Homework 1 Solutions

1. Let \( V \) be a vector space, \( S \) a set, and \( s \in S \). Let \( U = \{f : f : S \to V \} \) and \( W = \{f \in U \mid f(s) = 0 \} \). Is \( W \) a subspace of \( U \)?

**Solution.** Yes; we use the subspace test. We see \( 0 \in W \) so \( W \neq \emptyset \). Now, if \( f, g \in W \), \( c \in F \) (the field which \( V \) is a vector space over), then \((f + g)(s) = f(s) + g(s) = 0 + 0 = 0 \) and \((cf)(s) = c(f(s)) = c0 = 0\), so \( f + g \) and \( cf \) are both in \( W \).

2. Prove \( \text{span}\{(1, -1, 0), (0, 1, -1)\} \) coincides with the subspace of \( \mathbb{R}^3 \) consisting of all vectors \((a, b, c)\) with \( a + b + c = 0 \).

**Proof.** Let \( A = \text{span}\{(1, -1, 0), (0, 1, -1)\} \) and \( B = \{(a, b, c) \in \mathbb{R}^3 | a + b + c = 0\} \). We show \( A \subseteq B \) and \( B \subseteq A \). Suppose \( x \in A \). Then, \( x = c_1(1, -1, 0) + c_2(0, 1, -1) = (c_1, -c_1 + c_2, -c_2) \) for some \( c_1, c_2 \in \mathbb{R} \). Hence, \( c_1 + (-c_1 + c_2) + (-c_2) = 0 \), so \( x \in B \). Thus, \( A \subseteq B \). Now, suppose \( y = (a, b, c) \in B \). Then, \( a + b + c = 0 \), so \( b = -a - c \). Hence, \( y = (a, b, c) = (a, -a - c, c) = a(1, -1, 0) - c(0, 1, -1) \), so \( y \in A \). Thus, \( B \subseteq A \).

3. Let \( S \) be a linearly dependent subset of a vector space \( V \). Let \( S' \) be the subset of \( S \) consisting of all vectors in \( S \) that are linear combinations of the other vectors in \( S \). For any \( n > 0 \), find the smallest value of \( \text{card}(S') \) over all vector spaces \( V \) and all subsets \( S \) with \( \text{card}(S) = n \).

**Solution.** For each \( n \), let \( A(n) \) be the value sought. We show \( A(n) = 1 \) for each \( n \). Fix \( n > 0 \). Let \( V, S \) be as in the problem statement, say \( S = \{s_1, \ldots, s_n\} \). Then, as \( S \) is linearly dependent, for some \( k \in \{1, \ldots, n\} \) there is a relation \( \sum_{j \neq k} c_j s_j = s_k \) for some \( c_k \in \mathbb{R} \). Hence, \( \text{card}(S') \geq 1 \) for each possible pair \((V, S)\), so \( A(n) \geq 1 \). Now, let \( V = \mathbb{R}^n \) and \( S = \{0\} \cup \{e_k | 1 \leq k \leq n - 1\} \) where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{R}^n \). Then, \( S' = \{0\} \) as \( 0 = \sum_{j=1}^{n-1} 0e_j \) and \( \{e_1, \ldots, e_{n-1}\} \) is a linearly independent set. Hence, \( \text{card}(S') = 1 \), so \( A(n) \leq 1 \). Thus, \( A(n) = 1 \).

4. Find the dimension of \( \text{span}\{X^2 - 1, (X - 1)^2, X - 1\} \) in \( P_2[X] \).

**Solution.** Let \( V = \text{span}\{X^2 - 1, (X - 1)^2, X - 1\} \). Note \( \{X^2 - 1, X - 1\} \) is a linearly independent set, as if \( a(X^2 - 1) + b(X - 1) = 0 \), then we must have \( a = b = 0 \). Hence, \( \text{dim}(V) \geq 2 \). Now, \( \{X^2 - 1, (X - 1)^2, X - 1\} \) is a linearly dependent set. Indeed, \( (X^2 - 1) - (X - 1)^2 = 2X - 2 = 2(X - 1) \) (if \( \text{char}(F) = 2 \) where \( F \) is the base field, then \( (X^2 - 1) - (X - 1)^2 = 0 \)). Hence, \( \text{dim}(V) \leq 2 \), so \( \text{dim}(V) = 2 \).

5. Let \( T : V \to W \) be a linear map. Suppose \( v_1, \ldots, v_n \) span \( V \). Show \( T(v_1), \ldots, T(v_n) \) span \( R(T) \).

**Proof.** Let \( w \in R(T) \). Then, \( w = T(v) \) for some \( v \in V \). As \( v \in V = \text{span}\{v_1, \ldots, v_n\} \), we can write \( v = \sum_{k=1}^{n} c_k v_k \) for some \( c_k \in F \). Thus,

\[
 w = T(v) = T(\sum_{k=1}^{n} c_k v_k) = \sum_{k=1}^{n} c_k T(v_k)
\]

Hence, \( R(T) \subseteq \text{span}\{T(v_1), \ldots, T(v_n)\} \). As \( T(v_k) \in R(T) \) for each \( k \), we see \( \text{span}\{T(v_1), \ldots, T(v_n)\} \subseteq R(T) \), so \( R(T) = \text{span}\{T(v_1), \ldots, T(v_n)\} \).

6. Let \( T : V \to V \) be a linear map such that \( N(T^2) \neq 0 \). Show \( N(T) \neq 0 \).

**Proof.** As \( N(T^2) \neq 0 \), there is a \( v \in N(T^2) \) with \( v \neq 0 \). If \( T(v) = 0 \), then \( v \in N(T) \) and we are done. If \( T(v) \neq 0 \), then \( T(v) \in N(T) \) as \( T(T(v)) = T^2(v) = 0 \). Hence, we are done in this case as well.

7. Let \( V \) be a vector space of dimension \( n \) and \( W \subseteq V \) a subspace of dimension \( n - 1 \). Prove that there is an \( f \in V^* \) such that \( W = N(f) \).
Proof. Let $\alpha = \{v_1, \ldots, v_{n-1}\}$ be a basis for $W$. Extend $\alpha$ to a basis $\beta = \alpha \cup \{v_n\}$ for $V$. Let $f : V \to F$ be given by $f(\sum_{j=1}^{n} c_j v_j) = c_n$. We show $W = N(f)$. Let $w \in W$. Then, $w = \sum_{j=1}^{n-1} c_j v_j$ for some $c_j \in F$ as $\alpha$ is a basis for $W$. Then, $f(w) = f(\sum_{j=1}^{n-1} c_j v_j) = f(\sum_{j=1}^{n-1} c_j v_j + 0 v_n) = 0$. Hence, $W \subseteq N(f)$. Now, suppose $v = \sum_{j=1}^{n} c_j v_j \in N(f)$. Then, $f(v) = c_n = 0$, so $v = \sum_{j=1}^{n-1} c_j v_j \in W$ as $\alpha \subseteq W$. Thus, $N(f) \subseteq W$, so $W = N(f)$.

8. Let $\{f_1, \ldots, f_n\}$ be the dual basis of a basis $\{v_1, \ldots, v_n\}$. Find the dual basis of the basis $\{v_1 + v_2, v_2, \ldots, v_n\}$.

Solution. Let $g_k = f_k$ for $k \neq 2$ and let $g_2 = f_2 - f_1$. Then, for $k \neq 2$, $j \geq 2$, we see $g_k(v_j) = f_k(v_j) = \delta_k(j)$. Moreover, for $k \neq 2$, $g_k(v_1 + v_2) = g_k(v_1) + 0 = f_k(v_1) = \delta_k(j)$. Now, for $j \geq 3$, $g_2(v_j) = f_2(v_j) - f_1(v_j) = 0 - 0 = 0$. We see, $g_2(v_2) = f_2(v_2) - f_1(v_2) = 1 - 0 = 1$ and $g_2(v_1 + v_2) = f_2(v_1 + v_2) - f_1(v_1 + v_2) = 1 - 1 = 0$. Hence, $\{g_1, \ldots, g_n\}$ is the dual basis of $\{v_1 + v_2, v_2, \ldots, v_n\}$.

9. Let $T : V \to W$ be a linear map, $V, W$ finite dimensional vector spaces over $F$. Prove that $\dim N(T) + \dim R(T^*) = \dim(V)$.

Proof. By using the rank nullity theorem ($\dim N(T) + \dim R(T) = \dim(V)$), it suffices to show $\dim R(T) = \dim R(T^*)$. Let $\beta$ be a basis for $V$, $\gamma$ a basis for $W$, and let $\beta^*, \gamma^*$ be the corresponding dual bases. Then, $([T^*]_{\beta}^\gamma)^t = [T^*]_{\beta^*}^{\gamma^*}$. Hence, 

$$\dim(R(T)) = rank([T^*]_{\beta}^\gamma) = row\text{rank}([T^*]_{\beta}^\gamma) = rank((T^*)^t_{\beta^*}^{\gamma^*}) = dim(R(T^*))$$

by row rank = column rank (corollary 2 on page 158).

Proof 2. We claim $N(T^*) = R(T)^0$. Indeed, suppose $f \in N(T^*)$. Then, for $w = T(v) \in R(T)$, we have $f(w) = f(T(v)) = (T^*(f))(v) = 0$ as $T^*(f) = 0$. Hence, $N(T^*) \subseteq R(T)^0$. Now, suppose $f \in R(T)^0$. Then, for $v \in V$, $(T^*(f))(v) = f(T(v)) = 0$ as $T(v) \in R(T)$. Hence, $T^*(f) = 0$, so $N(T^*) \supseteq R(T)^0$. Hence, $N(T^*) = R(T)^0$. Now, if $V = dim V = dim R(T) + dim R(T)^0$ so 

$$\dim W = \dim R(T) + \dim R(T)^0 = \dim R(T) + \dim N(T^*)$$

and so 

$$\dim W = \dim R(T) + \dim W^* - \dim R(T^*)$$

Thus, $\dim R(T) = \dim R(T^*)$. By rank nullity, $\dim V = \dim R(T) + \dim N(T) = \dim R(T^*) + \dim N(T)$.

Proof 3. Let $\{w_1, \ldots, w_m\}$ be a basis for $R(T)$, say $w_k = T(v_k)$ for each $1 \leq k \leq m$. Extend this to a basis $\{w_1, \ldots, w_n\}$ of $W$. Let $\{f_1, \ldots, f_n\}$ be the dual basis of $\{w_1, \ldots, w_n\}$. We show $\{T^*(f_1), \ldots, T^*(f_m)\}$ is a basis for $R(T^*)$. Suppose $\sum_{j=1}^{m} c_j T^*(f_j)(v_k) = 0$. Then, for each $1 \leq k \leq m$, we have 

$$0 = (\sum_{j=1}^{m} c_j T^*(f_j))(v_k) = \sum_{j=1}^{m} c_j f_j(T(v_k)) = \sum_{j=1}^{m} c_j f_j(T(v_k)) = c_k$$

Hence, $\{T^*(f_1), \ldots, T^*(f_m)\}$ is a linearly independent set. To show $\{T^*(f_1), \ldots, T^*(f_m)\}$ spans $R(T^*)$, it suffices to show $T^*(f_k) = 0$ for $k > m$. Let $k > m$ and suppose $v \in V$. Then, $T^*(f_k)(v) = f_k(T(v)) = 0$ as $\{f_1, \ldots, f_m\}$ is a basis for $R(T)$ and $k > m$. Hence, $T^*(f_k)(v) = 0$ for all $v \in V$ so $T^*(f_k) = 0$.

10. Let $S = \{v_1, \ldots, v_m\}$ be a basis of a subspace $W$ of a vector space $V$. Let $S' = \{v_1, \ldots, v_n\}$ be an extension of this basis for $V$ and $\{f_1, \ldots, f_n\}$ the dual basis for $S'$. For each $i$, let $g_i$ be the restriction of $f_i$ to $W$ (i.e. $g_i = f_i|W$). Prove that $\{g_1, \ldots, g_m\}$ is the dual basis for $S$.

Proof. We note $g_i \in W^*$ for each $i$ and that for $1 \leq i, j \leq m$, $g_i(v_j) = f_i(v_j) = \delta_i(j)$, as desired.