

Selected Problems & Solutions from 7.5

Note: there are many hidden secrets of complex numbers. For more, take Math 132.

1 Write $z = 3 - 3i$ in polar form.

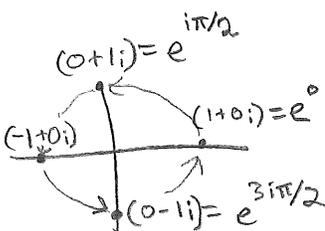
$$\rightarrow \begin{array}{c} 3 \\ | \\ \hline -3i \end{array} \quad r = \sqrt{z\bar{z}} = \sqrt{(3+3i)(3-3i)} = \sqrt{9-9i^2} \\ = \sqrt{9+9} = 3\sqrt{2}$$

$$\theta = \arctan\left(\frac{-3}{3}\right) \text{ in 2nd quadrant}$$

$$= -\frac{\pi}{4}, \text{ so } z = 3\sqrt{2} e^{-i\pi/4} = 3\sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$$

2 Solve $z^4 = 1$.

$$\rightarrow z = e^{2\pi ki/4}, \quad k=0,1,2,3.$$



3 Solve $z^n = 1$.

$$\rightarrow z = e^{2\pi ik/n}, \quad k=0,1,\dots,n-1.$$

4 Solve $w^2 = re^{i\theta}$.

$$\rightarrow w = \sqrt{r} e^{i\theta/2}, \quad \sqrt{r} e^{i(\theta/2 + \pi)} \\ = -\sqrt{r} e^{i\theta/2}.$$

5 Solve $w^n = re^{i\theta}$.

$$\rightarrow w = \sqrt[n]{r} e^{i(\theta + 2\pi k)/n}, \quad k=0,1,\dots,n-1.$$

6 If $z = re^{i\theta}$, represent \bar{z} and $\frac{1}{z}$ in polar form.

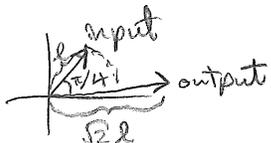
$$\rightarrow \bar{z} = \overline{re^{i\theta}} = \overline{(r\cos\theta + ir\sin\theta)} = r\cos\theta - ir\sin\theta = r(\cos(-\theta) + i\sin(-\theta)) \\ = re^{-i\theta}$$

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right) e^{-i\theta}, \text{ as long as } r \neq 0.$$

7 Describe the transformation $T: \mathbb{C} \rightarrow \mathbb{C}$, $T(z) = (1-i)z$.

$$\begin{aligned} \hookrightarrow T(z) &= (1-i)z = \sqrt{2} \left(\frac{1-i}{\sqrt{2}} \right) r e^{i\theta} \quad (\text{say } z = r e^{i\theta}) \\ &= \sqrt{2} e^{-i\pi/4} r e^{i\theta} = \sqrt{2} r e^{i(\theta - \pi/4)} \end{aligned}$$

so T rotates complex inputs by $\frac{\pi}{4}$ radians clockwise & scales them by a factor of $\sqrt{2}$.



8 Express $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

$$\hookrightarrow \cos(3\theta) = \frac{\cos(3\theta) + i\sin(3\theta)}{2} + \frac{\cos(3\theta) - i\sin(3\theta)}{2}$$

$$= \frac{\cos(3\theta) + i\sin(3\theta)}{2} + \frac{\cos(-3\theta) + i\sin(-3\theta)}{2}$$

$$= \frac{1}{2} e^{3\theta i} + \frac{1}{2} e^{-3\theta i} = \frac{1}{2} (e^{i\theta})^3 + \frac{1}{2} (e^{-i\theta})^3$$

for rigorous justification, take 132 or 246a

$$= \frac{1}{2} (\cos\theta + i\sin\theta)^3 + \frac{1}{2} (\cos(-\theta) + i\sin(-\theta))^3$$

$$= \frac{1}{2} (\cos\theta + i\sin\theta)^3 + \frac{1}{2} (\cos\theta - i\sin\theta)^3$$

$$\sin(3\theta) = \frac{\cos(3\theta) + i\sin(3\theta)}{2i} - \frac{\cos(-3\theta) + i\sin(-3\theta)}{2i}$$

$$= \frac{1}{2i} (e^{3\theta i} - e^{-3\theta i}) = \frac{1}{2i} ((e^{i\theta})^3 - (e^{-i\theta})^3)$$

$$= \frac{1}{2i} ((\cos\theta + i\sin\theta)^3 - (\cos\theta - i\sin\theta)^3)$$

12 If $f(\lambda)$ is a polynomial with real coefficients, show that if $\lambda_0 \in \mathbb{C}$ is a root of $f(\lambda)$, then so is $\bar{\lambda}_0$.

$$\hookrightarrow f(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0, \text{ so}$$

$$\begin{aligned} 0 = f(\lambda_0) &= a_n (\lambda_0)^n + \dots + a_0. \text{ Then } 0 = \overline{0} = \overline{a_n (\lambda_0)^n + \dots + a_0} = \overline{a_n} (\overline{\lambda_0})^n + \dots + \overline{a_0} \\ &= a_n (\overline{\lambda_0})^n + \dots + a_0, \text{ so } f(\overline{\lambda_0}) = 0. \end{aligned}$$

Note: $\overline{(a+bi)(c+di)} = \overline{(ac-bd) + i(bc+ad)}$

$$= ac-bd - i(bc+ad) = (a-bi)(c-di) = \overline{(a+bi)(c+di)}, \text{ and similarly for sums.}$$

For 20-26, find all e-val.

20 $\begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

$\hookrightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i.$

21 $\begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix}$

$\hookrightarrow \lambda^2 - 4\lambda + 13 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$

22 $\begin{pmatrix} 1 & 3 \\ -4 & 10 \end{pmatrix}$

$\hookrightarrow \lambda^2 - 11\lambda + 22 = 0 \Rightarrow \lambda = \frac{11 \pm \sqrt{121 - 88}}{2} = \frac{11 \pm \sqrt{33}}{2}$

23 $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\hookrightarrow -\lambda((- \lambda)^2) + 1 = -\lambda^3 + 1 = 0 \Rightarrow \lambda = e^{2\pi i k/3}, k=0,1,2.$

24 $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$

$\hookrightarrow -\lambda(-\lambda(3-\lambda)+7) + 5 = -\lambda^3 + 3\lambda^2 - 7\lambda + 5 = 0$

test $\lambda = \pm \frac{\text{factors of } 5}{\text{factors of } 1} \dots \lambda = 1$ works.

$\lambda - 1 \overline{) \begin{array}{r} -\lambda^2 + 2\lambda - 5 \\ -\lambda^3 + 3\lambda^2 - 7\lambda + 5 \\ \hline -\lambda^3 + \lambda^2 \\ \hline 2\lambda^2 \\ 2\lambda^2 - 2\lambda \\ \hline -5\lambda \\ -5\lambda + 5 \\ \hline 0 \end{array}}$, so $-(\lambda-1)(\lambda^2 - 2\lambda + 5) = 0$
 $= (\lambda-1)^2 + 4$, so $\lambda = 1, 1 \pm 2i.$

25 $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\hookrightarrow \lambda^4 - 1 = 0$, so $\lambda = \pm 1, \pm i$ ($e^{i\pi k/2}, k=0,1,2,3$).

26 $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

\hookrightarrow utilize block structure: $(\lambda^2 - 2\lambda + 1)((1-\lambda)^2 - 1) = 0 \Rightarrow \lambda = 1, 0, 2.$
 $= (\lambda-1)^2$ \downarrow
algebra = 2

27 Suppose a real 3×3 matrix A has only two distinct e-vals, $\text{tr}(A)=1$, and $\det(A)=3$. Find the e-vals of A .

\hookrightarrow Say the e-vals are $\lambda = a, b$. Then
 \downarrow \downarrow
 $\text{alnu } 1$ $\text{alnu } 2$

$\text{tr}(A) = 1 = a + b + b$, $\det(A) = 3 = abb$.

So neither a nor b is 0, and $a = \frac{3}{b^2}$. Thus

$1 = \frac{3}{b^2} + 2b$, or $2b^3 - b^2 + 3 = 0$. $b = -1$ works:

$$\begin{array}{r}
 b+1 \overline{) 2b^3 - b^2 + 3} \\
 \underline{2b^3 - b^2 + 0b + 3} \\
 2b^3 + 2b^2 \\
 \underline{-3b^2} \\
 -3b^2 - 3b \\
 \underline{3b} \\
 3b + 3 \\
 \underline{0}
 \end{array}$$

Note a & b are real, since complex roots must come in conjugate pairs.

$2b^2 - 3b + 3 = 0$ implies

$b = \frac{3 \pm \sqrt{9 - 24}}{4} = \text{complex!}$

$b = -1$. Thus $a = 3$.

28 Suppose A is 3×3 with e-val 2 and also two complex-conjugate e-vals. Also, $\det(A) = 50$ and $\text{tr}(A) = 8$. Find the complex e-vals.

\hookrightarrow say they are z and \bar{z} . Then $50 = \det(A) = 2z\bar{z}$ and

$8 = \text{tr}(A) = 2 + z + \bar{z}$. Say $z = a + bi$.

$\bar{z} = a - bi$, and $6 = a + bi + a - bi = 2a$, so $a = 3$.

$= |z|^2$, since $(a+bi)(a-bi) = a^2 + b^2$.

Also, $25 = |z|^2 = a^2 + b^2 = 9 + b^2$, so $b = \pm 4$. Thus the complex e-vals are $3 \pm 4i$.

29 If $A = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$, where $a, b, c, d > 0$, and A has three distinct real e-vals, what can you say about the sign of the e-vals? Is the eval with the largest absolute value positive or negative?

↳ $\text{tr}(A) = 0 + 0 + 0 = 0 = \lambda_1 + \lambda_2 + \lambda_3$
 $\det(A) = -c(-bd) = \underbrace{bcd}_{>0} = \lambda_1 \lambda_2 \lambda_3 > 0$

So either none or two of the e-vals are negative. Looking @ the trace, we know there must be exactly two negative e-vals (or else $\lambda_1 + \lambda_2 + \lambda_3 > 0$), and the last (positive) e-val must be the largest in magnitude (in order to cancel the sum of the negative e-vals in the trace equation).

30 a. If $2i$ is an e-val of a real 2×2 matrix; find A^2 .

b. Give an example of a real 2×2 matrix A s.t. all the entries of A are nonzero and $2i$ is an eval of A . Check that A^2 is what you got for part (a).

↳ a. e-vals of real matrices come in complex conjugate pairs, so $\lambda = \pm 2i$ are the e-vals. So A is diagonalizable:

$A = S \begin{pmatrix} 2i & \\ & -2i \end{pmatrix} S^{-1}$, so $A^2 = S \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}^2 S^{-1} = S \begin{pmatrix} -4 & \\ & -4 \end{pmatrix} S^{-1}$
 $= S(-4)I_2 S^{-1} = -4 S I_2 S^{-1} = -4 S S^{-1} = -4 I_2$.

b. $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & \\ & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -6 \\ 6 & -8 \end{pmatrix} = \begin{pmatrix} 36 & -50 \\ 26 & -36 \end{pmatrix}$ (works since $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ has e-vals $\pm 2i$).

Then $A^2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$, and $\lambda^2 - (\text{tr} A)\lambda + \det A = \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$.

