

## Selected Problems & Solutions from 7.2

For 1-13, find all eigenvalues.

1  $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$$\hookrightarrow \det\left(\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \lambda I_2\right) = \det\left(\begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix}\right) = (1-\lambda)(3-\lambda) = 0 \iff \lambda = 1, 3.$$

For any triangular matrix, the e-vals are the diagonal entries.

3  $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

$$\hookrightarrow \det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0 \iff \lambda = 1, 3$$

2  $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix}$

$$\hookrightarrow \lambda = 1, 2, \text{ where } \text{alnu}(1) = 2 = \text{alnu}(2).$$

In fact, the characteristic polynomial is  $(\lambda - 1)^2(\lambda - 2)^2$ .

4  $\begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$

$$\hookrightarrow \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \iff \lambda = 2, \text{ alnu}(2) = 2.$$

5  $\begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix}$

$$\hookrightarrow \lambda^2 - 4\lambda + 13 = 0 \iff (\lambda - 2)^2 = -9 \iff \lambda = 2 \pm 3i.$$

6  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\hookrightarrow \lambda^2 - 5\lambda - 2 = 0 \iff \lambda = \frac{5 \pm \sqrt{25 + 8}}{2} = \frac{5 \pm \sqrt{33}}{2}.$$

7  $I_3$

$$\hookrightarrow \lambda = 1, \text{ alnu}(1) = 3.$$

8  $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$

$\hookrightarrow$  easier to find e-vecs first:  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .  $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0} = A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  and  $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , so  $\text{genu}(0) \geq 2$  and  $\text{genu}(-3) \geq 1$ . So  $\lambda = 0, 3$  with  $\text{alnu}(0) \geq 2$  and  $\text{alnu}(-3) \geq 1$ .

9  $\begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}$

$\hookrightarrow (3-\lambda)(-\lambda(2-\lambda)) - 1((-2)(2-\lambda)) = -6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 - 4 + 2\lambda$

$= -\lambda^3 + 5\lambda^2 - 4\lambda - 4 = 0$ . Try:  $\lambda = \pm \frac{\text{(factor of 4)}}{\text{(factor of 1)}} \dots$

$\lambda = 2$  works:

$$\begin{array}{r} -\lambda^2 + 3\lambda + 2 \\ \lambda - 2 \overline{) -\lambda^3 + 5\lambda^2 - 4\lambda - 4} \\ \underline{-\lambda^3 + 2\lambda^2} \phantom{-4} \\ 3\lambda^2 \phantom{-4} \\ \underline{3\lambda^2 - 6\lambda} \phantom{-4} \\ 2\lambda \phantom{-4} \\ \underline{2\lambda - 4} \\ 0 \end{array}$$

so the characteristic equation is

$(\lambda - 2)(-\lambda^2 + 3\lambda + 2) = 0$

$\lambda = \frac{-3 \pm \sqrt{9 + 8}}{-2} = \frac{3}{2} \pm \frac{\sqrt{17}}{2}$

so the e-values are  $\lambda = 2, \frac{3 \pm \sqrt{17}}{2}$ , each with algebraic multiplicity 1.

10  $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

$\hookrightarrow (-3-\lambda)(-1-\lambda)(3-\lambda) - 2(-4(1-\lambda)) = (-3-\lambda)(\lambda^2 - 2\lambda - 3) - 8 - 8\lambda$

$= -\lambda^3 - \lambda^2 + 9\lambda + 9 - 8 - 8\lambda = -\lambda^3 - \lambda^2 + \lambda + 1 = 0$

$\lambda = 1$  works:

$$\begin{array}{r} -\lambda^2 - 2\lambda - 1 \\ \lambda - 1 \overline{) -\lambda^3 - \lambda^2 + \lambda + 1} \\ \underline{-\lambda^3 + \lambda^2} \phantom{+ \lambda + 1} \\ -2\lambda^2 \phantom{+ \lambda + 1} \\ \underline{-2\lambda^2 + 2\lambda} \phantom{+ 1} \\ -\lambda \phantom{+ 1} \\ \underline{-\lambda + 1} \\ 0 \end{array}$$

$= (\lambda - 1)(-\lambda^2 - 2\lambda - 1)$

$= -(\lambda - 1)(\lambda + 1)^2 = 0$

$\iff \lambda = 1, -1$

$\text{al mu}(1) = 1$

$\text{al mu}(-1) = 2$ .

11  $\begin{pmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{pmatrix}$

$\hookrightarrow (5-\lambda)(1-\lambda)(-7-\lambda) - 2(-7-\lambda+10) + 8(5(1-\lambda))$

$= (5-\lambda)(\lambda^2 + 6\lambda - 7) - 6 + 2\lambda + 40 - 40\lambda = -\lambda^3 - \lambda^2 - \lambda - 1 = 0$

$\lambda = -1$  works:

$$\begin{array}{r} -\lambda^2 + 0\lambda - 1 \\ \lambda + 1 \overline{) -\lambda^3 - \lambda^2 - \lambda - 1} \\ \underline{-\lambda^3 - \lambda^2} \phantom{- \lambda - 1} \\ 0 \phantom{- \lambda - 1} \\ \underline{0 + 0} \phantom{- \lambda - 1} \\ -\lambda \phantom{- 1} \\ \underline{-\lambda - 1} \\ 0 \end{array}$$

$= (\lambda + 1)(-\lambda^2 - 1)$

$= -(\lambda + 1)(\lambda^2 + 1)$

$= -(\lambda + 1)(\lambda - i)(\lambda + i)$

$\iff \lambda = -1, \pm i$ .

12 
$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{pmatrix}$$

↳ note if  $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_4 \end{pmatrix}$  is an evec for A, then

$A\vec{v} = \lambda\vec{v} \implies \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  and  $\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \lambda \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$ .

The reverse direction works as well, by appending zeros to evecs of  $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$  and  $\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$ .

$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}: \lambda^2 - \lambda = \lambda(\lambda - 1)$  .  $\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}: \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$ .

Thus  $\lambda = 0, 1, -1$ , and  $\text{almu}(0) = 1$ ,  
 $\text{almu}(1) = 2$ , and  $\text{almu}(-1) = 1$ .

13 
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

↳  $(-\lambda)(-\lambda)(-\lambda) + (1) = -\lambda^3 + 1 = -(\lambda^3 - 1) = -(\lambda - 1) \underbrace{(\lambda^2 + \lambda + 1)}_{(\lambda + \frac{1}{2})^2 + \frac{3}{4}} = 0$ ,

so  $\lambda = 1, -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ .

14 If  $4 \times 4$   $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$  and B, C, D are  $2 \times 2$ , what is the relationship b/w the e-vals of A, B, C, & D?

↳ by patterns, we know  $\det(A - \lambda I_4) = \det \begin{pmatrix} B - \lambda I_2 & C \\ 0 & D - \lambda I_2 \end{pmatrix}$   
 $= \det \begin{pmatrix} B - \lambda I_2 & 0 \\ 0 & D - \lambda I_2 \end{pmatrix} \stackrel{\text{again, by patterns}}{=} \det(B - \lambda I_2) \det(D - \lambda I_2)$ .

So the e-vals of A will be the combined e-vals of B & D with respective algebraic multiplicities equal to the sum of the corresponding algebraic multiplicities of that e-val in B & D.  
 C has no effect on A's e-vals.

15 How many e-vals does  $\begin{pmatrix} 1 & k \\ 1 & 1 \end{pmatrix}$  have?

$$\hookrightarrow \lambda^2 - 2\lambda + (1-k) = (\lambda-1)^2 - k = 0 \iff \lambda = 1 \pm \sqrt{k}$$

If  $k=0$ , get just  $\lambda=1$  with  $\text{almu}(1)=2$ . If  $k>0$ , get  $\lambda = 1 \pm \sqrt{k}$ . If  $k<0$ , get complex conjugate e-vals

$$\lambda = 1 \pm \sqrt{k} = 1 \pm i\sqrt{\frac{-k}{>0}}$$

16 When does  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  have 2 distinct e-vals?

$$\begin{aligned} \hookrightarrow \lambda^2 - (a+c)\lambda + (ac-b^2) &= \left(\lambda - \frac{a+c}{2}\right)^2 + ac - b^2 - \left(\frac{a+c}{2}\right)^2 \\ &= \left(\lambda - \frac{a+c}{2}\right)^2 - b^2 - \frac{(a-c)^2}{4} = 0 \end{aligned}$$

$$\iff \lambda = \frac{a+c}{2} \pm \frac{\sqrt{4b^2 + (a-c)^2}}{2}, \text{ so there are 2 distinct e-vals}$$

whenever either  $b \neq 0$ , or  $a \neq c$ , or both.

17 Find & Interpret all e-vals of  $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ .

$\hookrightarrow$  This is a reflection combined with a scaling, so our e-vals should be  $\pm c$  for some  $c \in \mathbb{R}$ .

$$\lambda^2 - a^2 - b^2 = 0 \iff \lambda = \pm \sqrt{a^2 + b^2}, \text{ which makes sense, since } \sqrt{a^2 + b^2} \text{ is the scaling factor.}$$

18 Find all e-vals of  $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ .

$$\hookrightarrow \lambda^2 - 2a\lambda + a^2 - b^2 = (\lambda-a)^2 - b^2 = 0 \iff \lambda = a \pm b$$

Note that  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  are easily seen to be e-vectors, w/ corr. e-vals  $a+b$  and  $a-b$ .

19 True or False? If  $2 \times 2$   $A$  has negative determinant, then  $A$  has two distinct e-vals.

$\hookrightarrow$  True!  $\lambda^2 - (\text{tr}(A))\lambda + \underbrace{\det(A)}_{<0} = 0$ . This quadratic dips below the x-axis and has two upward-pointing prongs, hence admits 2 roots.

20 If  $2 \times 2$   $A$  has two distinct e-vals,  $\lambda_1 \neq \lambda_2$ , show that  $A$  is diagonalizable.

$\hookrightarrow A\vec{v}_1 = \lambda_1 \vec{v}_1$  and  $A\vec{v}_2 = \lambda_2 \vec{v}_2$  for some  $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ , and  $\vec{v}_1, \vec{v}_2$  are LI by an easy argument that's given in greater generality in 7.3. So  $A[\vec{v}_1 \vec{v}_2] = [A\vec{v}_1 \ A\vec{v}_2] = [\lambda_1 \vec{v}_1 \ \lambda_2 \vec{v}_2] = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$ , so  $A = SDS^{-1}$ , where  $S = [\vec{v}_1 \ \vec{v}_2]$  and  $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ .

21 Prove that if  $n \times n$   $A$  has the e-vals  $\lambda_1, \dots, \lambda_n$  (listed with multiplicities), then  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$ .

$\hookrightarrow$  The characteristic eqn is  $\det(A - \lambda I_n) = (-1)^n \lambda^n + (-1)^{n-1} (\text{tr}(A)) \lambda^{n-1} + \dots + (\det(A)) = 0$ .

For each  $\lambda_i$ ,  $(\lambda - \lambda_i)$  is a factor of the char. poly., so

$$(-1)^n \lambda^n + (-1)^{n-1} (\text{tr}(A)) \lambda^{n-1} + \dots + \det(A) = c(\lambda - \lambda_1) \dots (\lambda - \lambda_n),$$

so  $c = (-1)^n$ , and thus (expanding the right side and looking at the coefficient of  $\lambda^{n-1}$ )

$$(-1)^{n-1} (\text{tr}(A)) = (-1)^n (-\lambda_1 - \dots - \lambda_n),$$

so  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$ .

22 If  $A$  is  $n \times n$ , how are the e-vals of  $A$  &  $A^T$  related?

$$\hookrightarrow \det(A^T - \lambda I_n) = \det(A^T - \lambda I_n^T) = \det((A - \lambda I_n)^T) = \det(A - \lambda I_n),$$

so  $\lambda$  is an e-val for  $A$  if and only if  $\lambda$  is an e-val for  $A^T$ .  $A$  and  $A^T$  have exactly the same characteristic polynomial, so they have the same e-vals with the same algebraic multiplicities (same geom's too, but that requires more arguing).

23 If  $A$  and  $B$  are similar, how are their e-vals related?

$\hookrightarrow$  Say  $A = SBS^{-1}$ .  $\det(A - \lambda I_n) = \det(SBS^{-1} - \lambda I_n)$   
 $= \det(S(B - \lambda I_n)S^{-1}) = \det(S) \det(B - \lambda I_n) \det(S^{-1})$   
 $= \underbrace{\det(SS^{-1})}_{=1} \det(B - \lambda I_n) = \det(B - \lambda I_n)$ , so  $A$  &  $B$

have the same characteristic equations, and thus also the same e-vals & corresponding almu's. In fact, more arguing will allow us to deduce that the genu's are also the same for  $A$  &  $B$  among each of the eigenvalues.

24 Find  $\lim_{n \rightarrow \infty} A^n$ , where  $A = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{pmatrix}$ .

$\hookrightarrow$  First, diagonalize  $A$ .  $\lambda^2 - \frac{5}{4}\lambda + \frac{1}{4} = (\lambda - \frac{5}{8})^2 - \frac{9}{64} = 0$

$\implies \lambda = \frac{5}{8} \pm \frac{3}{8} = \frac{1}{4}, 1$ .  $E_{1/4} = \ker(A - \frac{1}{4}I) = \ker \begin{pmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$E_1 = \ker(A - 1I) = \ker \begin{pmatrix} -1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \implies A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1}$

$\implies A^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1} = S \begin{pmatrix} (1/4)^2 & 0 \\ 0 & (1)^2 \end{pmatrix} S^{-1}$ . Continuing

in this manner, we get  $A^n = S \begin{pmatrix} (1/4)^n & 0 \\ 0 & (1)^n \end{pmatrix} S^{-1} = S \begin{pmatrix} 1/4^n & 0 \\ 0 & 1 \end{pmatrix} S^{-1}$ .

Then  $\lim_{n \rightarrow \infty} A^n = S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}$ .

29 Let  $A$  be an  $n \times n$  matrix such that the sum of the entries in each row is 1. Show that  $\vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  is an e-vec of  $A$ , and find the corresponding e-val.

$\hookrightarrow (A\vec{e})_i = \sum_{l=1}^n (A)_{i,l} (\vec{e})_l = \sum_{l=1}^n (A)_{i,l} = 1$ , by assumption.

So  $A\vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \vec{e}$ , so  $\vec{e}$  is an e-vec with corr. e-val 1.

35 Give an example of a  $4 \times 4$  matrix with no real eigenvalues.

$\hookrightarrow \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$ . We discussed for an earlier problem (14) why the e-vals of this matrix are the concatenation of all the e-vals of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

$\lambda^2 + 1 = 0 \iff \lambda = \pm i$ , so the matrix above has e-vals  $\lambda = i$  and  $\lambda = -i$ , both with algebraic multiplicity 2. No real e-vals.

36 Find a  $2n \times 2n$  matrix  $A$  s.t.  $A$  has no real e-vals.

$\hookrightarrow$  Similarly to the previous problem, put  $A = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$ . Then the e-vals are  $\pm i$ , with almu's  $n$  &  $n$ .

37 Let  $A$  be an  $n \times n$  matrix with e-val  $\lambda_0$  of algebraic multiplicity at least 2. Show that  $f'_A(\lambda_0) = 0$ , where  $f_A$  is the characteristic polynomial.

$\hookrightarrow f_A(\lambda) = (\lambda - \lambda_0)^2 P(\lambda)$ , where  $P$  is some polynomial of degree  $(n-2)$ . Then  $f'_A(\lambda) = 2(\lambda - \lambda_0)P(\lambda) + (\lambda - \lambda_0)^2 P'(\lambda)$ ,

so  $f'_A(\lambda_0) = 0$ , as desired.

38 If  $2 \times 2$   $A$  has  $\text{tr}(A) = 5$  and  $\det(A) = -14$ , what are the e-vals of  $A$ ?

$$\begin{aligned} \hookrightarrow \lambda^2 - (\text{tr}(A))\lambda + \det(A) &= \lambda^2 - 5\lambda - 14 \\ &= (\lambda - 7)(\lambda + 2) = 0 \implies \lambda = -2, 7. \end{aligned}$$

39 If  $A$  and  $B$  are  $2 \times 2$ , show that  $\text{tr}(AB) = \text{tr}(BA)$ .

$$\begin{aligned} \hookrightarrow \text{tr}(AB) &= (AB)_{11} + (AB)_{22} = (a_{11}b_{11} + a_{12}b_{21}) \\ &\quad + (a_{21}b_{12} + a_{22}b_{22}) \\ &= (BA)_{11} + (BA)_{22} = \text{tr}(BA) \end{aligned}$$

40 If  $A$  &  $B$  are  $n \times n$ , show that  $\text{tr}(AB) = \text{tr}(BA)$ .

$$\hookrightarrow \text{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji}a_{ij} = \sum_{j=1}^n (BA)_{jj} = \text{tr}(BA)$$

41 If  $A$  &  $B$  are similar, show that  $\text{tr}(A) = \text{tr}(B)$ .

↳  $A = SBS^{-1}$  for some invertible  $S$ , so

$$\text{tr}(A) = \text{tr}((SB)(S^{-1})) = \text{tr}((S^{-1})(SB)) = \text{tr}(B).$$

42 Consider two  $n \times n$  matrices  $A$  &  $B$  such that  $BA = 0$ . Show that  $\text{tr}((A+B)^2) = \text{tr}(A^2) + \text{tr}(B^2)$ .

$$\begin{aligned} \text{tr}((A+B)^2) &= \text{tr}((A+B)(A+B)) = \text{tr}(A^2 + AB + BA + B^2) \\ &= \text{tr}(A^2) + \text{tr}(B^2) + \text{tr}(AB) + \text{tr}(BA) \\ &= \text{tr}(A^2) + \text{tr}(B^2) + \underbrace{\text{tr}(BA)}_0 + \underbrace{\text{tr}(BA)}_0 \\ &= \text{tr}(A^2) + \text{tr}(B^2). \end{aligned}$$

43 Do there exist  $n \times n$  matrices  $A$  and  $B$  such that  $AB - BA = I_n$ ?

↳ no: if so, then  $\text{tr}(AB - BA) = \text{tr}(I_n) = n$   
 $= \text{tr}(AB) - \text{tr}(BA) = \text{tr}(BA) - \text{tr}(BA) = 0$ ,  
 a contradiction.

44 Do there exist invertible  $n \times n$  matrices  $A$  &  $B$  such that  $AB - BA = A$ ?

↳ no: Suppose so. Then  $ABA^{-1} - B = I_n$ , so  
 $n = \text{tr}(I_n) = \text{tr}(ABA^{-1} - B) = \text{tr}(ABA^{-1}) - \text{tr}(B)$   
 $= \text{tr}(BA^{-1}A) - \text{tr}(B) = \text{tr}(B) - \text{tr}(B) = 0$ , a contradiction.

45 For which value  $k$  does the matrix  $A = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}$  have 5 as an e-val?

$$\hookrightarrow (\lambda=5 \text{ is an e-val}) \iff (\det(A-5I_2)=0)$$



$$(k=3) \iff (12-4k=0) \iff (\det \begin{pmatrix} -6 & k \\ 4 & -2 \end{pmatrix} = 0)$$

46  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with e-vals  $\lambda_1, \lambda_2$ , possibly the same.

a. Show that  $\lambda_1^2 + \lambda_2^2 = a^2 + d^2 + 2bc$ .

b. Show that  $\lambda_1^2 + \lambda_2^2 \leq a^2 + b^2 + c^2 + d^2$ .

c. When does  $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$  hold?

$$\begin{aligned} \hookrightarrow \text{a. } \lambda_1^2 + \lambda_2^2 &= (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = (\text{tr}(A))^2 - 2(\det(A)) \\ &= (a+d)^2 - 2(ad-bc) = a^2 + 2ad + d^2 - 2ad + 2bc \\ &= a^2 + d^2 + 2bc \end{aligned}$$

$$\begin{aligned} \text{b. } \lambda_1^2 + \lambda_2^2 &= a^2 + d^2 + 2bc \leq a^2 + d^2 + 2bc + (c-b)^2 \\ &= a^2 + d^2 + 2bc + c^2 - 2bc + b^2 \\ &= a^2 + b^2 + c^2 + d^2 \end{aligned}$$

c. By the above (b), we get equality precisely when  $c-b=0$ , or when  $c=b$ .

47 For which  $2 \times 2$  matrices  $A$  does there exist a nonzero matrix  $M$  such that  $AM = MD$ , where  $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ ?

↳ Note that for the matrix multiplication to make sense,  $M$  must be  $2 \times 2$ . Say  $M = [\vec{v}_1, \vec{v}_2]$ .

$$\text{If } AM = MD, \text{ then } A[\vec{v}_1, \vec{v}_2] = [A\vec{v}_1, A\vec{v}_2] \\ = MD = [\vec{v}_1, \vec{v}_2] \begin{bmatrix} 2 & \\ & 3 \end{bmatrix} = [2\vec{v}_1, 3\vec{v}_2],$$

where not both  $\vec{v}_1, \vec{v}_2$  are zero vectors. So at least one of  $A\vec{v}_1 = 2\vec{v}_1$ ,  $A\vec{v}_2 = 3\vec{v}_2$  must hold for a nonzero  $\vec{v}_1$  or  $\vec{v}_2$ , so  $A$  must have either 2, or 3, or both, as an e-val.

Conversely, suppose  $A$  has 2 or 3 (or both) as an e-val, say  $\lambda = 2$  is an e-val (the argument for 3 is nearly identical). Then  $A\vec{v} = 2\vec{v}$  for some

$$\vec{v} \in \underbrace{\ker(A - 2I_2)}_{\text{nontrivial!}}, \text{ so } A \begin{bmatrix} \vec{v} & \vec{0} \\ \vec{0} & \vec{0} \end{bmatrix} = [A\vec{v} \quad A\vec{0}] = [2\vec{v} \quad \vec{0}] \\ = \begin{bmatrix} \vec{v} & \vec{0} \\ \vec{0} & \vec{0} \end{bmatrix} \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}, \text{ so } M = \begin{bmatrix} \vec{v} & \vec{0} \\ \vec{0} & \vec{0} \end{bmatrix}$$

works as our nonzero matrix.

Thus, the matrices  $A$  that satisfy the condition given are precisely the  $2 \times 2$  matrices with at least 2 or 3 as an e-val.

48 For which  $2 \times 2$   $A$  does there exist an invertible matrix  $S$  such that  $AS = SD$ , where  $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ ?

$$\hookrightarrow (AS = SD, S \text{ invertible})$$



(say  $S = [\vec{v}_1, \vec{v}_2]$ ,  
so  $\vec{v}_1 \neq \vec{0} \neq \vec{v}_2$ )

$$(A[\vec{v}_1, \vec{v}_2] = [\vec{v}_1, \vec{v}_2] \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}, \vec{v}_1 \neq \vec{0} \neq \vec{v}_2)$$



$$([A\vec{v}_1, A\vec{v}_2] = [2\vec{v}_1, 3\vec{v}_2], \vec{v}_1 \neq \vec{0} \neq \vec{v}_2)$$



$$\begin{pmatrix} A\vec{v}_1 = 2\vec{v}_1 \\ A\vec{v}_2 = 3\vec{v}_2 \end{pmatrix}, \vec{v}_1 \neq \vec{0} \neq \vec{v}_2$$



( $A$  has 2 & 3 as eigenvalues)

49 For which  $3 \times 3$  matrices  $A$  does there exist a non zero matrix  $M$  such that  $AM = MD$ , where  $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 4 \end{bmatrix}$ ?

$\hookrightarrow$  By a similar argument to that given for (47),  $A$  must have at least an e-val of 2, 3, or 4.