec{u}\| = \|\lambda\vec{u}\| = |\lambda| \|\vec{u}\| = |\lambda| \leq 4$ .

56 If  $\vec{u}$  is a nonzero vector in  $\mathbb{R}^n$ , then  $\vec{u}$  must be an eigenvector of matrix  $\vec{u}\vec{u}^T$ .

↪ T:  $(\vec{u}\vec{u}^T)\vec{u} = \vec{u}(\vec{u}^T\vec{u}) = \vec{u}(\vec{u} \cdot \vec{u}) = \vec{u}(\|\vec{u}\|)^2 = \|\vec{u}\|^2 \vec{u}$ , so  $\vec{u}$  is an e-vec w/ e-val  $\|\vec{u}\|^2$ . Note  $\vec{u}\vec{u}^T$  is a scaled projection.

B

57 If  $\vec{v}_1, \dots, \vec{v}_n$  is an eigenbasis for both A and B, then A & B must commute.

↪ T: Say  $S = [\vec{v}_1 \dots \vec{v}_n]$ . Then  $A = SD_1S^{-1}$  and

$B = SD_2S^{-1}$  for some diagonal  $D_1$  &  $D_2$ . Then

$$AB = SD_1S^{-1}SD_2S^{-1} = SD_1D_2S^{-1} = S\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix} S^{-1}$$

$$= S\begin{bmatrix} \lambda_1 u_1 & & \\ & \ddots & \\ & & \lambda_n u_n \end{bmatrix} S^{-1} = S\begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} S^{-1}$$

$= SD_2D_1S^{-1} = SD_2S^{-1}SD_1S^{-1} = BA$ . Note that all diagonal matrices commute with each other!

58 If  $\vec{v}$  is an e-vec of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then  $\vec{v}$  must be an e-vec of its classical adjoint  $\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  as well.

$$\hookrightarrow T: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

$$\implies \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} dv_1 - bv_2 \\ -cv_1 + av_2 \end{pmatrix} = \begin{pmatrix} dv_1 - bv_2 + av_1 + bv_2 - \lambda v_1 \\ -cv_1 + av_2 + cv_1 + dv_2 - \lambda v_2 \end{pmatrix}$$

$$\stackrel{\text{adj}(A)}{=} \begin{pmatrix} (d+a-\lambda)v_1 \\ (d+a-\lambda)v_2 \end{pmatrix} = (a+d-\lambda) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \text{ Alternatively,}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} = (a+d)\mathbb{I}, \text{ so } \text{adj}(A) = (a+d)\mathbb{I} - A.$$

Thus if  $\vec{v}$  is an e-vec of A, then  $A\vec{v} = \lambda\vec{v}$ , and

$$\text{adj}(A)\vec{v} = ((a+d)\mathbb{I} - A)\vec{v} = (a+d)\vec{v} - A\vec{v} = (a+d - \lambda)\vec{v}, \text{ so } \vec{v} \text{ is also an e-vec of } \text{adj}(A).$$

