

# Selected Problems & Solutions from 7.1

For 1-4,  $A$  is  $n \times n$  invertible,  $A\vec{v} = \lambda\vec{v}$ ,  $\vec{v} \neq \vec{0}$ .

1 Is  $\vec{v}$  an e-vec of  $A^3$ ?

→ Yes!  $A^3\vec{v} = A^2A\vec{v} = A^2(\lambda\vec{v}) \Rightarrow A^2\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v} = \lambda^3\vec{v}$ ,  
so  $\vec{v}$  is an e-vec w/ e-val  $\lambda^3$ , for  $A^3$ .

2 Is  $\vec{v}$  an e-vec of  $A^{-1}$ ?

→ Yes!  $A\vec{v} = \lambda\vec{v} \Rightarrow \vec{v} = \lambda A^{-1}\vec{v}$ . Since  $A$  is invertible,  
 $\ker(A) = \{\vec{0}\}$ , so  $A\vec{v} \neq \vec{0}$ , so  $\lambda \neq 0$ . Thus  $A^{-1}\vec{v} = \left(\frac{1}{\lambda}\right)\vec{v}$ .  
So  $\vec{v}$  is an e-vec of  $A^{-1}$  with e-val  $\left(\frac{1}{\lambda}\right)$ .

3 Is  $\vec{v}$  an e-vec of  $A + 2I_n$ ?

→ Yes!  $(A + 2I_n)\vec{v} = A\vec{v} + 2I_n\vec{v} = \lambda\vec{v} + 2\vec{v} = (\lambda + 2)\vec{v}$ ,  
so  $\vec{v}$  is an e-vec of  $(A + 2I_n)$  w/ e-val  $(\lambda + 2)$ .

4 Is  $\vec{v}$  an e-vec of  $7A$ ?

→ Yes!  $(7A)\vec{v} = 7\lambda\vec{v}$ , so  $\vec{v}$  is an e-vec of  $7A$   
w/ e-val  $(7\lambda)$ .

5 If  $\vec{v}$  is an e-vec for  $A$  &  $B$ , is it an e-vec for  $A+B$ ?

→ Yes!  $(A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda_1\vec{v} + \lambda_2\vec{v} = (\lambda_1 + \lambda_2)\vec{v}$ , so  
 $\vec{v}$  is an e-vec for  $A+B$ , with e-val  $(\lambda_1 + \lambda_2)$ .  
 $(\lambda_1 + \lambda_2)$  are the e-vals corr. to e-vec  $\vec{v}$ , for  $A$  &  $B$ , resp.).

6 If  $\vec{v}$  is an e-vec for  $A+B$ , is it an evec for  $AB$ ?

→ Yes!  $(AB)\vec{v} = A(B\vec{v}) = A(\lambda_2\vec{v}) = \lambda_2 A\vec{v} = (\lambda_2\lambda_1)\vec{v}$ ,  
so  $\vec{v}$  is an e-vec for  $AB$ , w/ e-val  $(\lambda_1\lambda_2)$ .

7 If  $\vec{v}$  is an evec of  $A$  w/ e-val  $\lambda$ , is  $\ker(A-\lambda I_n)$  trivial?

→ no,  $(A\vec{v}=\lambda\vec{v}) \Rightarrow ((A-\lambda I)\vec{v}=\vec{0}) \Rightarrow \vec{v} \in \ker(A-\lambda I)$   
so this also implies  $A-\lambda I$  is not invertible.

8 Find all  $2 \times 2$  matrices for which  $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is an e-vec w/ e-val 5.

→  $\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} a \\ c \end{pmatrix}, \text{ 1st col of } A} = 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ , so  $a=5, c=0$ . No restrictions on  $b, d$ .

So  $\left\{ \begin{pmatrix} 5 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{R} \right\}$  captures the full set of all possible such matrices.

9 Find all  $2 \times 2$  matrices for which  $\vec{e}_1$  is an e-vec.

$\hookrightarrow (A\vec{e}_1 = \lambda \vec{e}_1) \iff (A = \begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix} \text{ for some } b, d \in \mathbb{R})$ ,  
by similar reasoning to the above problem.

10 Find all  $2 \times 2$  matrices for which  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is an e-vec w/ e-val 5.

$\hookrightarrow \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{= \begin{pmatrix} a+2b \\ c+2d \end{pmatrix}} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \iff \begin{cases} a+2b=5 \\ c+2d=10 \end{cases} \iff \begin{cases} a=5-2b \\ c=10-2d \end{cases}$ ,

so  $\left\{ \begin{pmatrix} 5-2b & b \\ 10-2d & d \end{pmatrix} : b, d \in \mathbb{R} \right\}$  is the set of all such  $2 \times 2$  matrices.

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11 Find all  $2 \times 2$  matrices such that  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  is an e-vec w/ e-val -1.

$$\hookrightarrow \left( \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{= \begin{pmatrix} 2a+3b \\ 2c+3d \end{pmatrix}} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \right) \Leftrightarrow \begin{cases} a = -1 - \frac{3}{2}b \\ c = -\frac{3}{2} - \frac{3}{2}d \end{cases}, \text{ so } \left\{ \begin{pmatrix} -1 - \frac{3}{2}b & b \\ -\frac{3}{2} - \frac{3}{2}d & d \end{pmatrix} : b, d \in \mathbb{R} \right\} \text{ does the job.} \quad \text{b, d} \in \mathbb{R} \}$$

12 Diagonalize  $\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$ .

$$\hookrightarrow \text{e-vals: } \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 \\ 3 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - (3)(0) = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 2, 4. \quad E_2 = \ker(A - 2I) = \ker \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$$

$$E_4 = \ker(A - 4I) = \ker \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \text{ Thus}$$

$$A \begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} \text{ and } A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ so } A \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\text{so } A = SDS^{-1}, \text{ where } S = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \text{ and}$$

$$S^{-1} = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}^{-1} = \frac{1}{2-0} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}.$$

B Find all evecs for  $\begin{pmatrix} -6 & 6 \\ -15 & 13 \end{pmatrix}$  corresponding to e-val 4.

$$\hookrightarrow E_4 = \ker(A - 4I) = \ker \begin{pmatrix} -10 & 6 \\ -15 & 9 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}. \text{ Note that this}$$

implies 4 is an e-val, since  $\lambda=4$  causes  $(A - \lambda I)$  to have det. 0.

14 Find all  $4 \times 4$  matrices for which  $\vec{e}_2$  is an e-vec.

$$\rightarrow (A\vec{e}_2 = \lambda\vec{e}_2) \Leftrightarrow (2^{\text{nd}} \text{ col of } A \text{ is } \lambda\vec{e}_2), \text{ so it's all } 4 \times 4$$

matrices with  $\begin{pmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{pmatrix}$  as the 2<sup>nd</sup> col, where  $\lambda \in \mathbb{R}$  is arbitrary.

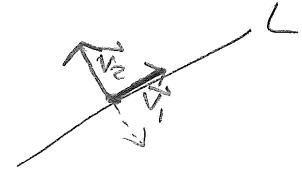
Diagonalize the linear transforms in 15-22.

15 Reflection about a line  $L$  in  $\mathbb{R}^2$ ,

→ Say  $L = \text{span}\{\vec{v}_1\}$ , and  $\vec{v}_2 \perp \vec{v}_1$ ,  $\vec{v}_2 \neq \vec{0}$ . Then

$$T(\vec{v}_1) = \vec{v}_1 \text{ and } T(\vec{v}_2) = -\vec{v}_2, \text{ so if}$$

$$T(\vec{x}) = A\vec{x}, \text{ then } A = (\vec{v}_1 \ \vec{v}_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\vec{v}_1 \ \vec{v}_2)^{-1}.$$



16 Rotation through an angle of  $180^\circ$  in  $\mathbb{R}^2$ .

$$\rightarrow A = \begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ already diagonal}$$

$$(A = (I_2)(-1 \ 0)(I_2)^{-1}).$$

17 → skip, until section 7.5! (diagonalizable, but e-vals are complex).

18 Reflection about a plane  $V$  in  $\mathbb{R}^3$ .

→ Say the plane is  $\text{span}\{\vec{v}_1, \vec{v}_2\}$  and  $\vec{v}_3 \neq \vec{0}$  is normal (read: orthogonal) to the plane. Then

$$A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2, \text{ and } A\vec{v}_3 = -\vec{v}_3, \text{ so } A = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3)^{-1}.$$

19 Orthogonal projection onto line  $L$  in  $\mathbb{R}^3$ .

→ Say  $L = \text{span}\{\vec{v}_1\}$  and  $L^\perp = \text{span}\{\vec{v}_2, \vec{v}_3\}$ , Then

$$T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{0} = 0\vec{v}_2, \text{ and } T(\vec{v}_3) = \vec{0} = 0\vec{v}_3, \text{ so}$$

$$A = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3)^{-1}.$$

20 Hold off until 7.5! Diagonalizable, but only with complex e-val

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21 Scaling by 5 in  $\mathbb{R}^3$ ,

$$\hookrightarrow A = 5I = \begin{pmatrix} 5 & & \\ & 5 & \\ & & 5 \end{pmatrix} \text{ ... already diagonal! } (A = I(5_{3 \times 3})I^{-1})$$

22 The linear transform  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $T(\vec{v}) = \vec{v}$  and  $T(\vec{w}) = \vec{v} + \vec{w}$ , where  $\vec{v}$  &  $\vec{w}$  are LI.

$$\hookrightarrow T(\vec{v}) = \vec{v} \text{ and } T(\vec{w}) = \vec{v} + \vec{w} \text{ imply}$$

$$A(\vec{v} \ \vec{w}) = (\vec{v} \ \vec{v} + \vec{w}) = (\vec{v} \ \vec{w}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \text{ Putting } S = (\vec{v} \ \vec{w}), \text{ we}$$

get  $A = S \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} S^{-1}$ , so  $A$  is similar to  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Since  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  isn't diagonalizable, neither is  $A$ , because they must have the same e-vals, and the same dimension eigenspaces corresponding to each e-val.

23 Prove that if  $S^{-1}AS = D$  = diagonal, then  $A$  is diagonalizable.

$$\hookrightarrow AS = SD = S \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = (\vec{v}_1, \dots, \vec{v}_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = (\lambda_1 \vec{v}_1, \dots, \lambda_n \vec{v}_n)$$

$$= A(\vec{v}_1, \dots, \vec{v}_n) = (A\vec{v}_1, \dots, A\vec{v}_n)$$

$$\text{so: } A\vec{v}_i = \lambda_i \vec{v}_i \text{ for } i=1, \dots, n,$$

so  $A$  is diagonalizable w/ e-vecs  $\vec{v}_i$  corresponding to e-vals  $\lambda_i$  (not necessarily distinct).

34 If  $\vec{v}$  is an e-vec of  $A$  w/ e-val  $\lambda$ , is  $\vec{v}$  an e-vec of  $A^2 + 2A + 3I$ ?

$$\rightarrow \text{Yes: } (A^2 + 2A + 3I)\vec{v} = A^2\vec{v} + 2A\vec{v} + 3I\vec{v} = \lambda^2\vec{v} + 2\lambda\vec{v} + 3\vec{v}$$

$$= (\lambda^2 + 2\lambda + 3)\vec{v}, \text{ so } \vec{v} \text{ is an e-vec}$$

$$\text{of } (A^2 + 2A + 3I) \text{ w/ e-val } (\lambda^2 + 2\lambda + 3) = 16 + 8 + 3 = 27.$$

35 Show that similar matrices have identical e-vals.

$\rightarrow$  if  $A = SBS^{-1}$  and  $A\vec{v} = \lambda\vec{v}$ , then  $SBS^{-1}\vec{v} = A\vec{v} = \lambda\vec{v}$ , so  $B(S^{-1}\vec{v}) = \lambda(S^{-1}\vec{v})$ . So  $\lambda$  is also an e-val of  $B$ . Then reverse the roles of  $A$  &  $B$  to get the reverse direction. Thus similar matrices have identical e-vals.

36 Find a  $2 \times 2$  matrix s.t.  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  &  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  are e-vects w/ evals 5 & 10, resp.

$$\rightarrow A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 15 & 10 \\ 5 & 20 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ -2 & 11 \end{pmatrix}.$$

37 Diagonalize  $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ .

$$\rightarrow \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} = (3-\lambda)(-3-\lambda) - (4)(4) = \lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$$

$$E_5 = \ker(A - 5I) = \ker \begin{pmatrix} 2 & 4 \\ 4 & -8 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$E_{-5} = \ker(A + 5I) = \ker \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$$\Rightarrow A = \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}_{= \frac{1}{4-1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}} \underbrace{\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}}_{= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1}}_{= \frac{1}{4-1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}.$$

38  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  is an e-vec of  $\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$  with which e-val?

$$\rightarrow \begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ so the e-val is 2.}$$

39 Find a basis of the subspace  $V$  of real  $2 \times 2$  matrices  $M_{2 \times 2}(\mathbb{R})$  s.t.  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is an e-vec.

$$\rightarrow \underbrace{\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \right)}_{= \begin{pmatrix} b \\ d \end{pmatrix}} \Leftrightarrow \begin{pmatrix} b=0 \\ d=\lambda \end{pmatrix}. \text{ Since } \lambda \text{ is arbitrary,}$$

$$V = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \text{all possible } 2 \times 2 \text{ matrices w/ } (1, 2) \text{ component equal to 0.}$$

$$\dim(V) = 3.$$

40 Find a basis of the subspace  $V$  of  $M_{2 \times 2}(\mathbb{R})$  s.t.  $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$  is an e-vec.

$$\rightarrow \left( \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{= \begin{pmatrix} a-3b \\ c-3d \end{pmatrix}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} \lambda \\ -3\lambda \end{pmatrix} \right) \Leftrightarrow \begin{cases} a = \lambda + 3b \\ c = -3\lambda + 3d \end{cases}, \text{ so the}$$

matrices in  $V$  may all be represented in the form

$$\begin{pmatrix} \lambda+3b & b \\ -3\lambda+3d & d \end{pmatrix}, \text{ where } \lambda, b, d \in \mathbb{R}. \text{ Thus } V = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right\},$$

and a basis for  $V$  is  $\left\{ \begin{pmatrix} 1 & 0 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right\}$ .  $\dim(V) = 3$ .

41 Find a basis for the subspace  $V \subset M_{2 \times 2}(\mathbb{R})$  of 2x2 real matrices with both  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  as e-vecs.

$$\rightarrow V$$
's matrices all take the form  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$   
 $= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 1 \end{pmatrix} \frac{1}{2-\lambda_1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2\lambda_1 & -\lambda_1 \\ -\lambda_2 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 & \lambda_2 \\ 2\lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix},$   
 $\text{so } V = \text{span} \left\{ \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \right\}.$   $\dim(V) = 2.$   

basis for  $V$

42 Find a basis of the subspace  $V \subset M_{3 \times 3}(\mathbb{R})$  of 3x3 real matrices w/ e-vecs  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  &  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ .

$$\rightarrow A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ so } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ so}$$
 $V = \text{span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$   

basis for  $V$

$\therefore \dim(V) = 5$ .

43 Determine the subspace  $V$  of  $M_{n \times n}(\mathbb{R})$  of  $n \times n$  real matrices for which  $\vec{e}_1, \dots, \vec{e}_n$  are e-vecs.

$$\hookrightarrow A \in V \iff A\vec{e}_i = \lambda_i \vec{e}_i \text{ for } i=1, \dots, n$$

$\iff$  the  $i^{\text{th}}$  col of  $A$  is  $\lambda_i \vec{e}_i$ ,

$$\text{so } V = \text{all } n \times n \text{ diagonal matrices} = \text{span} \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \right\}$$

$$\dim(V) = n.$$

44 Find the dimension of the subspace of  $M_{n \times n}(\mathbb{R})$  consisting of all matrices with  $\vec{e}_1, \dots, \vec{e}_m$  as e-vecs, where  $km \leq n$ .

$$\hookrightarrow (A\vec{e}_i = \lambda_i \vec{e}_i \text{ for } i=1, \dots, m) \iff (\text{the } i^{\text{th}} \text{ col of } A \text{ is } \lambda_i \vec{e}_i \text{ for } i=1, \dots, m)$$

Note that the other cols of  $A$  are arbitrary.

$$V = \text{span} \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}}, \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}}, \dots, \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}} \right\}$$

$$\text{so } \dim(V) = m + n(n-m) = n^2 - m(n-1),$$

45 If  $\vec{v} \in \mathbb{R}^2$  nonzero, what is the dimension of the subspace  $V$  of  $M_{2 \times 2}(\mathbb{R})$  of matrices with  $\vec{v}$  as an e-vec?

$\hookrightarrow$  find  $\vec{w} \in \mathbb{R}^2$  with  $\vec{0} \neq \vec{w} \perp \vec{v}$  (so  $\vec{v} \cdot \vec{w} = \vec{0}$ )  
 the 2<sup>nd</sup> component of  $\vec{v}$  is nonzero (at least one of the components will be ...). Then  $V = \text{span} \left\{ \begin{pmatrix} \vec{w}^\top & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \vec{w}^\top & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{v}^\top \\ 0 & 0 \end{pmatrix} \right\}$   
 since these matrices are  $\{I\}$  and  $V$  is not the full space  $M_{2 \times 2}(\mathbb{R})$ . So  $\dim(V) = 3$ . (note there are matrices st.  $\vec{v}$  is not an e-vec)

46 If  $\vec{v}$  is an e-vec of  $A$  w/ e-val 3, show that  $\vec{v} \in \text{im}(A)$ .

↪  $A\vec{v}=3\vec{v}$ , so  $A(\frac{1}{3}\vec{v})=\vec{v}$ , so  $\vec{v} \in \text{im}(A)$ .

47 If  $\vec{v}$  is an e-vec of  $A$ , show that  $\vec{v} \in \text{im}(A)$  or  $\vec{v} \in \ker(A)$ .

↪ if  $A\vec{v}=0\vec{v}=\vec{0}$ , then  $\vec{v} \in \ker(A)$ .

else  $A\vec{v}=\lambda\vec{v}$ ,  $\lambda \neq 0$ , so  $A(\frac{1}{\lambda}\vec{v})=\vec{v}$ , so  $\vec{v} \in \text{im}(A)$ .

48 If  $A$  is a rank 1 square matrix, show that any nonzero vector in  $\text{im}(A)$  is an e-vec for  $A$ .

↪ since  $\text{rank}(A)=1$ ,  $\text{im}(A)=\text{span}\{A's \text{ cols}\}=\text{span}\{\vec{v}\}$

for some  $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ , and  $A$ 's cols may be written as

$A = \begin{bmatrix} c_1\vec{v} & c_2\vec{v} & \cdots & c_n\vec{v} \end{bmatrix}$ . If  $\vec{w} \in \text{im}(A)$ , then  $\vec{w}=c\vec{v}$  for some

$c \in \mathbb{R}$ , and then  $A\vec{w}=cA\vec{v}=c(c_1\vec{v}, c_2\vec{v}, \dots, c_n\vec{v})$

$$= (cc_1\vec{v}, cc_2\vec{v}, \dots, cc_n\vec{v})\vec{v}$$

$$= (c_1\vec{v}, c_2\vec{v}, \dots, c_n\vec{v})(c\vec{v})$$

$$= (c_1\vec{v}, c_2\vec{v}, \dots, c_n\vec{v})\vec{w}$$

so  $\vec{w}$  is an e-vec for  $A$ .

49 Give an example of a rank 1 matrix that fails to be diagonalizable.

↪  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  fails to be diagonalizable, since if  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=SDS^{-1}$ , then

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = (SDS^{-1})(SDS^{-1}) = SDS^2S^{-1} \Rightarrow D^2 = S^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but  $D^2$  is the diag. matrix  $D$  but w/ its diag entries squared, so  $D=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , so  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , contr.

For 50-54, diagonalize  $A$ . Note: if  $A$  is rank 1 with  $\underbrace{\text{tr}(A) \neq 0}_{\text{sum of diag entries}}$ , then  $\ker(A)$  is  $(n-1)$ -dimensional, and  $\text{im}(A) = \text{span}\{\vec{v}\}$

for some  $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ , and by (48), we know  $A\vec{v} = (c_1 v_1 + \dots + c_n v_n)\vec{v}$   
 $= (\underbrace{\text{tr}(A)}_{\neq 0})\vec{v}$ ,

so  $\vec{v} \notin \ker(A)$  is an e-vec of  $A$  w/ e-val  $\text{tr}(A) \neq 0$ .

If a basis for  $\ker(A)$  is  $\{\vec{w}_1, \dots, \vec{w}_{n-1}\}$ , then  $\vec{v} \notin \ker(A)$  implies  $\{\vec{w}_1, \dots, \vec{w}_{n-1}, \vec{v}\}$  is an LI set, so  $A = (\vec{w}_1 \ \dots \ \vec{w}_{n-1} \ \vec{v}) \begin{pmatrix} 0 & & \\ & \ddots & \\ & & \text{tr}(A) \end{pmatrix} (\vec{w}_1 \ \dots \ \vec{w}_{n-1} \ \vec{v})'$ , so  $A$  is diagonalizable. If instead, we have  $\text{rank}(A)=1$  and  $\text{tr}(A)=0$ , then  $A\vec{v}=\vec{0}=c\vec{v}$ , so  $\vec{v} \in \ker(A)$ , and so if  $\vec{u} \in \mathbb{R}^n$  is an e-vec of  $A$  w/ e-val  $\lambda \neq 0$ , then  $A\vec{u} = \lambda\vec{u} = c\vec{v}$  for some  $c \in \mathbb{R}$ , so  $\vec{u} = \left(\frac{c}{\lambda}\right)\vec{v}$ ,

and  $A\vec{u} = \left(\frac{c}{\lambda}\right)A\vec{v} = \left(\frac{c}{\lambda}\right)\vec{0} = \vec{0}$ , contradicting that  $\lambda \neq 0$ . So all of

$A$ 's e-vals are 0. So if  $A$  were diagonalizable, then  $A = S \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} = \vec{0}$ ,

contradicting that  $A$  is rank 1. Thus: if  $A$  is  $n \times n$  rank 1, then  $(A \text{ is diagonalizable}) \iff (\text{tr}(A) \neq 0)$ .

50  $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ .

↪ note  $\text{im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ , so  $\text{rank}(A)=1$ , and  $\text{tr}(A)=7 \neq 0$ ,  $\ker(A) = \text{span}\left\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$ , so  $A = \underbrace{\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}^{-1}}_{S^{-1}}$ .

Note that  $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ , and

$$\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{1+6} \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}.$$

51  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .

$\hookrightarrow \text{tr}(A) = 2, \text{im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ .

$\ker(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}, \text{so } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ .

Note  $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .

52  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ .

$\hookrightarrow \text{tr}(A) = 3 \neq 0, \text{im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}, \ker(A) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

(note: these vecs are clearly in the kernel and  $\text{I}_3$ , and  $\dim(\ker(A)) = \text{nul}(A) = (\# \text{cols}) - \text{rank}(A) = 3 - 1 = 2$ )  
by rank-nullity, so this works as a basis for  $\ker(A)$ . ) . So

$$\text{since } A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ Note } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^{-1}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & -2 & -1 & | & -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & -1 & 0 \\ 0 & 0 & 3 & | & 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & -1 & 0 \\ 0 & 0 & 1 & | & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\text{so } \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

53  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ .

$\hookrightarrow \text{tr}(A) = 1 \quad (\neq 0), \text{im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}, \text{so } \text{rank}(A) = 1$ .

$\ker(A) = \text{span}\left\{\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right\}, \text{so } A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}^{-1}$

since  $A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 28 \\ 42 \end{pmatrix} = 14\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ . Find  $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}^{-1}$  by performing Gaussian elimination, as in (52).

$$54 \quad A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

$\hookrightarrow \text{tr}(A) = 3 \neq 0$ , and  $\text{Im}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ , so  $\text{rank}(A) = 1$ .

$$A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ and } \text{ker}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \text{ so}$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}, \text{ where } \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

can be found using Gaussian elimination, as before.