Polynomial Enumeration of the Standard Lattice

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1 Fueter writes:

At your request, I have written the following summary of work that I began, and that you ended so beautifully.

Reading Caratheodory's book on functions of a real variable, I wondered whether the function given on p. 29 was the only quadratic to satisfy the following properties:

1. $f(0,0) = 1$.
2. For nonnegative integer pairs x, y, the value $f(x, y)$ is a positive integer.
3. Given an arbitrary positive integer n, there is a pair of nonnegative integers x, y such that $f(x, y) = n$.

It is clear that that there are no linear functions with those characteristics.

Write

$$f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + 1$$

Then, because of properties 2 and 3, we must have

$$\begin{array}{ll} f(1,0) - 1 = A + D & f(0,1) - 1 = C + E \\ f(2,0) - 1 = 4A + 2D & f(0,2) - 1 = 4C + 2E \end{array}$$
$$f(1,1) - 1 = A + B + C + D + E$$

all positive integers. This is only possible if $2A, B, 2C, 2D, 2E$ are all integers. So let $A = \frac{a}{2}$, $B = b$, $C = \frac{c}{2}$, $D = \frac{d}{2}$, $E = \frac{e}{2}$; thus a, b, c, d, e are all integers, and furthermore so must

$$\frac{a+d}{2} \text{ and } \frac{c+e}{2}$$

be integers too.

Our function now stands as follows:

$$f(x,y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 + \frac{d}{2}x + \frac{e}{2}y + 1.$$

a and c are both nonnegative, because $f(x, y)$ is nonnegative for large x or y.

1.1 The parabolic case: $D = b^2 - ac = 0$

If b is less than zero, then certainly $a > 0$, $c > 0$, and we may write

$$f(x, y) = \frac{(ax + by)^2}{2a} + \frac{d}{2}x + \frac{e}{2}y + 1$$

We now analyze the infinite sequence of pairs

$$\begin{aligned} x &= -bt, \\ y &= at, \end{aligned} \qquad t = 1, 2, 3, 4, \ldots$$

All x, y are by hypothesis positive, so we have

$$f(-bt, at) = \frac{ea - bd}{2}t + 1$$

The coefficient $m = \frac{ea-bd}{2}$ is whole and bigger than 1, lest we find every integer in the form $f(-bt, at)$. On the other hand, $f(-bt, at) \quad (t = 1, 2, 3, \ldots)$ constitutes all positive integers $\equiv 1 \pmod m$. And $f(2m, 0)$ is such a number. Therefore there are two different points with image $f(2m, 0)$, which is contrary to condition 3.

If $b = 0$, then (without loss of generality) $c = 0$, and $\frac{e}{2}$ is a nonzero integer. So

$$\begin{aligned} f(0, y) &= \frac{e}{2}y + 1 \\ f(2e, 0) &= \frac{a}{2} \cdot 4e^2 + \frac{d}{2} \cdot 2e + 1, \end{aligned}$$

i.e. there is again an integer with two different preimages, *contra* condition 3.

If b is greater than zero, then

$$\begin{aligned} f(x+1, y) &= f(x, y) + ax + by + \frac{a+d}{2} \\ f(x, y+1) &= f(x, y) + bx + cy + \frac{c+e}{2} \end{aligned}$$

and, in particular,

$$\begin{aligned} f(1, 0) &= 1 + \frac{a+d}{2} = n_1, \\ f(0, 1) &= 1 + \frac{c+e}{2} = n_2 \end{aligned}$$

Well, both $\frac{a+d}{2}$ and $\frac{c+e}{2}$ exceed zero, and $f(x, y)$ grows in x and y. So one of $\frac{a+d}{2}$ and $\frac{c+e}{2}$ must be 1 — say, $\frac{c+e}{2}$. Because of

$$f(1, 0) = 1 + \frac{a+d}{2} \qquad f(0, 2) = 3 + c,$$

we have $\frac{a+d}{2} = 2$, or $c = 0$. In the second case $e = 2$, so that $f(0, y)$ would already give all numbers. So $\frac{a+d}{2} = 2$ and also: $f(1, 1) = 4 + b$, $f(2, 0) = 5 + a$, $f(0, 2) = 3 + c$; thus $c = 1$, $b = 1$, $a = 1$, $d = 3$, $e = 1$. The function $f(x, y)$ is thusly

$$f(x, y) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{3}{2}x + \frac{1}{2}y + 1.$$

Had we set $\frac{a+d}{2} = 1$ instead, we would obtain the same function, but with x and y reversed. This is the only solution in the parabolic case.

1.2 The hyperbolic case: $D = b^2 - ac > 0$

We let

$$\left.\begin{aligned} \nu &= be - cd \\ \mu &= bd - ae \end{aligned}\right\} \text{— equivalently —} \left\{\begin{aligned} a\nu + b\mu &= Dd \\ b\nu + c\mu &= De \end{aligned}\right.$$

and change coordinates to

$$\left.\begin{aligned} x_1 &= \nu + 2Dx \\ y_1 &= \mu + 2Dy \end{aligned}\right\} \text{— equivalently —} \left\{\begin{aligned} x &= \frac{x_1 - \nu}{2D} \\ y &= \frac{y_1 - \mu}{2D} \end{aligned}\right.$$

That done,

$$f(x,y) = \frac{1}{4D^2}\left(\frac{a}{2}x_1^2 + bx_1y_1 + \frac{c}{2}y_1^2\right) + \frac{1}{4D^2}\left(4D^2 - \frac{d}{2}D\nu - \frac{e}{2}D\mu\right)$$

As x, y runs through all nonnegative integer pairs, $f(x,y) - 1 = t$ runs through all integers $t = 0, 1, 2, 3, \ldots$, and each appears only once. We have:

$$ax_1^2 + 2bx_1y_1 + cy_1^2 = 8D^2t + D(d\nu + e\mu)$$

For s a real number > 1, the series

$$L(s) = \sum_{x,y=0}^{\infty}{}' \frac{1}{(ax_1^2 + 2bx_1y_1 + cy_1^2)^s} = \sum_{t=0}^{\infty}{}' \frac{1}{(8D^2t + D(d\nu + e\mu))^s}$$

converges absolutely, and can be expanded in a convergent Laurent series about 1:

$$L(s) = \sum_{x,y=0}^{\infty}{}' \frac{1}{(ax_1^2 + 2bx_1y_1 + cy_1^2)^s} = \frac{1}{8D^2(s-1)} + f(s), \qquad (s > 1)$$

where $f(s)$ is analytic at 1.

If D is not the square of a rational number n, and $b < 0$, then we can apply the theory of quadratic forms to see that the series $\sum_{x,y=0}^{\prime\infty} \frac{1}{(ax_1^2+2bx_1y_1+cy_1^2)^s}$ defining L never converges, so we have found a contradiction.

If $D = n^2$ for some positive rational n, then in the case $b < 0$, we have $a > 0$,

$$f(x,y) = \frac{(ax+by)^2 - n^2y^2}{2a} + \frac{d}{2}x + \frac{e}{2}y + 1$$

and $f((n-b)t, at) = 1 + mt$, where $m = \frac{d(n-b)+ea}{2}$, and $t = 1, 2, 3, \ldots$. Because $b < 0$, we have $n - b \neq 0$. One infers, as in the parabolic case, that $m > 1$, and $f(2m, 0)$ or $f(0, 2m)$ has two preimages.

If $b > 0$, then one can proceed as in section 1; but the only such solutions are quadratic forms of parabolic type.

So we may dispense with the forms of hyperbolic type.

1.3 The elliptic case: $D = b^2 - ac < 0$

The case $b > 0$ is solved as in section 1.

If, on the contrary, $b \leq 0$, then one can use the above coordinate transformation and write

$$L(s) = \sum_{x,y=0}^{\infty}{}' \frac{1}{(ax_1^2 + 2bx_1y_1 + cy_1^2)^s} = \frac{1}{8D^2(s-1)} + f(s), \qquad s > 1, f(s) \text{ analytic}$$

The series for L now converges, but we can go further by analyzing the coefficient of $(s-1)^{-1}$. The residue on the left ought to contain π or e, and that would give a contradiction, because the right-hand side has a rational residue. Given your great experience with such limits, can you conclude the argument?

2 Pólya writes:

The residue in your $L(s)$ described above depends essentially on the calculation of several areas. Your remarks immediately put me in mind of an areal method; with that technique instead, I will be able to solve your problem *and* do a little bit more.

A function $f(x, y)$, which satisfies the conditions 2 and 3 you formulated (condition 1 will be eliminated momentarily), I will call an "enumerative [*abzählende*]" function. Take $f(x, y)$ enumerative and polynomial of degree m, then we can write

$$f(x, y) = \phi_m(x, y) + \phi_{m-1}(x, y) + \cdots + \phi_0(x, y) \tag{1}$$

wherein $\phi_\mu(x, y)$ is a homogenous polynomial of degree μ, with $\mu = 0, 1, 2, \ldots, m$. For large values of x, y, the definition of f implies, in consequence of condition 2, that

$$\phi_m(x, y) \geq 0 \qquad \text{for} \qquad x \geq 0, y \geq 0 \tag{2}$$

2.1 Definite case

I call the "definite case [*definiter Fall*]" the case where

$$\phi_m(x, y) > 0 \qquad \text{for} \qquad x \geq 0, y \geq 0, x + y > 0 \tag{3}$$

Let N be a whole number, $N > 1$. In the region of the plane where the three inequalities

$$f(x, y) \leq N \qquad x \geq 0 \qquad y \geq 0 \tag{4}$$

simultaneously hold, there lies (in light of properties 2 and 3) exactly N lattice points (points for which x and y are whole numbers). The first inequality of (4) can, because of (1), be written like so:

$$\phi_m\left(xN^{-\frac{1}{m}}, yN^{-\frac{1}{m}}\right) + N^{-\frac{1}{m}}\phi_{m-1}\left(xN^{-\frac{1}{m}}, yN^{-\frac{1}{m}}\right) + N^{-\frac{2}{m}}\phi_{m-2}\left(xN^{-\frac{1}{m}}, yN^{-\frac{1}{m}}\right) + \cdots \leq 1 \tag{5}$$

I denote by F the area bounded by the inequalities

$$\phi_m(x, y) \leq 1 \qquad x \geq 0 \qquad y \geq 0 \tag{6}$$

and consider the points $xN^{-\frac{1}{m}}, yN^{-\frac{1}{m}}$, where x, y are whole numbers (they form a tightly-spaced lattice). Of the points in this lattice, there lie in (6) approximately $FN^{\frac{2}{m}}$ points; one sees from (3) and (5) that the number of points with mesh width 1 in region (4) is asymptotic to $FN^{\frac{2}{m}}$ as N grows to infinity. But the exact answer, as noted, is N. From

$$FN^{\frac{2}{m}} \propto N$$

it follows that

$$m = 2 \qquad F = 1 \tag{7}$$

As you detailed, in each of three different scenarios — the elliptic, the hyperbolic, and the parabolic — one has

$$\phi_2(x, y) = \frac{ax^2 + 2bxy + cy^2}{2}$$

with integer coefficients a, b, c. I denote by t_1, t_2 the two solutions to the equation

$$a + 2bt + ct^2 = 0$$

($c > 0$). In the elliptic case t_1, t_2 are imaginary; in the hyperbolic, both are < 0. We have

$$F = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2d\phi}{a\cos^2\phi + 2b\cos\phi\sin\phi + c\sin^2\phi} = \frac{1}{c}\frac{1}{t_1 - t_2}\log\left(\frac{t_2}{t_1}\right);$$

so

$$\frac{t_2}{t_1} = e^{c(t_1 - t_2)F}. \tag{8}$$

t_1 and t_2 are algebraic numbers; were $F = 1$, then (8) would show that the logarithm of an algebraic number were algebraic. This contradicts Lindemann's theorem, so we must have $F \neq 1$, and — for $f(x, y)$ in the elliptic or definite hyperbolic case — $f(x, y)$ is not enumerative. The solution indeed lies in the very direction that you with your mentions of π and e suggested.

In the parabolic case $2\phi_2(x, y) = (\sqrt{a}x + \sqrt{c}y)^2$ and the area (4) a right triangle with area

$$F = \frac{1}{2}\sqrt{\frac{2}{a}}\sqrt{\frac{2}{c}} = \frac{1}{\sqrt{ac}}.$$

[*Translator's note: the original has the middle formula as* $\frac{1}{2}\sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}$*; this is clearly a typo.*] $F = 1$ yields the whole-number solution $a = c = 1$; therefore $b = \pm 1$ and, because $\phi_2(1, 1) \neq 0$, $b = 1$. An enumerative function has, in the parabolic case, the form

$$f(x, y) = \frac{(x + y)(x + y + 1)}{2} + hx + ky + l$$

(h, k, l constants). Observe the formulas

$$\begin{aligned}
f(x+1, y) - f(x, y) &= x + y + 1 + h \\
f(x, y+1) - f(x, y) &= x + y + 1 + k \\
f(x+1, y-1) - f(x, y) &= h - k \\
f(0, s+1) - f(x, s-x) &= s + 1 + k - (h - k)x
\end{aligned}$$

and proceed in a discussion similar to the one you've carried out, towards the point that the only two enumerative functions of parabolic type are the ones you found above. This argument uses the restriction that, for $x \geq 0, y \geq 0, x + y > 0$, the highest degree of a homogenous component is > 0, rather than your condition 1.

2.2 The indefinite case

The indefinite case, i.e. the case, in which (3) fails, and only (2) holds, does not appear for functions of degree ≥ 3 to be easily accessible. For functions of degree 1 and 2 the above methods (enumeration of lattice points) can easily be brought to completion; one needs only rough estimates of sums covering all the pieces. So one arrives at the the result, that, aside from the two functions you highlighted above, enumerative polynomials of degree less than 3 do not exist.