LINEAR ALGEBRA

SHIN TAKAHASHI
IROHA INOUE
TREND-PRO CO., LTD.
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“The Manga Guides definitely have a place on my bookshelf.”
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“For parents trying to give their kids an edge or just for kids with a curiosity about their electronics, The Manga Guide to Electricity should definitely be on their bookshelves.”
—SACRAMENTO BOOK REVIEW

“This is a solid book and I wish there were more like it in the IT world.”
—SLASHDOT ON THE MANGA GUIDE TO DATABASES

“The Manga Guide to Electricity makes accessible a very intimidating subject, letting the reader have fun while still delivering the goods.”
—GEEKDAD BLOG, WIRED.COM

“If you want to introduce a subject that kids wouldn’t normally be very interested in, give it an amusing storyline and wrap it in cartoons.”
—MAKE ON THE MANGA GUIDE TO STATISTICS

“A clever blend that makes relativity easier to think about—even if you’re no Einstein.”
—STARDATE, UNIVERSITY OF TEXAS, ON THE MANGA GUIDE TO RELATIVITY

“This book does exactly what it is supposed to: offer a fun, interesting way to learn calculus concepts that would otherwise be extremely bland to memorize.”
—DAILY TECH ON THE MANGA GUIDE TO CALCULUS

“The art is fantastic, and the teaching method is both fun and educational.”
—ACTIVE ANIME ON THE MANGA GUIDE TO PHYSICS

“An awfully fun, highly educational read.”
—FRAZZLEDDAD ON THE MANGA GUIDE TO PHYSICS

“Makes it possible for a 10-year-old to develop a decent working knowledge of a subject that sends most college students running for the hills.”
—SKEPTICBLOG ON THE MANGA GUIDE TO MOLECULAR BIOLOGY

“This book is by far the best book I have read on the subject. I think this book absolutely rocks and recommend it to anyone working with or just interested in databases.”
—GEEK AT LARGE ON THE MANGA GUIDE TO DATABASES

“The book purposefully departs from a traditional physics textbook and it does it very well.”
—DR. MARINA MILNER-BOLOTIN, RYERSON UNIVERSITY ON THE MANGA GUIDE TO PHYSICS

“Kids would be, I think, much more likely to actually pick this up and find out if they are interested in statistics as opposed to a regular textbook.”
—GEEK BOOK ON THE MANGA GUIDE TO STATISTICS
THE MANGA GUIDE™ TO
LINEAR ALGEBRA

SHIN TAKAHASHI,
IROHA INOUE, AND
TREND-PRO CO., LTD.

Ohmsha

No Starch Press
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This book is for anyone who would like to get a good overview of linear algebra in a relatively short amount of time.

Those who will get the most out of The Manga Guide to Linear Algebra are:

- University students about to take linear algebra, or those who are already taking the course and need a helping hand
- Students who have taken linear algebra in the past but still don't really understand what it's all about
- High school students who are aiming to enter a technical university
- Anyone else with a sense of humor and an interest in mathematics!

The book contains the following parts:

Chapter 1: What Is Linear Algebra?
Chapter 2: The Fundamentals
Chapters 3 and 4: Matrices
Chapters 5 and 6: Vectors
Chapter 7: Linear Transformations
Chapter 8: Eigenvalues and Eigenvectors

Most chapters are made up of a manga section and a text section. While skipping the text parts and reading only the manga will give you a quick overview of each subject, I recommend that you read both parts and then review each subject in more detail for maximal effect. This book is meant as a complement to other, more comprehensive literature, not as a substitute.

I would like to thank my publisher, Ohmsha, for giving me the opportunity to write this book, as well as Iroha Inoue, the book’s illustrator. I would also like to express my gratitude towards re_akino, who created the scenario, and everyone at Trend Pro who made it possible for me to convert my manuscript into this manga. I also received plenty of good advice from Kazuyuki Hiraoka and Shizuka Hori. I thank you all.

SHIN TAKAHASHI
NOVEMBER 2008
PROLOGUE

LET THE TRAINING BEGIN!
Hanamichi University

nothing to be afraid of...

Okay!

SEYAAA!

Seyaaa!

Ei!

IT'S NOW OR NEVER!

* Ei! Hanamichi Karate Club

* Hanamichi Karate Club
Let the Training begin!

What do we have here?
I... I'M A FRESHMAN... MY NAME IS REIJI YURINO.

WOULD YOU BY ANY CHANCE BE TETSUO ICHINOSE, THE KARATE CLUB CAPTAIN?

INDEED.

WOW... THE HANAMICHI HAMMER HIMSELF!

U-UM...

I CAN'T BACK DOWN NOW!

I WANT TO JOIN THE KARATE CLUB!

ARE YOU SERIOUS? MY STUDENTS WOULD CHEW YOU UP AND SPIT YOU OUT.

I DON'T HAVE ANY EXPERIENCE, BUT I THINK I--
I WANT TO GET STRONGER!

PLEAS! I-

HAVE'NT I SEEN YOUR FACE SOMEWHERE?

AHA!

UHH...
Aren't you that guy? The one on my sister's math book?

Oh, you've seen my book?

So it is you!

I may not be the strongest guy...

But I've always been a whiz with numbers.

I see...

Hmm

I might consider letting you into the club...

Y-yes.

Wha—?
Let the Training begin!

really?!

...UNDER ONE CONDITION!

YOU HAVE TO TUTOR MY LITTLE SISTER IN MATH.

SHE'S NEVER BEEN THAT GOOD WITH NUMBERS, YOU SEE...

AND SHE COMPLAINED JUST YESTERDAY THAT SHE'S BEEN HAVING TROUBLE IN HER LINEAR ALGEBRA CLASS...

ERR

SO IF I TUTOR YOUR SISTER YOU'LL LET ME IN THE CLUB?
Would that be acceptable to you?

If you try to make a pass at her... even once...

Of course!

I suppose I should give you fair warning...

I wouldn't think of it!

We won't go easy on you, ya know.

Well start right away!

In that case... follow me.

Of course!

I'm in!
1

WHAT IS LINEAR ALGEBRA?
Okay! That's all for today!

Yurinooo!

Still alive, eh?

O-ossu...

You're free to start tutoring my sis after you've cleaned the room and put everything away, alright?

* OSSU! is an interjection often used in Japanese martial arts to enhance concentration and increase the power of one’s blows.

She's also a freshman here, but since there seem to be a lot of you this year, I somehow doubt you guys have met.

OSSU! Thank you!

Bow!

I told her to wait for you at...
I wonder what she'll be like...

Oww.

She looks a little like Tetsuo...

Umm...

Excuse me, are you Reiji Yurino?

WHA—

Pleased to meet you.

I'm Misa Ichinose.
SORRY ABOUT MY BROTHER ASKING YOU TO DO THIS ALL OF A SUDDEN.

NO PROBLEM! I DON'T MIND AT ALL!

BUT...

HOW COULD THIS GIRL POSSIBLY BE HIS SISTER?!

I HAD NO IDEA THAT I WENT TO SCHOOL WITH THE FAMOUS REIJI YURINO!

WELL, I WOULDN'T EXACTLY SAY FAMOUS...

I'VE BEEN LOOKING FORWARD TO THIS A LOT!

ERR... WOULD IT BE AWKWARD IF I ASKED YOU TO SIGN THIS?

LIKE...AN AUTOGRAPH?

NO, IT'S MY PLEASURE.

ONLY IF YOU WANT TO. IF IT'S TOO WEIRD—

GULP.
Sorry Misa, but I think I'll head home a bit early today.

Big brother!

You remember our little talk?

Yup!
Well then, when would you like to start?

Your brother said that you were having trouble with linear algebra?

Let's see...

Yes.

I don't really understand the concept of it all...

And the calculations seem way over my head.

It is true that linear algebra is a pretty abstract subject,

and there are some hard-to-understand concepts...

Linear independence

Subspace

Basis
The calculations aren't nearly as hard as they look!

And once you understand the basics, the math behind it is actually very simple.

Really?

I wouldn't say it's middle school level, but it's not far off.

Oh! Well, that's a relief...

You said it!

Um... err...

That's a tough question to answer properly.

Really? Why?

Err...

But I still don't understand... what is linear algebra exactly?

Um...

Well, it's pretty abstract stuff, but I'll give it my best shot.
BROADLY SPEAKING, LINEAR ALGEBRA IS ABOUT TRANSLATING SOMETHING RESIDING IN AN $m$-DIMENSIONAL SPACE INTO A CORRESPONDING SHAPE IN AN $n$-DIMENSIONAL SPACE.

WE'LL LEARN TO WORK WITH MATRICES...

AND VECTORS...

WITH THE GOAL OF UNDERSTANDING THE CENTRAL CONCEPTS OF:

- LINEAR TRANSFORMATIONS
- EIGENVALUES AND EIGENVECTORS

I SEE...
So...

What exactly is it good for? Outside of academic interest, of course.

You just had to ask me the dreaded question, didn't you?

While it is useful for a multitude of purposes indirectly, such as earthquake-proofing architecture, fighting diseases, protecting marine wildlife, and generating computer graphics...

It doesn't stand that well on its own, to be completely honest.

And mathematicians and physicists are the only ones who are really able to use the subject to its fullest potential.

Aww!
So even if I decide to study, it won't do me any good in the end?

That's not what I meant at all!

For example, for an aspiring chef to excel at his job, he has to know how to fillet a fish; it's just considered common knowledge.

The same relationship holds for math and science students and linear algebra; we should all know how to do it.

I see...

Like it or not, it's just one of those things you've got to know.

Best not to fight it. Just buckle down and study, and you'll do fine.

I'll try!
Regarding our future lessons...

I think it would be best if we concentrated on understanding linear algebra as a whole.

I'll try to avoid that as much as possible...

Most books and courses in the subject deal with long calculations and detailed proofs.

Whew.

Which leads to...

...and focus on explaining the basics as best I can.

Great!
We'll start with some of the fundamentals next time we meet.

Okay.

Oops, that's me!

It's from my brother.

Ring

Ring

Ring

Ring

Oh?

Hard to imagine...

Come home now x3?

Well, uh... tell him I said hello!
2
THE FUNDAMENTALS
Whump

You wish! After you're done with the pushups, I want you to start on your legs! That means squats! Go go go!

Hey... I thought we'd start off with...

Are you okay?

Reiji, you seem pretty out of it today.

I'll be fine. Take a look at thi--
Sorry, I guess I could use a snack...

Don't worry, pushing your body that hard has its consequences.

Just give me five minutes...

Well then, let's begin.

I don't mind. Take your time.

Take a look at this.
I took the liberty of making a diagram of what we're going to be talking about.

Wow!

I thought today we'd start on all the basic mathematics needed to understand linear algebra.

We'll start off slow and build our way up to the more abstract parts, okay?

Don't worry, you'll be fine.

Sure.
Complex numbers are written in the form
\[ a + b \cdot i \]
where \( a \) and \( b \) are real numbers and \( i \) is the imaginary unit, defined as \( i = \sqrt{-1} \).

<table>
<thead>
<tr>
<th>REAL NUMBERS</th>
<th>RATIONAL NUMBERS* (NOT INTEGERS)</th>
<th>IRRATIONAL NUMBERS</th>
<th>IMAGINARY NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>* Positive natural numbers</td>
<td>* Terminating decimal numbers like 0.3</td>
<td>* Numbers like ( \pi ) and ( \sqrt{2} ) whose decimals do not follow a pattern and repeat forever</td>
<td>Complex numbers without a real component, like ( 0 + bi ), where ( b ) is a nonzero real number</td>
</tr>
<tr>
<td>0</td>
<td>* Non-terminating decimal numbers like 0.333...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>* Negative natural numbers</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Numbers that can be expressed in the form \( q / p \) (where \( q \) and \( p \) are integers, and \( p \) is not equal to zero) are known as rational numbers. Integers are just special cases of rational numbers.

I don’t know for sure, but I suppose some mathematician made it up because he wanted to solve equations like \( x^2 + 5 = 0 \).
So...

Using this new symbol, these previously unsolvable problems suddenly became approachable.

\[ x^2 + 5 = x^2 - (-5) = (x + \sqrt{5}i)(x - \sqrt{5}i) = 0 \]

Why would you want to solve them in the first place? I don’t really see the point.

I understand where you’re coming from, but complex numbers appear pretty frequently in a variety of areas.

I’ll just have to get used to them, I suppose...

Don’t worry! I think it’d be better if we avoided them for now since they might make it harder to understand the really important parts.

Thanks!
A proposition is a declarative sentence that is either true or false, like...

"One plus one equals two" or "Japan's population does not exceed 100 people."

Let's look at a few examples.

A sentence like "Reiji Yurino is male" is a proposition.

But a sentence like "Reiji Yurino is handsome" is not.

To put it simply, ambiguous sentences that produce different reactions depending on whom you ask are not propositions.

"Reiji Yurino is female" is also a proposition, by the way.

My mom says I'm the most handsome guy in school...

That kind of makes sense.
Let's try to apply this knowledge to understand the concept of implication. The statement “If this dish is a schnitzel then it contains pork” is always true.

But if we look at its converse...

“If this dish contains pork then it is a schnitzel”...it is no longer necessarily true.

In situations where we know that "If P then Q" is true, but don't know anything about its converse "If Q then P"...

We say that "P entails Q" and that "Q could entail P."

When a proposition like "If P then Q" is true, it is common to write it with the implication symbol, like this: 

\[
P \Rightarrow Q
\]
If both "if P then Q" and "if Q then P" are true, then P and Q are equivalent.

That is, \( P \Rightarrow Q \) as well as \( Q \Rightarrow P \).

Then P and Q are equivalent.

Exactly! It's kind of like this.

Don't worry. You're due for a growth spurt...

So it's like the implication symbols point in both directions at the same time?

Reiji is shorter than Tetsuo.

Tetsuo is taller than Reiji.

And this is the symbol for equivalence.

\[ P \Leftrightarrow Q \]

Reiji is shorter than Tetsuo. \( \Leftrightarrow \) Tetsuo is taller than Reiji.
Another important field of mathematics is set theory.

Oh yeah... I think we covered that in high school.

Probably, but let's review it anyway.

Just as you might think, a set is a collection of things.

The things that make up the set are called its elements or objects.

Hehe, okay.

This might give you a good idea of what I mean.
**Example 1**

The set “Shikoku,” which is the smallest of Japan’s four islands, consists of these four elements:

- Kagawa-ken
- Ehime-ken
- Kouchi-ken
- Tokushima-ken

1. A Japanese ken is kind of like an American state.

**Example 2**

The set consisting of all even integers from 1 to 10 contains these five elements:

- 2
- 4
- 6
- 8
- 10
To illustrate, the set consisting of all even numbers between 1 and 10 would look like this:

These are two common ways to write out that set:

\[ \{2, 4, 6, 8, 10\} \quad \{2n \mid n = 1, 2, 3, 4, 5\} \]

It's also convenient to give the set a name, for example, \( X \).

With that in mind, our definition now looks like this:

\[ X = \{2, 4, 6, 8, 10\} \]
\[ X = \{2n \mid n = 1, 2, 3, 4, 5\} \]

This is a good way to express that "the element \( x \) belongs to the set \( X \)."

\[ x \in X \]

For example, \( \text{Ehime-ken} \in \text{Shikoku} \)
And then there are subsets.

Let's say that all elements of a set $X$ also belong to a set $Y$. $X$ is a subset of $Y$ in this case.

For example, Shikoku $\subset$ Japan

Set $Y$ (Japan)
- Hokkaidou
- Aomori-ken
- Iwate-ken
- Miyagi-ken
- Akita-ken
- Yamagata-ken
- Fukushima-ken
- Ibaraki-ken
- Tochigi-ken
- Gunma-ken
- Saitama-ken
- Chiba-ken
- Tokyo-to
- Kanagawa-ken
- Niigata-ken
- Toyama-ken
- Ishikawa-ken
- Fukui-ken
- Yamanashi-ken
- Nagano-ken
- Gifu-ken
- Shizuoka-ken
- Aichi-ken
- Mie-ken
- Shiga-ken
- Kyoto-fu
- Oita-ken
- Miyazaki-ken
- Fukuoka-ken
- Saga-ken
- Nagasaki-ken
- Kumamoto-ken
- Oita-ken
- Miyazaki-ken
- Kagoshima-ken
- Okinawa-ken

XCY

And it's written like this.

I see.
EXAMPLE 1
Suppose we have two sets $X$ and $Y$:

\[
X = \{4, 10\} \\
Y = \{2, 4, 6, 8, 10\}
\]

$X$ is a subset of $Y$, since all elements in $X$ also exist in $Y$.

EXAMPLE 2
Suppose we switch the sets:

\[
X = \{2, 4, 6, 8, 10\} \\
Y = \{4, 10\}
\]

Since all elements in $X$ don't exist in $Y$, $X$ is no longer a subset of $Y$.

EXAMPLE 3
Suppose we have two equal sets instead:

\[
X = \{2, 4, 6, 8, 10\} \\
Y = \{2, 4, 6, 8, 10\}
\]

In this case, both sets are subsets of each other. So $X$ is a subset of $Y$, and $Y$ is a subset of $X$.

EXAMPLE 4
Suppose we have the two following sets:

\[
X = \{2, 6, 10\} \\
Y = \{4, 8\}
\]

In this case neither $X$ nor $Y$ is a subset of the other.
I thought we'd talk about functions and their related concepts next.

It's all pretty abstract, but you'll be fine as long as you take your time and think hard about each new idea.

Got it.

Let's start by defining the concept itself.

Sounds good.
Imagine the following scenario:

Captain Ichinose, in a pleasant mood, decides to treat us freshmen to lunch.

So we follow him to restaurant A.

Follow me!

This is the restaurant menu.

- Udon 500 yen
- Curry 700 yen
- Breaded pork 1000 yen
- Broiled eel 1500 yen

But there is a catch, of course.

Since he’s the one paying, he gets a say in any and all orders.

What do you mean?

Kind of like this:
We wouldn’t really be able to say no if he told us to order the cheapest dish, right?

Or say, if he just told us all to order different things.
Even if he told us to order our favorites, we wouldn’t really have a choice. This might make us the most happy, but that doesn’t change the fact that we have to obey him.

You could say that the captain’s ordering guidelines are like a “rule” that binds elements of X to elements of Y.
AND THAT IS WHY...

WE DEFINE A "FUNCTION FROM X TO Y" AS THE RULE THAT BINDS ELEMENTS IN X TO ELEMENTS IN Y, JUST LIKE THE CAPTAIN'S RULES FOR HOW WE ORDER LUNCH!

THIS IS HOW WE WRITE IT:

\[
X \xrightarrow{f} Y \quad \text{or} \quad f : X \rightarrow Y
\]

\begin{align*}
\text{Club} & \quad \text{Rule} & \quad \text{Menu} \\
\text{Member} & \rightarrow & \text{Menu} \\
\text{Rule} & \quad \text{Rule} : & \quad \text{Club} \rightarrow \text{Menu}
\end{align*}

\[f\] IS COMPLETELY ARBITRARY. \(g\) OR \(h\) WOULD DO JUST AS WELL.

GOTCHA.

FUNCTIONS

A rule that binds elements of the set \(X\) to elements of the set \(Y\) is called "a function from \(X\) to \(Y\)." \(X\) is usually called the domain and \(Y\) the co-domain or target set of the function.
Let's assume that $x_i$ is an element of the set $X$.

The element in $Y$ that corresponds to $x_i$ when put through $f$ is called "$x_i$'s image under $f$ in $Y$.

Also, it's not uncommon to write "$x_i$'s image under $f$ in $Y" as $f(x_i)$. 

Okay!
AND IN OUR CASE...

\[\begin{align*}
  f(Yurino) &= \text{Udon} \\
  f(Yoshida) &= \text{Broiled Eel} \\
  f(Yajima) &= \text{Breaded Pork} \\
  f(Tomiyama) &= \text{Breaded Pork}
\end{align*}\]

I HOPE YOU LIKE UDON!

LIKE THIS:

This is the element in \(Y\) that corresponds to \(x_i\) of the set \(X\), when put through the function \(f\).
BY THE WAY, DO YOU REMEMBER THIS TYPE OF FORMULA FROM YOUR HIGH SCHOOL YEARS?

OH... YEAH, SURE.

\[ f(x) = 2x - 1 \]

DIDN'T YOU EVER WONDER WHY...

...THEY ALWAYS USED THIS WEIRD SYMBOL \( f(x) \) WHERE THEY COULD HAVE USED SOMETHING MUCH SIMPLER LIKE \( y \) INSTEAD?

"LIKE WHATEVER! ANYWAYS, SO IF I WANT TO SUBSTITUTE WITH 2 IN THIS FORMULA, I'M SUPPOSED TO WRITE \( f(2) \) AND..."

INSIDE MISA'S BRAIN

ACTUALLY... I HAVE!
Well, here’s why.

What $f(x) = 2x - 1$ really means is this:

The function $f$ is a rule that says:

"The element $x$ of the set $X$ goes together with the element $2x - 1$ in the set $Y$.”

Similarly, $f(2)$ implies this:

The image of 2 under the function $f$ is $2 \cdot 2 - 1$. So we were using functions in high school too?

I think I’m starting to get it. So that’s what it meant!

Exactly.
ON TO THE NEXT SUBJECT.

IN THIS CASE...

WE'RE GOING TO WORK WITH A SET

\{UDON, BREADED PORK, BROILED EEL\}

WHICH IS THE IMAGE OF THE SET X UNDER THE FUNCTION f.*

THIS SET IS USUALLY CALLED THE RANGE OF THE FUNCTION f, BUT IT IS SOMETIMES ALSO CALLED THE IMAGE OF f.

* THE TERM IMAGE IS USED HERE TO DESCRIBE THE SET OF ELEMENTS IN Y THAT ARE THE IMAGE OF AT LEAST ONE ELEMENT IN X.
AND THE SET $X$ IS DENOTED AS THE DOMAIN OF $f$.

WE COULD EVEN HAVE DESCRIBED THIS FUNCTION AS

$Y = \{ f(Yurino), f(Yoshida), f(Yajima), f(Tomiyama) \}$

IF WE WANTED TO.

RANGE AND CO-DOMAIN

The set that encompasses the function $f$'s image $\{ f(x_1), f(x_2), \ldots, f(x_n) \}$ is called the range of $f$, and the (possibly larger) set being mapped into is called its co-domain.

The relationship between the range and the co-domain $Y$ is as follows:

$\{ f(x_1), f(x_2), \ldots, f(x_n) \} \subset Y$

In other words, a function's range is a subset of its co-domain. In the special case where all elements in $Y$ are an image of some element in $X$, we have

$\{ f(x_1), f(x_2), \ldots, f(x_n) \} = Y$
Next we'll talk about onto and one-to-one functions.

Let's say our karate club decides to have a practice match with another club...

And that the captain's mapping function $f$ is "fight that guy."

You're already doing practice matches?

N—not really. This is just an example.

Still working on the basics!
Onto Functions

A function is onto if its image is equal to its co-domain. This means that all the elements in the co-domain of an onto function are being mapped onto.

One-to-One Functions

If \( x_i \neq x_j \) leads to \( f(x_i) \neq f(x_j) \), we say that the function is one-to-one. This means that no element in the co-domain can be mapped onto more than once.

One-to-One and Onto Functions

It's also possible for a function to be both onto and one-to-one. Such a function creates a "buddy system" between the elements of the domain and co-domain. Each element has one and only one "partner."
Now we have inverse functions.

Inverse?

This time we're going to look at the other team captain's orders as well.

We say that the function $g$ is $f$'s inverse when the two captains' orders coincide like this.

I see.
To specify even further...

If these two relations hold:

1. \( g(f(x_i)) = x_i \)
2. \( f(g(y_j)) = y_j \)

Oh, it's like the functions undo each other!

This is the symbol used to indicate inverse functions.

You raise it to \(-1\), right?

There is also a connection between one-to-one and onto functions and inverse functions. Have a look at this.

So if it's one-to-one and onto, it has an inverse, and vice versa. Got it!
I know it’s late, but I’d also like to talk a bit about linear transformations if you’re okay with it.

I know it’s late, but I’d also like to talk a bit about linear transformations if you’re okay with it.

Linear transformations? Oh right, one of the main subjects.

We’re already there?

Oh right, one of the main subjects.

No, we’re just going to have a quick look for now. We’ll go into more detail later on.

But don’t be fooled and let your guard down, it’s going to get pretty abstract from now on!

O-Okay!
LINEAR TRANSFORMATIONS

Let \( x_i \) and \( x_j \) be two arbitrary elements of the set \( X \), \( c \) be any real number, and \( f \) be a function from \( X \) to \( Y \). \( f \) is called a *linear transformation* from \( X \) to \( Y \) if it satisfies the following two conditions:

1. \( f(x_i) + f(x_j) = f(x_i + x_j) \)
2. \( cf(x_i) = f(cx_i) \)

Hmm... so that means...

I think we'd better draw a picture. What do you say?

That's a little easier to understand...

Condition 1 is when the sum of these two equals this.

And Condition 2 is when the product of this and a scalar \( c \) equals this.
An Example of a Linear Transformation

The function $f(x) = 2x$ is a linear transformation. This is because it satisfies both 1 and 2, as you can see in the table below.

| Condition 1  | $f(x_i) + f(x_j) = 2x_i + 2x_j$  
|             | $f(x_i + x_j) = 2(x_i + x_j) = 2x_i + 2x_j$ |
| Condition 2  | $cf(x_i) = c(2x_i) = 2cx_i$  
|             | $f(cx_i) = 2(cx_i) = 2cx_i$ |

An Example of a Function That Is Not a Linear Transformation

The function $f(x) = 2x - 1$ is not a linear transformation. This is because it satisfies neither 1 nor 2, as you can see in the table below.

| Condition 1  | $f(x_i) + f(x_j) = (2x_i - 1) + (2x_j - 1) = 2x_i + 2x_j - 2$  
|             | $f(x_i + x_j) = 2(x_i + x_j) - 1 = 2x_i + 2x_j - 1$ |
| Condition 2  | $cf(x_i) = c(2x_i - 1) = 2cx_i - c$  
|             | $f(cx_i) = 2(cx_i) - 1 = 2cx_i - 1$ |
Um...

Do you always eat lunch in the school cafeteria?

I live alone, and I'm not that good at cooking, so most of the time, yeah...

Well, next time we meet you won't have to. I'm making you lunch!

Don't worry about it! Tetsuo is still helping me out and all!

You don't want me to?
Oh... How lovely. I make a lot of them for my brother too, you know—stamina lunches.

On second thought, I'd love for you to...

...Make me lunch.

No, that's not it, it's just...

Uh...

I make a lot of them for my brother too, you know—stamina lunches.

Great!

Oh... How lovely.
Combinations and Permutations

I thought the best way to explain combinations and permutations would be to give a concrete example.

I'll start by explaining the **Problem**, then I'll establish a good **Way of Thinking**, and finally I'll present a **Solution**.

**Problem**

Reiji bought a CD with seven different songs on it a few days ago. Let's call the songs A, B, C, D, E, F, and G. The following day, while packing for a car trip he had planned with his friend Nemoto, it struck him that it might be nice to take the songs along to play during the drive. But he couldn't take all of the songs, since his taste in music wasn't very compatible with Nemoto's. After some deliberation, he decided to make a new CD with only three songs on it from the original seven.

Questions:

1. In how many ways can Reiji select three songs from the original seven?
2. In how many ways can the three songs be arranged?
3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

**Way of Thinking**

It is possible to solve question 3 by dividing it into these two subproblems:

1. Choose three songs out of the seven possible ones.
2. Choose an order in which to play them.

As you may have realized, these are the first two questions. The solution to question 3, then, is as follows:

<table>
<thead>
<tr>
<th>Solution to Question 1</th>
<th>Solution to Question 2</th>
<th>Solution to Question 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>In how many ways can Reiji select three songs from the original seven?</td>
<td>In how many ways can the three songs be arranged?</td>
<td>In how many ways can a CD be made, where three songs are chosen from a pool of seven?</td>
</tr>
</tbody>
</table>
1. In how many ways can Reiji select three songs from the original seven?

All 35 different ways to select the songs are in the table below. Feel free to look them over.

<table>
<thead>
<tr>
<th>Pattern 1</th>
<th>A and B and C</th>
<th>Pattern 15</th>
<th>A and F and G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 2</td>
<td>A and B and D</td>
<td>Pattern 16</td>
<td>B and C and D</td>
</tr>
<tr>
<td>Pattern 3</td>
<td>A and B and E</td>
<td>Pattern 17</td>
<td>B and C and E</td>
</tr>
<tr>
<td>Pattern 4</td>
<td>A and B and F</td>
<td>Pattern 18</td>
<td>B and C and F</td>
</tr>
<tr>
<td>Pattern 5</td>
<td>A and B and G</td>
<td>Pattern 19</td>
<td>B and C and G</td>
</tr>
<tr>
<td>Pattern 6</td>
<td>A and C and D</td>
<td>Pattern 20</td>
<td>B and D and E</td>
</tr>
<tr>
<td>Pattern 7</td>
<td>A and C and E</td>
<td>Pattern 21</td>
<td>B and D and F</td>
</tr>
<tr>
<td>Pattern 8</td>
<td>A and C and F</td>
<td>Pattern 22</td>
<td>B and D and G</td>
</tr>
<tr>
<td>Pattern 9</td>
<td>A and C and G</td>
<td>Pattern 23</td>
<td>B and E and F</td>
</tr>
<tr>
<td>Pattern 10</td>
<td>A and D and E</td>
<td>Pattern 24</td>
<td>B and E and G</td>
</tr>
<tr>
<td>Pattern 11</td>
<td>A and D and F</td>
<td>Pattern 25</td>
<td>B and F and G</td>
</tr>
<tr>
<td>Pattern 12</td>
<td>A and D and G</td>
<td>Pattern 26</td>
<td>C and D and E</td>
</tr>
<tr>
<td>Pattern 13</td>
<td>A and E and F</td>
<td>Pattern 27</td>
<td>C and D and F</td>
</tr>
<tr>
<td>Pattern 14</td>
<td>A and E and G</td>
<td>Pattern 28</td>
<td>C and D and G</td>
</tr>
<tr>
<td>Pattern 15</td>
<td>A and F and G</td>
<td>Pattern 29</td>
<td>C and E and F</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 30</td>
<td>C and E and G</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 31</td>
<td>C and F and G</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 32</td>
<td>D and E and G</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 33</td>
<td>D and E and G</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 34</td>
<td>D and F and G</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Pattern 35</td>
<td>E and F and G</td>
</tr>
</tbody>
</table>

Choosing \(k\) among \(n\) items without considering the order in which they are chosen is called a combination. The number of different ways this can be done is written by using the binominal coefficient notation:

\[
\binom{n}{k}
\]

which is read “\(n\) choose \(k\).”

In our case,

\[
\binom{7}{3} = 35
\]
2. In how many ways can the three songs be arranged?

Let's assume we chose the songs A, B, and C. This table illustrates the 6 different ways in which they can be arranged:

<table>
<thead>
<tr>
<th>Song 1</th>
<th>Song 2</th>
<th>Song 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

Suppose we choose B, E, and G instead:

<table>
<thead>
<tr>
<th>Song 1</th>
<th>Song 2</th>
<th>Song 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>E</td>
<td>G</td>
</tr>
<tr>
<td>B</td>
<td>G</td>
<td>E</td>
</tr>
<tr>
<td>E</td>
<td>B</td>
<td>G</td>
</tr>
<tr>
<td>E</td>
<td>G</td>
<td>B</td>
</tr>
<tr>
<td>G</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>G</td>
<td>E</td>
<td>B</td>
</tr>
</tbody>
</table>

Trying a few other selections will reveal a pattern: The number of possible arrangements does not depend on which three elements we choose—there are always six of them. Here's why:

Our result (6) can be rewritten as $3 \cdot 2 \cdot 1$, which we get like this:

1. We start out with all three songs and can choose any one of them as our first song.
2. When we're picking our second song, only two remain to choose from.
3. For our last song, we're left with only one choice.

This gives us $3 \cdot 2 \cdot 1 = 6$ possibilities.
3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

The different possible patterns are

The number of ways to choose three songs from seven

\[ \binom{7}{3} \cdot 6 \]

\[ = 35 \cdot 6 \]

\[ = 210 \]

This means that there are 210 different ways to make the CD.

Choosing three from seven items in a certain order creates a permutation of the chosen items. The number of possible permutations of \( k \) objects chosen among \( n \) objects is written as

\[ n^P_k \]

In our case, this comes to

\[ 7^P_3 = 210 \]

The number of ways \( n \) objects can be chosen among \( n \) possible ones is equal to \( n \)-factorial:

\[ n^P_n = n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1 \]

For instance, we could use this if we wanted to know how many different ways seven objects can be arranged. The answer is

\[ 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 \]
I've listed all possible ways to choose three songs from the seven original ones (A, B, C, D, E, F, and G) in the table below.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Song 1</th>
<th>Song 2</th>
<th>Song 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Pattern 2</td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>Pattern 3</td>
<td>A</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 30</td>
<td>A</td>
<td>G</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 31</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 60</td>
<td>B</td>
<td>G</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 61</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 90</td>
<td>C</td>
<td>G</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 91</td>
<td>D</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 120</td>
<td>D</td>
<td>G</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 121</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 150</td>
<td>E</td>
<td>G</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 151</td>
<td>F</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 180</td>
<td>F</td>
<td>G</td>
<td>E</td>
</tr>
<tr>
<td>Pattern 181</td>
<td>G</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Pattern 209</td>
<td>G</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>Pattern 210</td>
<td>G</td>
<td>F</td>
<td>E</td>
</tr>
</tbody>
</table>

We can, analogous to the previous example, rewrite our problem of counting the different ways in which to make a CD as $7 \cdot 6 \cdot 5 = 210$. Here’s how we get those numbers:

1. We can choose any of the 7 songs A, B, C, D, E, F, and G as our first song.
2. We can then choose any of the 6 remaining songs as our second song.
3. And finally we choose any of the now 5 remaining songs as our last song.
The definition of the binomial coefficient is as follows:

\[
\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-(r-1))}{r \cdot (r-1) \cdots 1} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 1}
\]

Notice that

\[
\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-(r-1))}{r \cdot (r-1) \cdots 1} = \frac{n \cdot (n-1) \cdots (n-(r-1)) \cdot (n-r) \cdot (n-r+1) \cdots 1}{r \cdot (r-1) \cdots 1 \cdot (n-r) \cdot (n-r+1) \cdots 1} = \frac{n!}{r! \cdot (n-r)!}
\]

Many people find it easier to remember the second version:

\[
\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}
\]

We can rewrite question 3 (how many ways can the CD be made?) like this:

\[
\binom{7}{3} = \binom{7}{3} \cdot 6 = \binom{7}{3} \cdot 3! = \frac{7!}{3! \cdot 4!} \cdot 3! = \frac{7!}{4!} \cdot 3! = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 = 210
\]
We talked about the three commands “Order the cheapest one!” “Order different stuff!” and “Order what you want!” as functions on pages 37–38. It is important to note, however, that “Order different stuff!” isn’t actually a function in the strictest sense, because there are several different ways to obey that command.
3
INTRO TO MATRICES
EI!

EI!

YURINO!

Put your backs into it!

Don’t rely on your hands.

Use your waist!

Ossu!

I thought he’d quit right away...

I guess I was wrong.

Heheh
You must be really tired after all that exercise!

Wow! But...I could never eat something so beautiful!

Hehe, don't be silly.

I don't know what to say...thank you!

OM NOM NOM

Awesome!

So good!

Thanks.

Misa, really...thank you.

Don't worry about it.
What is a Matrix?

A matrix is a collection of numbers arranged in m rows and n columns, bounded by parentheses, like this.

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
$$

These are called subscripts.
A matrix with \( m \) rows and \( n \) columns is called an "\( m \) by \( n \) matrix."

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
-3 \\
0 \\
8 \\
-7
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

2×3 matrix 4×1 matrix \( m \times n \) matrix

Ah.

The items inside a matrix are called its elements.

I've marked the (2, 1) elements of these three matrices for you.

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
\text{row 1} \\
\text{row 2} \\
\text{row 3} \\
\text{row 4}
\end{pmatrix}
\begin{pmatrix}
\text{COL 1} & \text{COL 2} & \text{COL N}
\end{pmatrix}
\begin{pmatrix}
-3 \\
0 \\
8 \\
-7
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

Ah.

A matrix that has an equal number of rows and columns is called a square matrix.

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

Square matrix with two rows Square matrix with \( n \) rows

Uh huh...

The grayed out elements in this matrix are part of what is called its main diagonal.
HMM... MATRICES AREN'T AS EXCITING AS THEY SEEM IN THE MOVIES.

Yeah. Just numbers, no Keanu...

EXCITING OR NOT, MATRICES ARE VERY USEFUL!

- They're great for writing linear systems more compactly.
- Since they make for more compact systems, they also help to make mathematical literature more compact.
- And they help teachers write faster on the blackboard during class.

Well, these are some of the advantages.

WHY IS THAT?

So people use them because they're practical, huh?

Yep.
Instead of writing this linear system like this...

\[
\begin{align*}
1x_1 + 2x_2 &= -1 \\
3x_1 + 4x_2 &= 0 \\
5x_1 + 6x_2 &= 5
\end{align*}
\]

We could write it like this, using matrices.

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
=
\begin{pmatrix}
-1 \\
0 \\
5
\end{pmatrix}
\]

It does look a lot cleaner.

Exactly!

So this...

\[
\begin{align*}
1x_1 + 2x_2 \\
3x_1 + 4x_2 \\
5x_1 + 6x_2
\end{align*}
\]

Becomes this?

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Not bad!

---

**Writing Systems of Equations as Matrices**

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

is written

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
& \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix}
\]

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n
\end{align*}
\]

is written

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
& \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]
Matrix Calculations

Now let's look at some calculations.

The four relevant operators are:
- Addition
- Subtraction
- Scalar multiplication
- Matrix multiplication

**Addition**

Let's add the $3 \times 2$ matrix

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\]

to this $3 \times 2$ matrix

\[
\begin{pmatrix}
6 & 5 \\
4 & 3 \\
2 & 1
\end{pmatrix}
\]

That is:

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
+ \begin{pmatrix}
6 & 5 \\
4 & 3 \\
2 & 1
\end{pmatrix}
= \begin{pmatrix}
1+6 & 2+5 \\
3+4 & 4+3 \\
5+2 & 6+1
\end{pmatrix}
\]

The elements would be added elementwise, like this:

**Examples**

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
+ \begin{pmatrix}
6 & 5 \\
4 & 3 \\
2 & 1
\end{pmatrix}
= \begin{pmatrix}
7 & 7 \\
7 & 7 \\
7 & 7
\end{pmatrix}
\]

\[
(10, 10) + (-3, -6) = (10 + (-3), 10 + (-6)) = (7, 4)
\]

\[
\begin{pmatrix}
10 \\
10
\end{pmatrix}
+ \begin{pmatrix}
-3 \\
-6
\end{pmatrix}
= \begin{pmatrix}
10 + (-3) \\
10 + (-6)
\end{pmatrix}
= \begin{pmatrix}
7 \\
4
\end{pmatrix}
\]

Note that addition and subtraction work only with matrices that have the same dimensions.
SUBTRACTION

LET’S SUBTRACT THE 3×2 MATRIX

\[
\begin{bmatrix}
6 & 5 \\
4 & 3 \\
2 & 1 \\
\end{bmatrix}
\]

FROM THIS 3×2 MATRIX

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{bmatrix}
\]

THAT IS:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{bmatrix}
- \begin{bmatrix}
6 & 5 \\
4 & 3 \\
2 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 - 6 & 2 - 5 \\
3 - 4 & 4 - 3 \\
5 - 2 & 6 - 1 \\
\end{bmatrix}
\]

THE ELEMENTS WOULD SIMILARLY BE SUBTRACTED ELEMENTWISE, LIKE THIS:

EXAMPLES

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{bmatrix}
- \begin{bmatrix}
6 & 5 \\
4 & 3 \\
2 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 - 6 & 2 - 5 \\
3 - 4 & 4 - 3 \\
5 - 2 & 6 - 1 \\
\end{bmatrix}
= \begin{bmatrix}
-5 & -3 \\
-1 & 1 \\
3 & 5 \\
\end{bmatrix}
\]

\[
(10, 10) - (-3, -6) = (10 - (-3), 10 - (-6)) = (13, 16)
\]

\[
\begin{bmatrix}
10 \\
10 \\
\end{bmatrix}
- \begin{bmatrix}
-3 \\
-6 \\
\end{bmatrix}
= \begin{bmatrix}
10 - (-3) \\
10 - (-6) \\
\end{bmatrix}
= \begin{bmatrix}
13 \\
16 \\
\end{bmatrix}
\]
## Scalar Multiplication

Let's multiply the $3 \times 2$ matrix

$$
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{pmatrix}
$$

by 10. That is:

$$
10 \begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{pmatrix}
$$

The elements would each be multiplied by 10, like this:

$$
\begin{pmatrix}
10 \cdot 1 & 10 \cdot 2 \\
10 \cdot 3 & 10 \cdot 4 \\
10 \cdot 5 & 10 \cdot 6 \\
\end{pmatrix}
$$

### Examples

- $10 \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 \cdot 1 & 10 \cdot 2 \\ 10 \cdot 3 & 10 \cdot 4 \\ 10 \cdot 5 & 10 \cdot 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 30 & 40 \\ 50 & 60 \end{pmatrix}$

- $2 (3, 1) = (2 \cdot 3, 2 \cdot 1) = (6, 2)$

- $2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
Matrix Multiplication

The product

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 & y_1 \\
x_2 & y_2
\end{pmatrix}
= 
\begin{pmatrix}
1x_1 + 2x_2 & 1y_1 + 2y_2 \\
3x_1 + 4x_2 & 3y_1 + 4y_2 \\
5x_1 + 6x_2 & 5y_1 + 6y_2
\end{pmatrix}
\]

Can be derived by temporarily separating the two columns

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

forming the two products

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
1x_1 + 2x_2 \\
3x_1 + 4x_2 \\
5x_1 + 6x_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= 
\begin{pmatrix}
1y_1 + 2y_2 \\
3y_1 + 4y_2 \\
5y_1 + 6y_2
\end{pmatrix}
\]

and then rejoining the resulting columns:

\[
\begin{pmatrix}
1x_1 + 2x_2 & 1y_1 + 2y_2 \\
3x_1 + 4x_2 & 3y_1 + 4y_2 \\
5x_1 + 6x_2 & 5y_1 + 6y_2
\end{pmatrix}
\]

Example

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 & y_1 \\
x_2 & y_2
\end{pmatrix}
= 
\begin{pmatrix}
1x_1 + 2x_2 & 1y_1 + 2y_2 \\
3x_1 + 4x_2 & 3y_1 + 4y_2 \\
5x_1 + 6x_2 & 5y_1 + 6y_2
\end{pmatrix}
\]

There's more!
As you can see from the example below, changing the order of factors usually results in a completely different product.

\[
\begin{bmatrix}
8 & -3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}
= 
\begin{bmatrix}
8 \cdot 3 + (-3) \cdot 1 & 8 \cdot 1 + (-3) \cdot 2 \\
2 \cdot 3 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 2
\end{bmatrix}
= 
\begin{bmatrix}
24 - 3 & 8 - 6 \\
6 + 1 & 2 + 2
\end{bmatrix}
= 
\begin{bmatrix}
21 & 2 \\
7 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
8 & -3 \\
2 & 1
\end{bmatrix}
= 
\begin{bmatrix}
3 \cdot 8 + 1 \cdot 2 & 3 \cdot (-3) + 1 \cdot 1 \\
1 \cdot 8 + 2 \cdot 2 & 1 \cdot (-3) + 2 \cdot 1
\end{bmatrix}
= 
\begin{bmatrix}
24 + 2 & -9 + 1 \\
8 + 4 & -3 + 2
\end{bmatrix}
= 
\begin{bmatrix}
26 & -8 \\
12 & -1
\end{bmatrix}
\]

And you have to watch out.

\[
\begin{bmatrix}
1 & 2 & \cdots & n
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
\chi_{11} & \chi_{12} & \cdots & \chi_{1p} \\
\chi_{21} & \chi_{22} & \cdots & \chi_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{n1} & \chi_{n2} & \cdots & \chi_{np}
\end{bmatrix}
\]

An \( m \times n \) matrix times an \( n \times p \) matrix yields an \( m \times p \) matrix.

Matrices can be multiplied only if the number of columns in the left factor matches the number of rows in the right factor.
This means we wouldn't be able to calculate the product if we switched the two matrices in our first example.

Huh, really?

Well, nothing stops us from trying.

Oops...

One more thing. It's okay to use exponents to express repeated multiplication of square matrices.

\[
\begin{pmatrix}
\mathbf{a}_{11} & \mathbf{a}_{12} & \ldots & \mathbf{a}_{1n} \\
\mathbf{a}_{21} & \mathbf{a}_{22} & \ldots & \mathbf{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{a}_{n1} & \mathbf{a}_{n2} & \ldots & \mathbf{a}_{nn}
\end{pmatrix}^p
\begin{pmatrix}
\mathbf{a}_{11} & \mathbf{a}_{12} & \ldots & \mathbf{a}_{1n} \\
\mathbf{a}_{21} & \mathbf{a}_{22} & \ldots & \mathbf{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{a}_{n1} & \mathbf{a}_{n2} & \ldots & \mathbf{a}_{nn}
\end{pmatrix}
\]

\[
P \text{ factors}
\]
Like this. 

OH, SO... 

THIS IS ALL RIGHT THEN? 

YEAH.

UH...BUT HOW AM I SUPPOSED TO CALCULATE THREE OF THEM IN A ROW? 

PRETTY CONFUSING 

WELL...

THE EASIEST WAY WOULD BE TO JUST MULTIPLY THEM FROM LEFT TO RIGHT, LIKE THIS: 

$$
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}^3 = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} = \begin{pmatrix}
1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\
3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4
\end{pmatrix} \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
$$

$$
\begin{pmatrix}
7 & 10 \\
15 & 22
\end{pmatrix} \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} = \begin{pmatrix}
7 \cdot 1 + 10 \cdot 3 & 7 \cdot 2 + 10 \cdot 4 \\
15 \cdot 1 + 22 \cdot 3 & 15 \cdot 2 + 22 \cdot 4
\end{pmatrix} = \begin{pmatrix}
37 & 54 \\
81 & 118
\end{pmatrix}
$$
There are many special types of matrices. To explain them all would take too much time...

So we'll look at only these eight today.

1. Zero matrices
2. Transpose matrices
3. Symmetric matrices
4. Upper triangular matrices
5. Lower triangular matrices
6. Diagonal matrices
7. Identity matrices
8. Inverse matrices

Let's look at them in order.

Okay!

1. Zero matrices

A zero matrix is a matrix where all elements are equal to zero.

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
The easiest way to understand transpose matrices is to just look at an example.

If we transpose the 2×3 matrix
\[
\begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}
\]
we get the 3×2 matrix
\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\]

As you can see, the transpose operator switches the rows and columns in a matrix.

The transpose of the \(n \times m\) matrix
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
is consequently
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}^T
\]

The most common way to indicate a transpose is to add a small \(T\) at the top-right corner of the matrix.

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}^T
\]

For example:
\[
\begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}^T = \begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\]
Symmetric matrices are square matrices that are symmetric around their main diagonals.

\[
\begin{bmatrix}
1 & 5 & 6 & 7 \\
5 & 2 & 8 & 9 \\
6 & 8 & 3 & 10 \\
7 & 9 & 10 & 4
\end{bmatrix}
\]

Because of this characteristic, a symmetric matrix is always equal to its transpose.

Triangular matrices are square matrices in which the elements either above the main diagonal or below it are all equal to zero.

This is an upper triangular matrix, since all elements below the main diagonal are zero.

\[
\begin{bmatrix}
1 & 5 & 6 & 7 \\
0 & 2 & 8 & 9 \\
0 & 0 & 3 & 10 \\
0 & 0 & 0 & 4
\end{bmatrix}
\]

This is a lower triangular matrix—all elements above the main diagonal are zero.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
5 & 2 & 0 & 0 \\
6 & 8 & 3 & 0 \\
7 & 9 & 10 & 4
\end{bmatrix}
\]
A diagonal matrix is a square matrix in which all elements that are not part of its main diagonal are equal to zero.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4 \\
\end{pmatrix}
\]

For example, \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4 \\
\end{pmatrix}
\]
is a diagonal matrix.

Note that this matrix could also be written as \text{diag}(1,2,3,4).
SEE FOR YOURSELF!

\[
\begin{pmatrix}
    a_{11} & 0 & \cdots & 0 \\
    0 & a_{22} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{nn}
\end{pmatrix}^p =
\begin{pmatrix}
    a_{11}^p & 0 & \cdots & 0 \\
    0 & a_{22}^p & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{nn}^p
\end{pmatrix}
\]

UH...

TRY CALCULATING

\[
\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 \quad \text{AND} \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^3
\]

TO SEE WHY.

\[
\begin{align*}
\cdot (2 \ 0)^2 &= (2 \ 0)(2 \ 0) = \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 2 + 0 \cdot 0 & 2 \cdot 0 + 0 \cdot 3 \end{pmatrix} \\
&= \begin{pmatrix} 2^2 & 0 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\cdot (2 \ 0)^3 &= (2 \ 0)^2(2 \ 0) = \begin{pmatrix} 2 \ 0 \end{pmatrix}^2 \begin{pmatrix} 2 \ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 2 + 0 \cdot 0 & 2 \cdot 0 + 0 \cdot 3 \end{pmatrix} \\
&= \begin{pmatrix} 2^2 & 0 \end{pmatrix}
\end{align*}
\]

LIKE THIS?

YOU'RE RIGHT!

\[
\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^p = \begin{pmatrix} 2^p & 0 \\ 0 & 3^p \end{pmatrix}
\]

WEIRD, HUH?
Identity matrices are in essence diag(1,1,1,...,1). In other words, they are square matrices with \( n \) rows in which all elements on the main diagonal are equal to 1 and all other elements are 0.

For example, an identity matrix with \( n = 4 \) would look like this:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Multiplying with the identity matrix yields a product equal to the other factor.

It's like the number 1 in ordinary multiplication.

\[ 1 \cdot 50 = 50 \]

\[ 1 \cdot \mathbf{x} = \mathbf{x} \]

Try multiplying

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

if you'd like.

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2
\end{pmatrix} =
\begin{pmatrix}
1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 \\
0 \cdot \mathbf{x}_1 + 1 \cdot \mathbf{x}_2
\end{pmatrix} =
\begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2
\end{pmatrix}
\]

It stays the same, just like you said!
Let's try a few other examples.

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
1 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n \\
0 \cdot x_1 + 1 \cdot x_2 + \cdots + 0 \cdot x_n \\
\vdots \\
0 \cdot x_1 + 0 \cdot x_2 + \cdots + 1 \cdot x_n
\end{pmatrix}
= 
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{21} & \cdots & x_{n1} \\
x_{12} & x_{22} & \cdots & x_{n2}
\end{pmatrix}
= 
\begin{pmatrix}
1 \cdot x_{11} + 0 \cdot x_{12} & 1 \cdot x_{21} + 0 \cdot x_{22} & \cdots & 1 \cdot x_{n1} + 0 \cdot x_{n2} \\
0 \cdot x_{11} + 1 \cdot x_{12} & 0 \cdot x_{21} + 1 \cdot x_{22} & \cdots & 0 \cdot x_{n1} + 1 \cdot x_{n2}
\end{pmatrix}
= 
\begin{pmatrix}
x_{11} & x_{21} & \cdots & x_{n1} \\
x_{12} & x_{22} & \cdots & x_{n2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22} \\
\vdots & \vdots \\
x_{n1} & x_{n2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
x_{11} \cdot 1 + x_{12} \cdot 0 & x_{11} \cdot 0 + x_{12} \cdot 1 \\
x_{21} \cdot 1 + x_{22} \cdot 0 & x_{21} \cdot 0 + x_{22} \cdot 1 \\
\vdots & \vdots \\
x_{n1} \cdot 1 + x_{n2} \cdot 0 & x_{n1} \cdot 0 + x_{n2} \cdot 1
\end{pmatrix}
= 
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22} \\
\vdots & \vdots \\
x_{n1} & x_{n2}
\end{pmatrix}
\]

Were you able to follow? Want another look? No way! Piece of cake!
LET'S TAKE A BREAK. WE STILL HAVE INVERSE MATRICES LEFT TO LOOK AT, BUT THEY'RE A BIT MORE COMPLEX THAN THIS.

FINE BY ME.

NO, I WASN'T TRYING TO GET YOU TO—

Fine by me.

I'LL MAKE YOU ANOTHER TOMORROW IF YOU'D LIKE.
GATHER 'EM UP...

AND SWEEP...
Inverse matrices are very important and have a lot of different applications. Explaining them without using examples isn’t all that easy, but let’s start with the definition anyway.

If the product of two square matrices is an identity matrix, then the two factor matrices are inverses of each other.

This means that  
\[
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{pmatrix}
\]  
is an inverse matrix to  
\[
\begin{pmatrix}
  1 & 2 \\
  3 & 4
\end{pmatrix}
\]  
if  
\[
\begin{pmatrix}
  1 & 2 \\
  3 & 4
\end{pmatrix}
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}
\]
And that's it.

DIDN'T YOU SAY SOMETHING ABOUT EXAMPLES? WE'RE DONE ALREADY?

Don't worry, that was only the definition!

Huh?

Didn't you say something about examples? We're done already?

Don't worry, that was only the definition!

Since they're so important, I thought we'd go into more detail on this one.

I'll teach you how to identify whether an inverse exists or not—and also how to calculate one.

Should we get right down to business?

Sure!
Calculating Inverse Matrices

There are two main ways to calculate an inverse matrix:

Using cofactors or using Gaussian elimination.

The calculations involved in the cofactor method can very easily become cumbersome, so...

Ignore it as long as you're not expecting it on a test.

Can do.

In contrast, Gaussian elimination is easy both to understand and to calculate.

In fact, it's as easy as sweeping the floor!

Anyway, I won't talk about cofactors at all today.

Gotcha.

In addition to finding inverse matrices, Gaussian elimination can also be used to solve linear systems.

Let's have a look at that.

Cool!

* The Japanese term for Gaussian elimination is Hakidashi/hou, which roughly translates to “the sweeping out method.” Keep this in mind as you're reading this chapter!
### Problem

Solve the following linear system:

\[
\begin{align*}
3x_1 + 1x_2 &= 1 \\
1x_1 + 2x_2 &= 0
\end{align*}
\]

### Solution

#### The Common Method

<table>
<thead>
<tr>
<th>THE COMMON METHOD</th>
<th>THE COMMON METHOD EXPRESSED WITH MATRICES</th>
<th>GAUSSIAN ELIMINATION</th>
</tr>
</thead>
</table>
| \[
\begin{align*}
3x_1 + 1x_2 &= 1 \\
1x_1 + 2x_2 &= 0
\end{align*}
\] | \[
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] | \[
\begin{bmatrix}
3 & 1 & 1 \\
1 & 2 & 0
\end{bmatrix}
\] |
| Start by multiplying the top equation by 2. | Subtract the bottom equation from the top equation. | |
| \[
\begin{align*}
6x_1 + 2x_2 &= 2 \\
1x_1 + 2x_2 &= 0
\end{align*}
\] | \[
\begin{align*}
6 & 2 & x_1 \\
1 & 2 & x_2
\end{align*}
= 
\begin{align*}
2 \\
0
\end{align*}
\] | \[
\begin{bmatrix}
6 & 2 & 2 \\
1 & 2 & 0
\end{bmatrix}
\] |
| Subtract the bottom equation from the top equation. | Multiply the bottom equation by 5. | |
| \[
\begin{align*}
5x_1 + 0x_2 &= 2 \\
1x_1 + 2x_2 &= 0
\end{align*}
\] | \[
\begin{align*}
5 & 0 & x_1 \\
1 & 2 & x_2
\end{align*}
= 
\begin{align*}
2 \\
0
\end{align*}
\] | \[
\begin{bmatrix}
5 & 0 & 2 \\
1 & 2 & 0
\end{bmatrix}
\] |
| Multiply the bottom equation by 5. | Subtract the top equation from the bottom equation. | |
| \[
\begin{align*}
5x_1 + 0x_2 &= 2 \\
5x_1 + 10x_2 &= 0
\end{align*}
\] | \[
\begin{align*}
5 & 0 & x_1 \\
5 & 10 & x_2
\end{align*}
= 
\begin{align*}
2 \\
0
\end{align*}
\] | \[
\begin{bmatrix}
5 & 0 & 2 \\
5 & 10 & 0
\end{bmatrix}
\] |
| Subtract the top equation from the bottom equation. | Divide the top equation by 5 and the bottom by 10. | |
| \[
\begin{align*}
5x_1 + 0x_2 &= 2 \\
0x_1 + 10x_2 &= -2
\end{align*}
\] | \[
\begin{align*}
5 & 0 & x_1 \\
0 & 10 & x_2
\end{align*}
= 
\begin{align*}
2 \\
-2
\end{align*}
\] | \[
\begin{bmatrix}
5 & 0 & 2 \\
0 & 10 & -2
\end{bmatrix}
\] |
| Divide the top equation by 5 and the bottom by 10. | And we're done! | |
| \[
\begin{align*}
x_1 + 0x_2 &= \frac{2}{5} \\
0x_1 + 1x_2 &= -\frac{1}{5}
\end{align*}
\] | | \[
\begin{bmatrix}
1 & 0 & \frac{2}{5} \\
0 & 1 & -\frac{1}{5}
\end{bmatrix}
\] |
Problem

Find the inverse of the 2×2 matrix \[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\]

Think about it like this.

We're trying to find the inverse of \[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\]

We need to find the matrix \[
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{pmatrix}
\]
that satisfies \[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

or \[
\begin{pmatrix}
x_{11} \\
x_{21}
\end{pmatrix}
\text{ and } \begin{pmatrix}
x_{12} \\
x_{22}
\end{pmatrix}
\]
that satisfy
\[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x_{11} \\
x_{21}
\end{pmatrix} = \begin{pmatrix}
1 \\
0
\end{pmatrix}
\text{ and } \begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x_{12} \\
x_{22}
\end{pmatrix} = \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

We need to solve the systems
\[
\begin{cases}
3x_{11} + 1x_{21} = 1 \\
x_{11} + 2x_{21} = 0
\end{cases}
\text{ and } \begin{cases}
3x_{12} + 1x_{22} = 0 \\
x_{12} + 2x_{22} = 1
\end{cases}
\]
### SOLUTION

<table>
<thead>
<tr>
<th>THE COMMON METHOD</th>
<th>THE COMMON METHOD EXPRESSED WITH MATRICES</th>
<th>GAUSSIAN ELIMINATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3x_{11} + 1x_{21} = 1) (1x_{11} + 2x_{21} = 0)</td>
<td>(\begin{pmatrix}3 &amp; 1 \1 &amp; 2\end{pmatrix}\begin{pmatrix}x_{11} &amp; \quad x_{12} \x_{21} &amp; \quad x_{22}\end{pmatrix} = \begin{pmatrix}1 &amp; \quad 0 \0 &amp; \quad 1\end{pmatrix})</td>
<td>(\begin{pmatrix}3 &amp; 1 &amp; 1 &amp; 0 \1 &amp; 2 &amp; 0 &amp; 1\end{pmatrix})</td>
</tr>
</tbody>
</table>

Multiply the top equation by 2.

\[
\begin{align*}
6x_{11} + 2x_{21} &= 2 \\
1x_{11} + 2x_{21} &= 0
\end{align*}
\]

Subtract the bottom equation from the top.

\[
\begin{align*}
5x_{11} + 0x_{21} &= 2 \\
x_{11} + 2x_{21} &= 0
\end{align*}
\]

Multiply the bottom equation by 5.

\[
\begin{align*}
5x_{11} + 0x_{21} &= 2 \\
x_{11} + 2x_{21} &= 0
\end{align*}
\]

Subtract the bottom equation from the top.

\[
\begin{align*}
5x_{11} + 0x_{21} &= 2 \\
x_{11} + 10x_{21} &= 0
\end{align*}
\]

Divide the top by 5 and the bottom by 10.

\[
\begin{align*}
x_{11} + 0x_{21} &= \frac{2}{5} \\
x_{11} + 10x_{21} &= -\frac{2}{5}
\end{align*}
\]

This is our inverse matrix; we're done!

---

**HUFF**

**YO!**

**YOU!**

**THAT WAS A LOT EASIER THAN I THOUGHT IT WOULD BE...**

**GREAT, BUT...**

---

**SO THE INVERSE WE WANT IS**

\[
\begin{pmatrix}
2 & -1 \\
5 & 5 \\
-1 & 3 \\
5 & 5
\end{pmatrix}
\]

**YAY!**
Let's make sure that the product of the original and calculated matrices really is the identity matrix.

The product of the original and inverse matrix is

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 3 \cdot \frac{2}{5} + 1 \cdot \left(-\frac{1}{5}\right) & 3 \cdot \left(-\frac{1}{5}\right) + 1 \cdot \frac{3}{5} \\ 1 \cdot \frac{2}{5} + 2 \cdot \left(-\frac{1}{5}\right) & 1 \cdot \left(-\frac{1}{5}\right) + 2 \cdot \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product of the inverse and original matrix is

$$\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \cdot 3 + \left(-\frac{1}{5}\right) \cdot 1 & \frac{2}{5} \cdot 1 + \left(-\frac{1}{5}\right) \cdot 2 \\ -\frac{1}{5} \cdot 3 + \frac{3}{5} \cdot 1 & -\frac{1}{5} \cdot 1 + \frac{3}{5} \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It seems like they both become the identity matrix...

That's an important point: the order of the factors doesn't matter. The product is always the identity matrix! Remembering this test is very useful. You should use it as often as you can to check your calculations.

By the way...

The symbol used to denote inverse matrices is the same as any inverse in mathematics, so...

The inverse of

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{-1}$$

To the power of minus one, got it.
Actually...we also could have solved
\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}^{-1}
\]
with...

\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix}
  a_{22} & -a_{12} \\
  -a_{21} & a_{11}
\end{pmatrix}
\]

...this formula right here.

Huh?

Let's apply the formula to our previous example:

\[
\begin{pmatrix}
  3 & 1 \\
  1 & 2
\end{pmatrix}
\]

We got the same answer as last time.

Why even bother with the other method?

Ah, well...

This formula only works on 2×2 matrices.

If you want to find the inverse of a bigger matrix, I'm afraid you're going to have to settle for Gaussian elimination.

Hmm

That's too bad...
Next, I thought I'd show you how to determine whether a matrix has an inverse or not.

So...some matrices lack an inverse?

Yeah. Try to calculate the inverse of this one with the formula I just showed you.

\[
\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}^{-1}
\]

\[
= \frac{1}{3 \cdot 2 - 6 \cdot 1} \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}
\]

Oh, the denominator becomes zero. I guess you're right.

One last thing: The inverse of an invertible matrix is, of course, also invertible.

Invertible

\[
\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Not invertible

Makes sense!
Determinants

Now for the test to see whether a matrix is invertible or not.

We'll be using this function.

It's also written with straight bars, like this:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Does a given matrix have an inverse?

$$\text{det} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \neq 0$$ means that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1}$$ exists.

The inverse of a matrix exists as long as its determinant isn't zero.

Hmm.
There are several different ways to calculate a determinant. Which one's best depends on the size of the matrix.

Let's start with the formula for two-dimensional matrices and work our way up.

To find the determinant of a $2 \times 2$ matrix, just substitute the expression like this.

$$\text{det}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Holding your fingers like this makes for a good trick to remember the formula.

Oh, cool!
Let’s see whether \[
\begin{pmatrix}
3 & 0 \\
0 & 2 \\
\end{pmatrix}
\] has an inverse or not.

\[
\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 3 \cdot 2 - 0 \cdot 0 = 6
\]

It does, since \[
\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.
\]

Incidentally, the area of the parallelogram spanned by the following four points...

- The origin
- The point \((a_{11}, a_{21})\)
- The point \((a_{12}, a_{22})\)
- The point \((a_{11} + a_{12}, a_{21} + a_{22})\)

...coincides with the absolute value of

\[
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]

So \[
\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}
\]
looks like this?

6
To find the determinant of a $3\times3$ matrix, just use the following formula.

$$\det\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

This is sometimes called Sarrus' Rule.

I'm supposed to memorize this? Don't worry, there's a nice trick for this one too.

**Sarrus' Rule**

Write out the matrix, and then write its first two columns again after the third column, giving you a total of five columns. Add the products of the diagonals going from top to bottom (indicated by the solid lines) and subtract the products of the diagonals going from bottom to top (indicated by dotted lines). This will generate the formula for Sarrus' Rule, and it's much easier to remember!
Let's see if \[ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} \] has an inverse.

\[
\begin{align*}
\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} &= 1 \cdot 1 \cdot 3 + 0 \cdot (-1) \cdot (-2) + 0 \cdot 1 \cdot 0 + 0 \cdot 1 \cdot (-2) - 0 \cdot 1 \cdot 3 - 1 \cdot (-1) \cdot 0 \\
&= 3 + 0 + 0 - 0 - 0 \\
&= 3
\end{align*}
\]

So this one has an inverse too!

And the volume of the parallelepiped spanned by the following eight points...

- The origin
- The point \((a_{11}, a_{21}, a_{31})\)
- The point \((a_{12}, a_{22}, a_{32})\)
- The point \((a_{13}, a_{23}, a_{33})\)
- The point \((a_{11} + a_{12}, a_{21} + a_{22}, a_{31} + a_{32})\)
- The point \((a_{11} + a_{13}, a_{21} + a_{23}, a_{31} + a_{33})\)
- The point \((a_{12} + a_{13}, a_{22} + a_{23}, a_{32} + a_{33})\)
- The point \((a_{11} + a_{12} + a_{13}, a_{21} + a_{22} + a_{23}, a_{31} + a_{32} + a_{33})\)

...also coincides with the absolute value of

\[
\begin{align*}
\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} 
\end{align*}
\]

Each pair of opposite faces on the parallelepiped are parallel and have the same area.

* A parallelepiped is a three-dimensional figure formed by six parallelograms.
So next up are $4 \times 4$ matrices, I suppose...

YEP.

I'm afraid not... The grim truth is that the formulas used to calculate determinants of dimensions four and above are very complicated.

Nope!

So how do we calculate them?

To be able to do that...
You'll have to learn the three rules of determinants.

Yep, the terms in the determinant formula are formed according to certain rules.

Take a closer look at the term indexes.

Pay special attention to the left index in each factor.

\[
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

\[
\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}
\]
Oh, they all go from one to the number of dimensions!

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}
\]

Exactly.

And that's rule number one!

Now for the right indexes.

Hmm... They seem a bit more random.

Actually, they're not. Their orders are all permutations of 1, 2, and 3—like in the table to the right. This is rule number two.

I see it now!

<table>
<thead>
<tr>
<th>PERMUTATIONS OF 1-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>PATTERN 1</td>
</tr>
<tr>
<td>PATTERN 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PERMUTATIONS OF 1-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PATTERN 1</td>
</tr>
<tr>
<td>PATTERN 2</td>
</tr>
<tr>
<td>PATTERN 3</td>
</tr>
<tr>
<td>PATTERN 4</td>
</tr>
<tr>
<td>PATTERN 5</td>
</tr>
<tr>
<td>PATTERN 6</td>
</tr>
</tbody>
</table>
Calculating Determinants

The third rule is a bit tricky, so don’t lose concentration.

Okay!

Let’s start by making an agreement.

We will say that the right index is in its natural order if

\[
\begin{align*}
\begin{array}{c}
\text{Pattern 1} \\
\text{Pattern 2}
\end{array}
\end{align*}
\]

\[
\begin{align*}
a_1 \quad a_2 \\
a_2 \quad a_1
\end{align*}
\]

That is, indexes have to be in an increasing order.

The next step is to find all the places where two terms aren’t in the natural order—meaning the places where two indexes have to be switched for them to be in an increasing order.

Then we count how many switches we need for each term.

Squeeze

We gather all this information into a table like this.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Permutations of 1-2</th>
<th>Corresponding Term in the Determinant</th>
<th>Switches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1</td>
<td>1 2</td>
<td>a₁₁a₂₂</td>
<td>2 and 1</td>
</tr>
<tr>
<td>Pattern 2</td>
<td>2 1</td>
<td>a₁₂a₂₁</td>
<td>2 and 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Permutations of 1-3</th>
<th>Corresponding Term in the Determinant</th>
<th>Switches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1</td>
<td>1 2 3</td>
<td>a₁₁a₂₂a₃₃</td>
<td>3 and 2</td>
</tr>
<tr>
<td>Pattern 2</td>
<td>1 3 2</td>
<td>a₁₁a₂₃a₃₂</td>
<td>3 and 2</td>
</tr>
<tr>
<td>Pattern 3</td>
<td>2 1 3</td>
<td>a₁₂a₂₁a₃₃</td>
<td>3 and 2</td>
</tr>
<tr>
<td>Pattern 4</td>
<td>2 3 1</td>
<td>a₁₂a₂₃a₃₁</td>
<td>3 and 2</td>
</tr>
<tr>
<td>Pattern 5</td>
<td>3 1 2</td>
<td>a₁₃a₂₁a₃₂</td>
<td>3 and 2</td>
</tr>
<tr>
<td>Pattern 6</td>
<td>3 2 1</td>
<td>a₁₃a₂₂a₃₁</td>
<td>3 and 2</td>
</tr>
</tbody>
</table>

If the number is even, we write the term as positive. If it is odd, we write it as negative.
### Table: Comparing Determinant Formulas

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Permutations of 1-2</th>
<th>Corresponding Term in the Determinant</th>
<th>Switches</th>
<th>Number of Switches</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1</td>
<td>1 2</td>
<td>a₁₁ a₂₂</td>
<td></td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>Pattern 2</td>
<td>2 1</td>
<td>a₁₂ a₂₁</td>
<td>2 and 1</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

#### Permutations of 1-3

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Permutations of 1-3</th>
<th>Corresponding Term in the Determinant</th>
<th>Switches</th>
<th>Number of Switches</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1</td>
<td>1 2 3</td>
<td>a₁₁ a₂₂ a₃₃</td>
<td></td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>Pattern 2</td>
<td>1 3 2</td>
<td>a₁₁ a₂₃ a₃₂</td>
<td></td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>Pattern 3</td>
<td>2 1 3</td>
<td>a₁₂ a₂₁ a₃₃</td>
<td></td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>Pattern 4</td>
<td>2 3 1</td>
<td>a₁₂ a₂₃ a₃₁</td>
<td></td>
<td>2</td>
<td>+</td>
</tr>
<tr>
<td>Pattern 5</td>
<td>3 1 2</td>
<td>a₁₃ a₂₁ a₃₂</td>
<td></td>
<td>2</td>
<td>+</td>
</tr>
<tr>
<td>Pattern 6</td>
<td>3 2 1</td>
<td>a₁₃ a₂₂ a₃₁</td>
<td></td>
<td>3</td>
<td>-</td>
</tr>
</tbody>
</table>

**Like This.**

**HMM...**

**Try comparing our earlier determinant formulas with the columns "corresponding term in the determinant" and "sign."**

**Ah!**

![Image](image_url)

\[
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \\
\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32}
\]

**Wow, they're the same!**

**Exactly, and that's the third rule.**

104 Chapter 4 More Matrices
These three rules can be used to find the determinant of any matrix.

Cool!

So, say we wanted to calculate the determinant of this 4×4 matrix:

\[
\begin{vmatrix}
\alpha_{11} & \alpha_{22} & \alpha_{33} & \alpha_{44} \\
\alpha_{12} & \alpha_{23} & \alpha_{34} & \alpha_{41} \\
\alpha_{13} & \alpha_{24} & \alpha_{31} & \alpha_{42} \\
\alpha_{14} & \alpha_{21} & \alpha_{32} & \alpha_{43}
\end{vmatrix}
\]

Using this information, we could calculate the determinant if we wanted to.

So, say we wanted to calculate the determinant of this 4×4 matrix:

\[
\begin{vmatrix}
a_{11} & a_{22} & a_{33} & a_{44} \\
a_{21} & a_{23} & a_{34} & a_{41} \\
a_{13} & a_{24} & a_{31} & a_{42} \\
a_{14} & a_{21} & a_{32} & a_{43}
\end{vmatrix}
\]
If this is on the test, I'm done for...

Don't worry, most teachers will give you problems involving only $2 \times 2$ and $3 \times 3$ matrices.

I think that's enough for today. We got through a ton of new material.

Thanks, Reiji. You're the best!

Time really flew by, though...

Maybe I can...

Phew
Jeez. Look at the time...

There shouldn't be anyone left in there at this hour.

Pow!

Do you need a bag for that?

Smack!

Biff!
CALCULATING INVERSE MATRICES USING COFACTORS

There are two practical ways to calculate inverse matrices, as mentioned on page 88.

- Using cofactors
- Using Gaussian elimination

Since the cofactor method involves a lot of cumbersome calculations, we avoided using it in this chapter. However, since most books seem to introduce the method, here’s a quick explanation.

To use this method, you first have to understand these two concepts:

- The \((i, j)\)-minor, written as \(M_{ij}\)
- The \((i, j)\)-cofactor, written as \(C_{ij}\)

So first we’ll have a look at these.

\(M_{ij}\)

The \((i, j)\)-minor is the determinant produced when we remove row \(i\) and column \(j\) from the \(n\times n\) matrix \(A\):

\[
M_{ij} = \det \begin{pmatrix}
a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\
a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn}
\end{pmatrix}
\]

All the minors of the 3×3 matrix \[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
\] are listed on the next page.
Calculating inverse Matrices Using Cofactors

If we multiply the \((i, j)\)-minor by \((-1)^{i+j}\), we get the \((i, j)\)-cofactor. The standard way to write this is \(C_{ij}\). The table below contains all cofactors of the 3×3 matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{bmatrix}
\]

\[
C_{ij}
\]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_{11}) (1, 1)</td>
<td>(M_{12}) (1, 2)</td>
<td>(M_{13}) (1, 3)</td>
</tr>
<tr>
<td>(\det \begin{bmatrix} 1 &amp; -1 \ 0 &amp; 3 \end{bmatrix} = 3)</td>
<td>(\det \begin{bmatrix} 1 &amp; -1 \ -2 &amp; 3 \end{bmatrix} = 1)</td>
<td>(\det \begin{bmatrix} 1 &amp; 1 \ -2 &amp; 0 \end{bmatrix} = 2)</td>
</tr>
<tr>
<td>(M_{21}) (2, 1)</td>
<td>(M_{22}) (2, 2)</td>
<td>(M_{23}) (2, 3)</td>
</tr>
<tr>
<td>(\det \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 3 \end{bmatrix} = 0)</td>
<td>(\det \begin{bmatrix} 1 &amp; 0 \ -2 &amp; 3 \end{bmatrix} = 3)</td>
<td>(\det \begin{bmatrix} 1 &amp; 0 \ -2 &amp; 0 \end{bmatrix} = 0)</td>
</tr>
<tr>
<td>(M_{31}) (3, 1)</td>
<td>(M_{32}) (3, 2)</td>
<td>(M_{33}) (3, 3)</td>
</tr>
<tr>
<td>(\det \begin{bmatrix} 0 &amp; 0 \ 1 &amp; -1 \end{bmatrix} = 0)</td>
<td>(\det \begin{bmatrix} 1 &amp; 0 \ 1 &amp; -1 \end{bmatrix} = -1)</td>
<td>(\det \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 1 \end{bmatrix} = 1)</td>
</tr>
</tbody>
</table>

\[
C_{11} (1, 1) = (-1)^{1+1} \cdot \det \begin{bmatrix} 1 & -1 \\
0 & 3 \end{bmatrix} = 1 \cdot 3 = 3
\]

\[
C_{12} (1, 2) = (-1)^{1+2} \cdot \det \begin{bmatrix} 1 & -1 \\
-2 & 3 \end{bmatrix} = (-1) \cdot 1 = -1
\]

\[
C_{13} (1, 3) = (-1)^{1+3} \cdot \det \begin{bmatrix} 1 & 1 \\
-2 & 0 \end{bmatrix} = 1 \cdot 2 = 2
\]

\[
C_{21} (2, 1) = (-1)^{2+1} \cdot \det \begin{bmatrix} 0 & 0 \\
0 & 3 \end{bmatrix} = (-1) \cdot 0 = 0
\]

\[
C_{22} (2, 2) = (-1)^{2+2} \cdot \det \begin{bmatrix} 1 & 0 \\
-2 & 3 \end{bmatrix} = 1 \cdot 3 = 3
\]

\[
C_{23} (2, 3) = (-1)^{2+3} \cdot \det \begin{bmatrix} 1 & 0 \\
-2 & 0 \end{bmatrix} = (-1) \cdot 0 = 0
\]

\[
C_{31} (3, 1) = (-1)^{3+1} \cdot \det \begin{bmatrix} 0 & 0 \\
1 & -1 \end{bmatrix} = 1 \cdot 0 = 0
\]

\[
C_{32} (3, 2) = (-1)^{3+2} \cdot \det \begin{bmatrix} 1 & 0 \\
1 & -1 \end{bmatrix} = (-1) \cdot (-1) = 1
\]

\[
C_{33} (3, 3) = (-1)^{3+3} \cdot \det \begin{bmatrix} 1 & 0 \\
1 & 1 \end{bmatrix} = 1 \cdot 1 = 1
\]
The \( n \times n \) matrix

\[
\begin{pmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{pmatrix}
\]

which at place \((i, j)\) has the \((j, i)\)-cofactor\(^1\) of the original matrix is called a \textit{cofactor matrix}.

The sum of any row or column of the \( n \times n \) matrix

\[
\begin{pmatrix}
a_{11}C_{11} & a_{21}C_{21} & \cdots & a_{n1}C_{n1} \\
a_{12}C_{12} & a_{22}C_{22} & \cdots & a_{n2}C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n}C_{1n} & a_{2n}C_{2n} & \cdots & a_{nn}C_{nn}
\end{pmatrix}
\]

is equal to the determinant of the original \( n \times n \) matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

\textbf{CALCULATING INVERSE MATRICES}

The inverse of a matrix can be calculated using the following formula:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}^{-1} = \frac{1}{\text{det}} \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

\[
\begin{pmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{pmatrix}
\]

\(^1\) This is not a typo. \((j, i)\)-cofactor is the correct index order. This is the transpose of the matrix with the cofactors in the expected positions.
For example, the inverse of the 3×3 matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
\]
is equal to
\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}} \begin{pmatrix}
3 & 0 & 0 \\
-1 & 3 & 1 \\
2 & 0 & 1
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
3 & 0 & 0 \\
-1 & 3 & 1 \\
2 & 0 & 1
\end{pmatrix}
\]

**USING DETERMINANTS**

The method presented in this chapter only defines the determinant and does nothing to explain what it is used for. A typical application (in image processing, for example) can easily reach determinant sizes in the \(n = 100\) range, which with the approach used here would produce insurmountable numbers of calculations.

Because of this, determinants are usually calculated by first simplifying them with Gaussian elimination–like methods and then using these three properties, which can be derived using the definition presented in the book:

- If a row (or column) in a determinant is replaced by the sum of the row (column) and a multiple of another row (column), the value stays unchanged.
- If two rows (or columns) switch places, the values of the determinant are multiplied by \(-1\).
- The value of an upper or lower triangular determinant is equal to the product of its main diagonal.

The difference between the two methods is so extreme that determinants that would be practically impossible to calculate (even using modern computers) with the first method can be done in a jiffy with the second one.

**SOLVING LINEAR SYSTEMS WITH CRAMER’S RULE**

Gaussian elimination, as presented on page 89, is only one of many methods you can use to solve linear systems. Even though Gaussian elimination is one of the best ways to solve them by hand, it is always good to know about alternatives, which is why we’ll cover the Cramer’s rule method next.
Use Cramer's rule to solve the following linear system:

\[
\begin{align*}
3x_1 + 1x_2 &= 1 \\
1x_1 + 2x_2 &= 0
\end{align*}
\]

## Solution
### Step 1
Rewrite the system

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
& \quad \cdots \cdots \\
\end{align*}
\]

like so:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{pmatrix}
\]

If we rewrite

\[
\begin{align*}
3x_1 + 1x_2 &= 1 \\
1x_1 + 2x_2 &= 0
\end{align*}
\]

we get

\[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

### Step 2
Make sure that

\[
\det
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \neq 0
\]

We have

\[
\det
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix} = 3 \cdot 2 - 1 \cdot 1 \neq 0
\]

### Step 3
Replace each column with the solution vector to get the corresponding solution:

\[
x_i = \frac{\det
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn}
\end{pmatrix}}{\det
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}}
\]

\[
\cdot x_1 = \frac{\det
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}}{\det
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}} = \frac{1 \cdot 2 - 1 \cdot 0}{5} = \frac{2}{5}
\]

\[
\cdot x_2 = \frac{\det
\begin{pmatrix}
3 & 1 \\
1 & 0
\end{pmatrix}}{\det
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}} = \frac{3 \cdot 0 - 1 \cdot 1}{5} = \frac{-1}{5}
\]
5

INTRODUCTION TO VECTORS
One more minute!

OSSU!

You can do better than that!

Put your back into it!

Great!

Let's leave it at that for today.

Wheeze

M-more!
I CAN DO MORE!

YOU CAN BARELY EVEN STAND!

THUMP

UWA!!

JUST ONE MORE ROUND!

I HAVEN'T GOTTEN STRONGER AT ALL YET!

HEHEH, FINE BY ME!

OSSU!
What Are Vectors?

They appear quite frequently in Linear Algebra, so pay close attention. Of course!

Reiji, are you okay?

Do you want to meet tomorrow instead?

No, I'm okay. Just give me five minutes to digest this delicious lunch, and I'll be great!

Sorry about that! Ready?

Vectors are actually just a special interpretation of matrices.

Really?
What kind of interpretation? I think it'll be easier to explain using an example...

I think it'll be easier to explain using an example...

Minigolf should make an excellent metaphor.

Minigolf?

Don't be intimidated—my putting skills are pretty rusty.

Don't be intimidated—my putting skills are pretty rusty.

Yeah—mine too!

Our course will look like this.

We'll use coordinates to describe where the ball and hole are to make explaining easier.

That means that the starting point is at (0, 0) and that the hole is at (7, 4), right?
I went first.
I played conservatively and put the ball in with three strokes.

**Stroke Information**

<table>
<thead>
<tr>
<th></th>
<th>First stroke</th>
<th>Second stroke</th>
<th>Third stroke</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ball position</strong></td>
<td>Point (3, 1)</td>
<td>Point (4, 3)</td>
<td>Point (7, 4)</td>
</tr>
<tr>
<td><strong>Ball position</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>relative to its</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>last position</td>
<td>3 to the right and 1 up relative to (0, 0)</td>
<td>1 to the right and 2 up relative to (3, 1)</td>
<td>3 to the right and 1 up relative to (4, 3)</td>
</tr>
<tr>
<td><strong>Ball movement</strong></td>
<td>(3, 1)</td>
<td>(3, 1) + (1, 2) = (4, 3)</td>
<td>(3, 1) + (1, 2) + (3, 1) = (7, 4)</td>
</tr>
<tr>
<td>expressed in the</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>form (to the right, up)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
You gave the ball a good wallop and put the ball in with two strokes.

**REPLAY**

**FIRST STROKE**

- **Point** (10, 10)

**SECOND STROKE**

- **Point** (7, 4)

**STROKE INFORMATION**

<table>
<thead>
<tr>
<th></th>
<th>First stroke</th>
<th>Second stroke</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ball position</strong></td>
<td>Point (10, 10)</td>
<td>Point (7, 4)</td>
</tr>
<tr>
<td><strong>Ball position</strong></td>
<td>10 to the right and 10 up relative to (0, 0)</td>
<td>−3 to the right and −6 up relative to (10, 10)</td>
</tr>
<tr>
<td><strong>Ball movement</strong></td>
<td>(10, 10)</td>
<td>(10, 10) + (−3, −6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>= (7, 4)</td>
</tr>
</tbody>
</table>
AND YOUR BROTHER GOT A HOLE-IN-ONE...OF COURSE.

**Player 3**

**TETSUO ICHINOSE**

**OBSEVE MY EXERCISE IN SKILL!**

**YAY BIG BRO!**

**REPLAY**

**FIRST STROKE**

- **Graph**
  - **Axes**: $x_1$, $x_2$
  - **Points**: $(4, 0)$, $(7, 4)$

**STROKE INFORMATION**

<table>
<thead>
<tr>
<th>First stroke</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ball position</td>
</tr>
<tr>
<td>Point $(7, 4)$</td>
</tr>
<tr>
<td>Ball position relative to its last position</td>
</tr>
<tr>
<td>7 to the right and 4 up relative to $(0, 0)$</td>
</tr>
<tr>
<td>Ball movement expressed in the form (to the right, up)</td>
</tr>
<tr>
<td>$(7, 4)$</td>
</tr>
</tbody>
</table>
WELL, AT LEAST WE ALL MADE IT IN!

TRY TO REMEMBER THE MINIGOLF EXAMPLE WHILE WE TALK ABOUT THE NEXT FEW SUBJECTS.

VECTORS CAN BE INTERPRETED IN FOUR DIFFERENT WAYS. LET ME GIVE YOU A QUICK WALK-THROUGH OF ALL OF THEM.

I’LL USE THE $1 \times 2$ MATRIX $(7, 4)$
AND THE $2 \times 1$ MATRIX $egin{pmatrix} 7 \\ 4 \end{pmatrix}$
TO MAKE THINGS SIMPLER.

$1 \times n$ matrices $(a_1, a_2, \ldots, a_n)$ and $n \times 1$ matrices $egin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$
The point (7, 4)

Interpretation 1

(7, 4) and \[
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]
are sometimes interpreted as a point in space.

Interpretation 2

In other cases, (7, 4) and \[
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]
are interpreted as the “arrow” from the origin to the point (7, 4).

Interpretation 3

And in yet other cases, (7, 4) and \[
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]
can mean the sum of several arrows equal to (7, 4).
FINALLY, (7, 4) AND \begin{pmatrix} 7 \\ 4 \end{pmatrix} CAN ALSO BE INTERPRETED AS ANY OF THE ARROWS ON MY LEFT, OR ALL OF THEM AT THE SAME TIME!

HANG ON A SECOND. I WAS WITH YOU UNTIL THAT LAST ONE...

HOW COULD ALL OF THEM BE REPRESENTATIONS OF (7, 4) AND \begin{pmatrix} 7 \\ 4 \end{pmatrix} WHEN THEY START IN COMPLETELY DIFFERENT PLACES?

While they do start in different places, they're all the same in that they go "seven to the right and four up," right?

Yeah, I guess that's true!
VECTORS ARE KIND OF A MYSTERIOUS CONCEPT, DON'T YOU THINK?

WELL, THEY MAY SEEM THAT WAY AT FIRST.

BUT...

ONCE YOU'VE GOT THE BASICS DOWN, YOU'LL BE ABLE TO APPLY THEM TO ALL SORTS OF INTERESTING PROBLEMS.

FOR EXAMPLE, THEY'RE FREQUENTLY USED IN PHYSICS TO DESCRIBE DIFFERENT TYPES OF FORCES.

COOL!

PUTTING FORCE

GRAVITATIONAL FORCE
Even though vectors have a few special interpretations, they’re all just 1×n and n×1 matrices...

And they’re calculated in the exact same way.

**Addition**
- \((10, 10) + (-3, -6) = (10 + (-3), 10 + (-6)) = (7, 4)\)
- \[
\begin{bmatrix}
10 \\
10
\end{bmatrix} + 
\begin{bmatrix}
-3 \\
-6
\end{bmatrix} = 
\begin{bmatrix}
10 + (-3) \\
10 + (-6)
\end{bmatrix} = 
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]

**Subtraction**
- \((10, 10) - (3, 6) = (10 - 3, 10 - 6) = (7, 4)\)
- \[
\begin{bmatrix}
10 \\
10
\end{bmatrix} - 
\begin{bmatrix}
3 \\
6
\end{bmatrix} = 
\begin{bmatrix}
10 - 3 \\
10 - 6
\end{bmatrix} = 
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]

**Scalar Multiplication**
- \(2(3, 1) = (2 \cdot 3, 2 \cdot 1) = (6, 2)\)
- \[
2 
\begin{bmatrix}
3 \\
1
\end{bmatrix} = 
\begin{bmatrix}
2 \cdot 3 \\
2 \cdot 1
\end{bmatrix} = 
\begin{bmatrix}
6 \\
2
\end{bmatrix}
\]

**Matrix Multiplication**
- \[
\begin{bmatrix}
3 \\
1
\end{bmatrix}
(1, 2) = 
\begin{bmatrix}
3 & 3 & 2 \\
1 & 1 & 2
\end{bmatrix} = 
\begin{bmatrix}
3 & 6 \\
1 & 2
\end{bmatrix}
\]
- \[
(3, 1)
\begin{bmatrix}
1 \\
2
\end{bmatrix} = (3 \cdot 1 + 1 \cdot 2) = 5
\]
- \[
\begin{bmatrix}
8 & -3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
1
\end{bmatrix} = (8 \cdot 3 + (-3) \cdot 1, 2 \cdot 3 + 1 \cdot 1) = (21, 7) = 
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]

Simple!
Horizontal vectors like this one are called row vectors.

AND VERTICAL VECTORS ARE CALLED COLUMN VECTORS.

Makes sense.

We also call the set of all \( n \times 1 \) matrices \( \mathbb{R}^n \).

Sure, why not...

When writing vectors by hand, we usually draw the leftmost line double, like this.

\[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \]

\( \mathbb{R}^2 \) ALL 2x1 VECTORS

\( \mathbb{R}^3 \) ALL 3x1 VECTORS

\( \mathbb{R}^n \) ALL nx1 VECTORS

\( \mathbb{R}^n \) appears a lot in linear algebra, so make sure you remember it.

No problem.
A POINT
Let's say that $c$ is an arbitrary real number. Can you see how the point $(0, c)$ and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are related?

YUP.

AN AXIS
Do you understand this notation?

$\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$ $c$ is an arbitrary real number

"\(\|$ can be read as "where."

Yeah.

A STRAIGHT LINE
Even the straight line $x_1 = 3$ can be expressed as:

$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $c$ is an arbitrary real number

No problem.
AND THE $x_1x_2$ PLANE $\mathbb{R}^2$ CAN BE EXPRESSED AS:
\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} c_1, c_2 \text{ are arbitrary real numbers}
\]

IT CAN ALSO BE WRITTEN ANOTHER WAY:
\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_1, c_2 \text{ are arbitrary real numbers}
\]

A PLANE

SOME ENOUGH!

ANOTHER PLANE

SURE

HMM...
SO IT'S LIKE A WEIRD, SLANTED DRAWING BOARD.
The three-dimensional space $\mathbb{R}^3$ is the natural next step. It is spanned by $x_1, x_2,$ and $x_3$ like this:

\[
\begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix} c_1 +
\begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix} c_2 +
\begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix} c_3 =
\begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{pmatrix}
\]

$c_1, c_2, c_3$ are arbitrary real numbers.

Now try to imagine the $n$-dimensional space $\mathbb{R}^n$, spanned by $x_1, x_2, \ldots, x_n$:

\[
\begin{pmatrix}
  1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix} c_1 +
\begin{pmatrix}
  0 \\
  1 \\
  \vdots \\
  0
\end{pmatrix} c_2 +
\ldots +
\begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
  1
\end{pmatrix} c_n =
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{pmatrix}
\]

$c_1, c_2, \ldots, c_n$ are arbitrary real numbers.
WHOO, I'M BEAT!

LET'S TAKE A BREAK, THEN.

GOOD IDEA!

BY THE WAY, REIJI...

WHY DID YOU DECIDE TO JOIN THE KARATE CLUB, ANYWAY?

UM, WELL...

NO SPECIAL REASON REALLY...

EH?

WE'D BETTER GET BACK TO WORK!

Eh?
we're linearly... independent!

They're bases.

Definitely bases.

Yea, bases!
WELL THEN, LET’S HAVE A LOOK AT LINEAR INDEPENDENCE AND BASES.

THE TWO ARE PRETTY SIMILAR...

BUT WE DON’T WANT TO MIX THEM UP, OKAY?

I’LL TRY NOT TO.

LINEAR INDEPENDENCE

WHY DON’T WE START OFF TODAY WITH A LITTLE QUIZ?

SURE.

QUESTION ONE.

PROBLEM 1

Find the constants $c_1$ and $c_2$ satisfying this equation:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Correct!
\[
\begin{align*}
\begin{cases}
  c_1 = 0 \\
  c_2 = 0
\end{cases}
\end{align*}
\]

Well then, question two.

Problem 2?

Find the constants \( c_1 \) and \( c_2 \) satisfying this equation:

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

Isn't that also?

Problem 3?

Find the constants \( c_1, c_2, c_3, \) and \( c_4 \) satisfying this equation:

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

That's easy.

\[
\begin{align*}
  c_2 = 0 \\
  c_1 = 0
\end{align*}
\]

Is that also?  Problem 2  Problem 3

Last one.
\[ c_1 = 0 \]
\[ c_2 = 0 \]
\[ c_3 = 0 \]
\[ c_4 = 0 \]

Again?

Not quite.

\[ \begin{align*}
  c_1 &= 0 \\
  c_2 &= 0 \\
  c_3 &= 0 \\
  c_4 &= 0
\end{align*} \]

Isn't wrong, but...

\[ \begin{align*}
  c_1 &= 1 \\
  c_2 &= 2 \\
  c_3 &= 0 \\
  c_4 &= -1
\end{align*} \]

\[ \begin{align*}
  c_1 &= 1 \\
  c_2 &= -3 \\
  c_3 &= -1 \\
  c_4 &= 2
\end{align*} \]

Are other possible answers.

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

Try to keep this in mind while we move on to the main problem.
Linear independence is sometimes called one-dimensional independence...

Ah...

As long as there is only one unique solution to problems such as the first or second examples:

\[
\begin{align*}
0 &= c_1 a_{11} + c_2 a_{21} + \cdots + c_n a_{n1} \\
0 &= c_1 a_{12} + c_2 a_{22} + \cdots + c_n a_{n2} \\
&\vdots \\
0 &= c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn}
\end{align*}
\]

Linear independence

We say that its vectors are linearly independent.

Linear dependence

As for problems like the third example, where there are solutions other than the origin:

\[
\begin{align*}
c_1 &= 0 \\
c_2 &= 0 \\
&\vdots \\
c_n &= 0
\end{align*}
\]

We can never return to the origin.

If we add all work together, we can get back to the origin!

Linear dependence is similarly sometimes called one-dimensional dependence.

Weee!
Here are some examples. Let's look at linear independence first.

**Example 1**

The vectors \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

give us the equation
\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = c_1 \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + c_2 \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} + c_3 \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

which has the unique solution
\[
\begin{align*}
c_1 &= 0 \\
c_2 &= 0 \\
c_3 &= 0
\end{align*}
\]

The vectors are therefore linearly independent.
**Example 2**

The vectors \[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\quad \text{and} \quad \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

give us the equation

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = c_1 \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + c_2 \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

which has the unique solution \[
\begin{cases}
c_1 = 0 \\
c_2 = 0
\end{cases}
\]

These vectors are therefore also linearly independent.
AND NOW WE’LL LOOK AT LINEAR DEPENDENCE.

EXAMPLE 1

The vectors \[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]
give us the equation
\[ c_1 + c_2 + c_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
which has several solutions, for example, \[ \begin{cases} c_1 = 0 \\ c_2 = 0 \text{ and } c_2 = 1 \\ c_3 = 0 \text{ or } c_3 = -1 \end{cases} \]
This means that the vectors are linearly dependent.
Example 2

Suppose we have the vectors \[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \] as well as the equation \[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \]

The vectors are linearly dependent because there are several solutions to the system—

\[
\begin{cases}
  c_1 = 0 \\
  c_2 = 0 \\
  c_3 = 0 \\
  c_4 = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
  c_1 = a_1 \\
  c_2 = a_2 \\
  c_3 = a_3 \\
  c_4 = -1
\end{cases}
\]

The vectors \[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \] are similarly linearly dependent because there are several solutions to the equation

\[
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + c_m \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} + c_{m+1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}
\]

Among them is

\[
\begin{cases}
  c_1 = 0 \\
  c_2 = 0 \\
  \vdots \\
  c_m = 0 \\
  c_{m+1} = 0
\end{cases}
\quad \text{but also} \quad
\begin{cases}
  c_1 = a_1 \\
  c_2 = a_2 \\
  \vdots \\
  c_m = a_m \\
  c_{m+1} = -1
\end{cases}
\]
Here are three more problems.

MHMM.

FIRST ONE.

IT KINDA LOOKS LIKE THE OTHER PROBLEMS...

Problem 4

Find the constants $c_1$ and $c_2$ satisfying this equation:

\[
\begin{pmatrix} 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
\begin{cases} 
   c_1 = 7 \\
   c_2 = 4 
\end{cases}
\]

SHOULD WORK.

CORRECT!
Problem 5

Find the constants $c_1$ and $c_2$ satisfying this equation:

\[
\begin{pmatrix}
7 \\
4
\end{pmatrix} = c_1 \begin{pmatrix}
3 \\
1
\end{pmatrix} + c_2 \begin{pmatrix}
1 \\
2
\end{pmatrix}
\]

\[
\begin{pmatrix}
7 \\
4
\end{pmatrix} = 2 \begin{pmatrix}
3 \\
1
\end{pmatrix} + 1 \begin{pmatrix}
1 \\
2
\end{pmatrix}
\]

\[
\begin{cases}
c_1 = 2 \\
c_2 = 1
\end{cases}
\]

Correct again!

You're really good at this!

Well those were pretty easy...
### Problem 6

Find the constants $c_1$, $c_2$, $c_3$, and $c_4$ satisfying this equation:

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

There's $c_1 = 7$, $c_2 = 4$, $c_3 = 0$, $c_4 = 0$ and of course...

AND

$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 2 \\ c_4 = 1 \end{cases}$

AND OF COURSE

$\begin{cases} c_1 = 5 \\ c_2 = -5 \\ c_3 = -1 \\ c_4 = 5 \end{cases}$

There's...

That's enough.
Linear dependence and independence are closely related to the concept of a basis. Have a look at the following equation:

\[
\begin{pmatrix}
y_1 \\ y_2 \\ \vdots \\ y_m
\end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}
\]

Where the left side of the equation is an arbitrary vector in \( \mathbb{R}^m \) and the right side is a number of \( n \) vectors of the same dimension, as well as their coefficients.

If there's only one solution \( c_1 = c_2 = \ldots = c_n = 0 \) to the equation, then our vectors make up a basis for \( \mathbb{R}^n \).

Does that mean that the solution
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
for problem 4

and the solution
\[
\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}
\]
for problem 5 are bases, but the solution
\[
\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}
\]
for problem 6 isn't?

Here are some examples of what is and what is not a basis.
All these vector sets make up bases for their graphs.

In other words, a basis is a minimal set of vectors needed to express an arbitrary vector in $\mathbb{R}^n$. Another important feature of bases is that they're all linearly independent.
The vectors of the following set do not form a basis.

The set \( \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} \)

To understand why they don’t form a basis, have a look at the following equation:

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

where \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) is an arbitrary vector in \( \mathbb{R}^2 \).

\( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) can be formed in many different ways (using different choices for \( c_1, c_2, c_3, \) and \( c_4 \)).

Because of this, the set does not form “a minimal set of vectors needed to express an arbitrary vector in \( \mathbb{R}^m \).”
Neither of the two vector sets below is able to describe the vector \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \), and if they can’t describe that vector, then there’s no way that they could describe “an arbitrary vector in \( \mathbb{R}^3 \).” Because of this, they’re not bases.

The set \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \), The set \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \),

Just because a set of vectors is linearly independent doesn’t mean that it forms a basis.

For instance, the set \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) forms a basis,

while the set \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) does not, even though they’re both linearly independent.
Since bases and linear independence are confusingly similar, I thought I'd talk a bit about the differences between the two.

**Linear Independence**

We say that a set of vectors \([ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}, \ldots, \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \] is linearly independent if there's only one solution to the equation

\[
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}
\]

where the left side is the zero vector of \(R^m\).

**Bases**

A set of vectors \([ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}, \ldots, \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \] forms a basis if there's only one solution to the equation

\[
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}
\]

where the left side is an arbitrary vector \(\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}\) in \(R^m\). And once again, a basis is a minimal set of vectors needed to express an arbitrary vector in \(R^m\).
While linear independence is about finding a clear-cut path back to the origin, bases are about finding clear-cut paths to any vector in a given space $R^m$?

We're linearly independent! They're bases. Yes, they are. Exactly!

Not a lot of people are able to grasp the difference between the two that fast! I must say I'm impressed!

That's all for today—

Ah, wait a sec!

No big deal!
You know, I've been thinking.

It's kind of obvious that a basis is made up of two vectors when in $R^2$ and three vectors when in $R^3$.

But why is it that the basis of an $m$-dimensional space consists of $n$ vectors and not $m$?

To answer that, we'll have to take a look at another, more precise definition of bases.

There's also a more precise definition of vectors, which can be hard to understand.

Oh, wow... I didn't think you'd notice...

I'm up for it!
You sure?
I— I think so.
It’s actually not that hard—just a little abstract.
Let’s have a look, since you asked and all.

But first we have to tackle another new concept: subspaces.

Subspace
So let’s talk about them.

Subspaces
It’s kinda like this.

So it’s another word for subset?
No, not quite. Let me try again.
WHAT IS A SUBSPACE?

Let $c$ be an arbitrary real number and $W$ be a nonempty subset of $\mathbb{R}^m$ satisfying these two conditions:

1. An element in $W$ multiplied by $c$ is still an element in $W$. (Closed under scalar multiplication.)

   If \[
   \begin{pmatrix}
   a_{1i} \\
   a_{2i} \\
   \vdots \\
   a_{mi}
   \end{pmatrix}
   \in W,
   \text{ then } c \begin{pmatrix}
   a_{1i} \\
   a_{2i} \\
   \vdots \\
   a_{mi}
   \end{pmatrix}
   \in W
   \]

2. The sum of two arbitrary elements in $W$ is still an element in $W$. (Closed under addition.)

   If \[
   \begin{pmatrix}
   a_{1i} \\
   a_{2i} \\
   \vdots \\
   a_{mi}
   \end{pmatrix}
   \in W \quad \text{and} \quad \begin{pmatrix}
   a_{1j} \\
   a_{2j} \\
   \vdots \\
   a_{mj}
   \end{pmatrix}
   \in W,
   \text{ then } \begin{pmatrix}
   a_{1i} \\
   a_{2i} \\
   \vdots \\
   a_{mi}
   \end{pmatrix} + \begin{pmatrix}
   a_{1j} \\
   a_{2j} \\
   \vdots \\
   a_{mj}
   \end{pmatrix}
   \in W
   \]

If both of these conditions hold, then $W$ is a subspace of $\mathbb{R}^m$. 

THIS PICTURE ILLUSTRATES THE RELATIONSHIP.
IT'S PRETTY ABSTRACT, SO YOU MIGHT HAVE TO READ IT A FEW TIMES BEFORE IT STARTS TO SINK IN.

ANOTHER, MORE CONCRETE WAY TO LOOK AT ONE-DIMENSIONAL SUBSPACES IS AS LINES THROUGH THE ORIGIN. TWO-DIMENSIONAL SUBSPACES ARE SIMILARLY PLANES THROUGH THE ORIGIN. OTHER SUBSPACES CAN ALSO BE VISUALIZED, BUT NOT AS EASILY.

I MADE SOME EXAMPLES OF SPACES THAT ARE SUBSPACES—AND OF SOME THAT ARE NOT. HAVE A LOOK!

THIS IS A SUBSPACE

Let's have a look at the subspace in \( \mathbb{R}^3 \) defined by the set

\[
\begin{pmatrix}
  \alpha \\
  0 \\
  0
\end{pmatrix}
\]

\( \alpha \) is an arbitrary real number, in other words, the x-axis.

\[\begin{vmatrix}
  a_1 \\
  0 \\
  0
\end{vmatrix}
\]

\( \alpha \) is an arbitrary real number

\[\begin{vmatrix}
  a_1 \\
  a_2 \\
  0
\end{vmatrix}
\]

\( \alpha + \beta \) is an arbitrary real number

If it really is a subspace, it should satisfy the two conditions we talked about before.

1. \[c \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ca_1 \\ 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \]

2. \[\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1 + a_2 \\ 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \]

It seems like they do! This means it actually is a subspace.
THIS IS NOT A SUBSPACE

Let's use our conditions to see why:

1. \[ \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} ca_1 \\ (ca_1)^2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} a \\ a^2 \\ 0 \end{bmatrix} \]

2. \[ \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \alpha + a_2 \alpha \\ (a_1 + a_2)^2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} a + \alpha \\ a^2 + \alpha \\ 0 \end{bmatrix} \]

The set doesn't seem to satisfy either of the two conditions, and therefore it is not a subspace!
The following subspaces are called linear spans and are a bit special.

**WHAT IS A LINEAR SPAN?**

We say that a set of $m$-dimensional vectors
\[
\begin{pmatrix}
    a_{11} \\
    a_{21} \\
    \vdots \\
    a_{m1}
\end{pmatrix}, \quad
\begin{pmatrix}
    a_{12} \\
    a_{22} \\
    \vdots \\
    a_{m2}
\end{pmatrix}, \quad \ldots, \quad
\begin{pmatrix}
    a_{1n} \\
    a_{2n} \\
    \vdots \\
    a_{mn}
\end{pmatrix}
\]
span the following subspace in $\mathbb{R}^m$:

\[
\begin{pmatrix}
    c_1 a_{11} \\
    c_2 a_{21} \\
    \vdots \\
    c_m a_{m1}
\end{pmatrix} + \begin{pmatrix}
    c_1 a_{12} \\
    c_2 a_{22} \\
    \vdots \\
    c_m a_{m2}
\end{pmatrix} + \ldots + \begin{pmatrix}
    c_1 a_{1n} \\
    c_2 a_{2n} \\
    \vdots \\
    c_m a_{mn}
\end{pmatrix}
\]

This set forms a subspace and is called the linear span of the $n$ original vectors.

**EXAMPLE 1**

The $x_1x_2$-plane is a subspace of $\mathbb{R}^2$ and can, for example, be spanned by using the two vectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ like so:

\[
\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1 \text{ and } c_2 \text{ are arbitrary numbers}
\]
EXAMPLE 2

The \( x_1, x_2 \)-plane could also be a subspace of \( R^3 \), and we could span it using the vectors \(
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \(
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), creating this set:

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix}
\]

\[
\{ c_1 \text{ and } c_2 \text{ are arbitrary numbers} \}
\]

\( R^n \) is also a subspace of itself, as you might have guessed from Example 1.

All subspaces contain the zero factor, which you could probably tell from looking at the example on page 152. Remember, they must pass through the origin!
Suppose that $W$ is a subspace of $\mathbb{R}^m$ and that it is spanned by the linearly independent vectors $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$, $\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$, and $\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

This could also be written as follows:

$$W = \left\{ c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \left| c_1, c_2, \ldots, c_n \text{ are arbitrary numbers} \right. \right\}$$

When this equality holds, we say that the set forms a basis to the subspace $W$.

The dimension of the subspace $W$ is equal to the number of vectors in any basis for $W$.

"The dimension of the subspace $W$" is usually written as $\dim W$. 
This example might clear things up a little.

**Example**

Let's call the $x_1x_2$-plane $W$, for simplicity's sake. So suppose that $W$ is a subspace of $\mathbb{R}^3$ and is spanned by the linearly independent vectors

$$
\begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}.
$$

We have this:

$$
W = \left\{ c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ for } c_1, c_2 \text{ arbitrary numbers} \right\}
$$

The fact that this equality holds means that the vector set

$$
\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}
$$

is a basis of the subspace $W$. Since the base contains two vectors, $\dim W = 2$. 
What do you think? Were you able to follow?

Sure! it's like this, right?

It's like this, right?

You got it!

That's enough for today.

Thanks for all the help.

Let's talk about linear transformations next time.

It's also an important subject, so come prepared!

Of course!

Oh, looks like we're going the same way...

Three dimensions

If the subspace's basis has two vectors, then the dimension of the subspace has to be two.
Umm, I didn't really answer your question before...

I joined the karate club because...

I'm tired of being such a wimp.

I want to get stronger. That way I can...

Well...
I GUESS YOU'LL BE NEEDING...

...A LOT MORE HOMEMADE LUNCHES, THEN!

TO SURVIVE MY BROTHER'S REIGN OF TERROR, THAT IS. DON'T WORRY, I'LL MAKE YOU MY SUPER SPECIAL STAMINA-LUNCH EXTRAVAGANZA EVERY WEEK FROM NOW ON!

THANK YOU, MISA.

HEHE, DON'T WORRY 'BOUT IT!
Coordinates in linear algebra are a bit different from the coordinates explained in high school. I'll try explaining the difference between the two using the image below.

When working with coordinates and coordinate systems at the high school level, it's much easier to use only the trivial basis:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\vdots & \vdots & \ddots \\
0 & 0 & 1
\end{pmatrix}
\]

In this kind of system, the relationship between the origin and the point in the top right is interpreted as follows:

\[
\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
It is important to understand that the trivial basis is only one of many bases when we move into the realm of linear algebra—and that using other bases produces other relationships between the origin and a given point. The image below illustrates the point \((2, 1)\) in a system using the nontrivial basis consisting of the two vectors \(u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\) and \(u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\).

\[
\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

This alternative way of thinking about coordinates is very useful in factor analysis, for example.
7
LINEAR TRANSFORMATIONS
A practice match with Nanhou University?

That's right! We go head-to-head in two weeks.

A match, huh? I guess I'll be sitting it out.

You're in.

Yurino!

What?!

For real?

Um, sensei? Isn't it a bit early for...

Are you telling me what to do?

Oh no! Of course not!
I'll be looking forward to seeing what you've learned so far!

OSSU! Of course!

Understood?

Great. Dismissed!

What am I going to do... Stop thinking like that! He's giving me this opportunity.

A match... Shudder

I have to do my best.
Let $x_i$ and $x_j$ be two arbitrary elements, $c$ an arbitrary real number, and $f$ a function from $X$ to $Y$.

We say that $f$ is a linear transformation from $X$ to $Y$ if it satisfies the following two conditions:

1. $f(x_i) + f(x_j)$ and $f(x_i + x_j)$ are equal
2. $cf(x_i)$ and $f(cx_i)$ are equal

BUT THIS DEFINITION IS ACTUALLY INCOMPLETE.
**Linear Transformations**

Let \( \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \) and \( \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} \) be two arbitrary elements from \( \mathbb{R}^n \), \( c \) an arbitrary real number, and \( f \) a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

We say that \( f \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) if it satisfies the following two conditions:

1. \( f \left( \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \right) + f \left( \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} \right) \) and \( f \left( \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{pmatrix} \right) \) are equal.

2. \( cf \left( \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \right) \) and \( f \left( \begin{pmatrix} cx_{1i} \\ cx_{2i} \\ \vdots \\ cx_{ni} \end{pmatrix} \right) \) are equal.

A linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is sometimes called a **linear map** or **linear operation**.
AND IF $f$ IS A LINEAR TRANSFORMATION FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$...

Then it shouldn't be a surprise to hear that $f$ can be written as an $m \times n$ matrix. Um...it shouldn't?

Have a look at the following equations.
We'll verify the first rule first: 

\[
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix}
+ 
\begin{pmatrix}
  x_{ij} \\
  x_{2j} \\
  \vdots \\
  x_{nj}
\end{pmatrix}
= 
\begin{pmatrix}
  x_{1i} + x_{1j} \\
  x_{2i} + x_{2j} \\
  \vdots \\
  x_{ni} + x_{nj}
\end{pmatrix}
\]

We just replace \( f \) with a matrix, then simplify:

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_{1i} \\
x_{2i} \\
\vdots \\
x_{ni}
\end{pmatrix}
+ 
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_{ij} \\
x_{2j} \\
\vdots \\
x_{nj}
\end{pmatrix}
= 
\begin{pmatrix}
a_{11}x_{1i} + a_{12}x_{2i} + \ldots + a_{1n}x_{ni} \\
a_{21}x_{1i} + a_{22}x_{2i} + \ldots + a_{2n}x_{ni} \\
\vdots \\
a_{m1}x_{1i} + a_{m2}x_{2i} + \ldots + a_{mn}x_{ni}
\end{pmatrix}
+ 
\begin{pmatrix}
a_{11}x_{1j} + a_{12}x_{2j} + \ldots + a_{1n}x_{nj} \\
a_{21}x_{1j} + a_{22}x_{2j} + \ldots + a_{2n}x_{nj} \\
\vdots \\
a_{m1}x_{1j} + a_{m2}x_{2j} + \ldots + a_{mn}x_{nj}
\end{pmatrix}
= 
\begin{pmatrix}
a_{11}(x_{1i} + x_{1j}) + a_{12}(x_{2i} + x_{2j}) + \ldots + a_{1n}(x_{ni} + x_{nj}) \\
a_{21}(x_{1i} + x_{1j}) + a_{22}(x_{2i} + x_{2j}) + \ldots + a_{2n}(x_{ni} + x_{nj}) \\
\vdots \\
a_{m1}(x_{1i} + x_{1j}) + a_{m2}(x_{2i} + x_{2j}) + \ldots + a_{mn}(x_{ni} + x_{nj})
\end{pmatrix}
\]

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_{1i} + x_{1j} \\
x_{2i} + x_{2j} \\
\vdots \\
x_{ni} + x_{nj}
\end{pmatrix}
\]
Now for the second rule:

\[
\begin{pmatrix}
  x_{1f} \\
  x_{2f} \\
  \vdots \\
  x_{nf}
\end{pmatrix} = \begin{pmatrix}
  c
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix}
\]

Again, just replace \( f \) with a matrix and simplify:

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix} =
\begin{pmatrix}
  c
\end{pmatrix}
\begin{pmatrix}
  a_{11} x_{1i} + a_{12} x_{2i} + \cdots + a_{1n} x_{ni} \\
  a_{21} x_{1i} + a_{22} x_{2i} + \cdots + a_{2n} x_{ni} \\
  \vdots \\
  a_{m1} x_{1i} + a_{m2} x_{2i} + \cdots + a_{mn} x_{ni}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  c x_{1i} \\
  c x_{2i} \\
  \vdots \\
  c x_{ni}
\end{pmatrix} =
\begin{pmatrix}
  c
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix} =
\begin{pmatrix}
  c
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix} =
\begin{pmatrix}
  c
\end{pmatrix}
\begin{pmatrix}
  x_{1i} \\
  x_{2i} \\
  \vdots \\
  x_{ni}
\end{pmatrix}
\]

Oh, I see!
We can demonstrate the same thing visually. We'll use the $2 \times 2$ matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ as $f$.

We'll show that the first rule holds:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \end{pmatrix}$$

If you multiply first...

IF YOU
MULTIPLY
FIRST...

THEN ADD...

IF YOU
ADD FIRST...

THEN MULTIPLY...

YOU GET THE SAME FINAL RESULT!
And the second rule, too: 

\[
c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c x_{1i} \\ c x_{2i} \end{pmatrix}
\]

You get the same final result!
So when $f$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can also say that $f$ is equivalent to the $m \times n$ matrix that defines the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$
\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

So... What are linear transformations good for, exactly?

They seem pretty important. I guess we'll be using them a lot from now on?

Well, it's not really a question of importance...

Now I get it!

Well...

That's exactly what I wanted to talk about next.

Why We Study Linear Transformations
Consider the linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) defined by the following \( m \times n \) matrix:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

If \( \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \) is the image of \( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \) under this linear transformation, then the following equation is true:

\[
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_m
\end{pmatrix} =
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix}
\]

Yep, here's a definition. We talked a bit about this before, didn't we?

Yeah, in Chapter 2.

Flip!

Images

Suppose \( x_i \) is an element from \( X \).

The element in \( Y \) corresponding to \( x_i \) under \( f \) is called "\( x_i \)'s image under \( f \)."

We talked a bit about this before, didn't we?

Yeah, in Chapter 2.
but that definition is a bit vague. Take a look at this. 

Okay. 

Doesn't it kind of look like a common one-dimensional equation \( y = ax \) to you?

\[
\begin{pmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_m
\end{pmatrix} = 
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

Okay. 

Maybe if I squint... 

What if I put it like this?

I guess that makes sense. 

Multiplying an \( n \)-dimensional space by an \( m \times n \) matrix... 

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

turns it \( m \)-dimensional!
WE STUDY LINEAR TRANSFORMATIONS IN AN EFFORT TO BETTER UNDERSTAND THE CONCEPT OF IMAGE, USING MORE VISUAL MEANS THAN SIMPLE FORMULAE.

I HAVE TO LEARN THIS STUFF BECAUSE OF...THAT?

OOH, BUT "THAT" IS A LOT MORE SIGNIFICANT THAN YOU MIGHT THINK!

TAKE THIS LINEAR TRANSFORMATION FROM THREE TO TWO DIMENSIONS, FOR EXAMPLE.

\[
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} =
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

YOU COULD WRITE IT AS THIS LINEAR SYSTEM OF EQUATIONS INSTEAD, IF YOU WANTED TO.

\[
\begin{align*}
  y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
  y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3
\end{align*}
\]

BUT YOU HAVE TO AGREE THAT THIS DOESN'T REALLY CONVEY THE FEELING OF "TRANSFORMING A THREE-DIMENSIONAL SPACE INTO A TWO-DIMENSIONAL ONE," RIGHT?
Ah... I think I'm starting to get it.

Linear transformations are definitely one of the harder-to-understand parts of linear algebra. I remember having trouble with them when I started studying the subject.
I wouldn't want you thinking that linear transformations lack practical uses, though. Computer graphics, for example, rely heavily on linear algebra and linear transformations in particular.

Yeah. As we're already on the subject, let's have a look at some of the transformations that let us do things like scaling, rotation, translation, and 3-D projection.

Let \((x_1, x_2)\) be some point on the drawing. The top of the dorsal fin will do!
Let's say we decide to

\[
\begin{align*}
    y_1 &= \alpha x_1 \\
    y_2 &= \beta x_2
\end{align*}
\]

This gives rise to the interesting relationship

\[
\begin{align*}
    y_1 &= \alpha x_1 \\
    y_2 &= \beta x_2
\end{align*}
\]

Could be rewritten like this, right?

Yeah, sure.

Uh-huh...

So that means that applying the set of rules

\[
\begin{align*}
    y_1 &= \alpha x_1 \\
    y_2 &= \beta x_2
\end{align*}
\]

onto an arbitrary image is basically the same thing as passing the image through a linear transformation in \(\mathbb{R}^2\) equal to the following matrix!

\[
\begin{pmatrix}
    \alpha & 0 \\
    0 & \beta
\end{pmatrix}
\]

Oh, it's a one-to-one onto mapping!
• Rotating \[ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \] by \( \theta \) degrees gets us \[ \begin{bmatrix} x_1 \cos \theta \\ x_1 \sin \theta \end{bmatrix} \] the point \((x_1 \cos \theta, x_1 \sin \theta)\)

• Rotating \[ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \] by \( \theta \) degrees gets us \[ \begin{bmatrix} -x_2 \sin \theta \\ x_2 \cos \theta \end{bmatrix} \] the point \((-x_2 \sin \theta, x_2 \cos \theta)\)

• Rotating \[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \] that is \[ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \], by \( \theta \) degrees gets us

\[
\begin{bmatrix} x_1 \cos \theta \\ x_1 \sin \theta \end{bmatrix} + \begin{bmatrix} -x_2 \sin \theta \\ x_2 \cos \theta \end{bmatrix} = \begin{bmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{bmatrix}
\]

the point \((x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta)\)

* \( \theta \) is the Greek letter \( \theta \)eta.
So if we wanted to rotate the entire picture by \( \theta \) degrees, we'd get...

Due to this relationship.

\[
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  x_1 \cos \theta - x_2 \sin \theta \\
  x_1 \sin \theta + x_2 \cos \theta
\end{pmatrix} = \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

Aha.

Rotating an arbitrary image by \( \theta \) degrees consequently means we're using a linear transformation in \( \mathbb{R}^2 \) equal to this matrix:

\[
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\]

Another one-to-one onto mapping!
If we instead decide to translate all $x_1$ values by $b_1$ and all $x_2$ values by $b_2$, we get another interesting relationship:

\[
\begin{align*}
y_1 &= x_1 + b_1 \\
y_2 &= x_2 + b_2
\end{align*}
\]

And this can also be rewritten like so:

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ x_2 + b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

That's true.

If we wanted to, we could also rewrite it like this:

\[
\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ x_2 + b_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}
\]

Seems silly, but okay.
SO APPLYING THE SET OF RULES \[
\begin{align*}
\text{Translate all } x_1 \text{ values by } b_1 \\
\text{Translate all } x_2 \text{ values by } b_2
\end{align*}
\]
ONTO AN ARBITRARY IMAGE IS BASICALLY THE SAME THING AS PASSING THE IMAGE THROUGH A LINEAR TRANSFORMATION IN $\mathbb{R}^3$ EQUAL TO THE FOLLOWING MATRIX:

\[
\begin{pmatrix}
1 & 0 & b_1 \\
0 & 1 & b_2 \\
0 & 0 & 1
\end{pmatrix}
\]

HEY, WAIT A MINUTE! WHY ARE YOU DRAGGING ANOTHER DIMENSION INTO THE DISCUSSION ALL OF A SUDDEN?

AND WHAT WAS THE POINT OF THAT WEIRD REWRITE?

"$y = ax_l$"
We'd like to express translations in the same way as rotations and scale operations, with

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

instead of with

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

The first formula is more practical than the second, especially when dealing with computer graphics.

A computer stores all transformations as 3×3 matrices...

...even rotations and scaling operations.

NOT TOO DIFFERENT, I GUESS.

<table>
<thead>
<tr>
<th></th>
<th>CONVENTIONAL LINEAR TRANSFORMATIONS</th>
<th>LINEAR TRANSFORMATIONS USED BY COMPUTER GRAPHICS SYSTEMS</th>
</tr>
</thead>
</table>
| **SCALING** | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\] | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\] |
| **ROTATION** | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\] | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\] |
| **TRANSLATION** | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\] | \[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\] |

*Note: This one isn't actually a linear transformation. You can verify this by setting \( b_1 \) and \( b_2 \) to 1 and checking that both linear transformation conditions fail.*

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3-D Projection

Next we'll very briefly talk about a 3-D projection technique called perspective projection.

Don't worry too much about the details.

Perspective projection provides us with a way to project three-dimensional objects onto a near plane by tracing our way from each point on the object toward a common observation point and noting where these lines intersect with the near plane.

The math is a bit more complex than what we've seen so far.

So I'll cheat a little bit and skip right to the end!

The linear transformation we use for perspective projection is in $\mathbb{R}^4$ and can be written as the following matrix:

$$
\frac{1}{x_3 - s_3} \begin{pmatrix}
-s_3 & 0 & s_1 & 0 \\
0 & -s_3 & s_2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & -s_3
\end{pmatrix}
$$
And that's what transformations are all about!

Yeah... but that's enough for today, I think.

Final lesson? So soon?

So much to learn...

We'll be talking about eigenvectors and eigenvalues in our next and final lesson.

Don't worry, we'll cover all the important topics.

Hehe, why would I worry? You're such a good teacher.

You shouldn't worry either, you know.

Oh, you heard?

Hm?

About the match.

Yeah, my brother told me.

Heh. Thanks. I'm going to the gym after this, actually. I hope I don't lose too badly...
DON'T SAY THAT!

YOU'VE GOT TO STAY POSITIVE!

TH- THANKS...

I KNOW YOU CAN DO IT.

I'LL DO MY BEST!
Some Preliminary Tips

Before we dive into kernel, rank, and the other advanced topics we're going to cover in the remainder of this chapter, there's a little mathematical trick that you may find handy while working some of these problems out.

The equation

\[
\begin{pmatrix}
  y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

can be rewritten like this:

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = x_1 \begin{pmatrix} 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \ldots + x_n \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

As you can see, the product of the matrix \( M \) and the vector \( \mathbf{x} \) can be viewed as a linear combination of the columns of \( M \) with the entries of \( \mathbf{x} \) as the weights.

Also note that the function \( f \) referred to throughout this chapter is the linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) corresponding to the following \( m \times n \) matrix:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
Kernel, image, and the Dimension Theorem for Linear Transformations

The set of vectors whose images are the zero vector, that is

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]

is called the \textit{kernel} of the linear transformation \(f\) and is written \(\text{Ker} f\).

The \textit{image} of \(f\) (written \(\text{Im} f\)) is also important in this context. The image of \(f\) is equal to the set of vectors that is made up of all of the possible output values of \(f\), as you can see in the following relation:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_m
\end{bmatrix} =
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

(This is a more formal definition of image than what we saw in Chapter 2, but the concept is the same.)

An important observation is that \(\text{Ker} f\) is a subspace of \(\mathbb{R}^n\) and \(\text{Im} f\) is a subspace of \(\mathbb{R}^m\). The \textit{dimension theorem for linear transformations} further explores this observation by defining a relationship between the two:

\[
\dim \text{Ker} f + \dim \text{Im} f = n
\]

Note that the \(n\) above is equal to the first vector space's dimension (\(\dim \mathbb{R}^n\)).

* If you need a refresher on the concept of dimension, see “Basis and Dimension” on page 156.
**Example 1**

Suppose that \( f \) is a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) equal to the matrix \[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\].

Then:

\[
\text{Ker} \ f = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \{0\}
\]

\[
\text{Im} \ f = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \mathbb{R}^2
\]

And:

\[
\begin{cases}
\quad n = 2 \\
\dim \text{Ker} \ f = 0 \\
\dim \text{Im} \ f = 2
\end{cases}
\]

**Example 2**

Suppose that \( f \) is a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) equal to the matrix \[
\begin{pmatrix}
3 & 6 \\
1 & 2
\end{pmatrix}
\].

Then:

\[
\text{Ker} \ f = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} c \\ -2 \\ 1 \end{pmatrix} \quad \text{c is an arbitrary number}
\]

\[
\text{Im} \ f = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} c \\ 3 \\ 1 \end{pmatrix} \quad \text{c is an arbitrary number}
\]

And:

\[
\begin{cases}
\quad n = 2 \\
\dim \text{Ker} \ f = 1 \\
\dim \text{Im} \ f = 1
\end{cases}
\]
**Example 3**

Suppose $f$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ equal to the $3 \times 2$ matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]

Then:

\[
\text{Ker} \, f = \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = x_1 \begin{pmatrix}
1 \\
0
\end{pmatrix} + x_2 \begin{pmatrix}
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\text{Im} \, f = \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = y_1 \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + y_2 \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad c_1 \text{ and } c_2 \text{ are arbitrary numbers}
\]

\[
\begin{cases}
n = 2 \\
\dim \text{Ker} \, f = 0 \\
\dim \text{Im} \, f = 2
\end{cases}
\]
EXAMPLE 4

Suppose that $f$ is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ equal to

the $2 \times 4$ matrix $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. Then:

\[
\text{Ker } f = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = x_1 + 3x_3 + x_4 = 0, \ x_2 + x_3 + 2x_4 = 0
\]

\[
\text{Im } f = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbb{R}^2
\]

And:

\[
\begin{cases} n & = 4 \\ \dim \text{Ker } f & = 2 \\ \dim \text{Im } f & = 2 \end{cases}
\]
**RANK**

The number of linearly independent vectors among the columns of the matrix $M$ (which is also the dimension of the $\mathbb{R}^n$ subspace $\text{Im} f$) is called the rank of $M$, and it is written like this: rank $M$.

**EXAMPLE 1**

The linear system of equations \[
\begin{cases}
3x_1 + 1x_2 = y_1 \\
1x_1 + 2x_2 = y_2
\end{cases}
\]

that is \[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
3x_1 + 1x_2 \\
1x_1 + 2x_2
\end{pmatrix},
\]

can be rewritten as follows:
\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
3x_1 + 1x_2 \\
1x_1 + 2x_2
\end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

The two vectors \[
\begin{pmatrix} 3 \\ 1 \\
\end{pmatrix}
\]
and \[
\begin{pmatrix} 1 \\ 2
\end{pmatrix}
\]
are linearly independent, as can be seen on pages 133 and 135, so the rank of \[
\begin{pmatrix} 3 & 1 \\ 1 & 2
\end{pmatrix}
\]
is 2.

Also note that \[
\det \begin{pmatrix} 3 & 1 \\ 1 & 2
\end{pmatrix} = 3 \cdot 2 - 1 \cdot 1 = 5 \neq 0.
\]

**EXAMPLE 2**

The linear system of equations \[
\begin{cases}
3x_1 + 6x_2 = y_1 \\
1x_1 + 2x_2 = y_2
\end{cases}
\]

that is \[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
3x_1 + 6x_2 \\
1x_1 + 2x_2
\end{pmatrix},
\]

can be rewritten as follows:
\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
3x_1 + 6x_2 \\
1x_1 + 2x_2
\end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 6 \\ 2 \end{pmatrix}
\]

\[
= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\]

\[
= [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1
\end{pmatrix}
\]

So the rank of \[
\begin{pmatrix} 3 & 6 \\ 1 & 2
\end{pmatrix}
\]
is 1.

Also note that \[
\det \begin{pmatrix} 3 & 6 \\ 1 & 2
\end{pmatrix} = 3 \cdot 2 - 6 \cdot 1 = 0.
\]
**Example 3**

The linear system of equations
\[
\begin{align*}
1x_1 + 0x_2 &= y_1 \\
0x_1 + 1x_2 &= y_2 \\
0x_1 + 0x_2 &= y_3
\end{align*}
\]
that is
\[
\begin{align*}
y_1 &= 1x_1 + 0x_2 \\
y_2 &= 0x_1 + 1x_2 \\
y_3 &= 0x_1 + 0x_2
\end{align*}
\]
can be rewritten as:
\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

The two vectors \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) are linearly independent, as we discovered on page 137, so the rank of \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \) is 2.

The system could also be rewritten like this:
\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

Note that \( \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \)
**EXAMPLE 4**

The linear system of equations
\[
\begin{align*}
1x_1 + 0x_2 + 3x_3 + 1x_4 &= y_1 \\
0x_1 + 1x_2 + 1x_3 + 2x_4 &= y_2
\end{align*}
\]
that is
\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
1x_1 + 0x_2 + 3x_3 + 1x_4 \\
0x_1 + 1x_2 + 1x_3 + 2x_4
\end{pmatrix},
\]
can be rewritten as follows:

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

The rank of \( \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \) is equal to 2, as we'll see on page 203.

The system could also be rewritten like this:

\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} = \begin{pmatrix}
1x_1 + 0x_2 + 3x_3 + 1x_4 \\
0x_1 + 1x_2 + 1x_3 + 2x_4 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
\]

Note that \( \det \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \).

The four examples seem to point to the fact that
\[
\det \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix} = 0 \] is the same as rank \( \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \neq n \).

This is indeed so, but no formal proof will be given in this book.
CALCULATING THE RANK OF A MATRIX

So far, we’ve only dealt with matrices where the rank was immediately apparent or where we had previously figured out how many linearly independent vectors made up the columns of that matrix. Though this might seem like “cheating” at first, these techniques can actually be very useful for calculating ranks in practice.

For example, take a look at the following matrix:

\[
\begin{pmatrix}
1 & 4 & 4 \\
2 & 5 & 8 \\
3 & 6 & 12
\end{pmatrix}
\]

It’s immediately clear that the third column of this matrix is equal to the first column times 4. This leaves two linearly independent vectors (the first two columns), which means this matrix has a rank of 2.

Now look at this matrix:

\[
\begin{pmatrix}
1 & 0 \\
0 & 3 \\
0 & 5
\end{pmatrix}
\]

It should be obvious right from the start that these vectors form a linearly independent set, so we know that the rank of this matrix is also 2.

Of course there are times when this method will fail you and you won’t be able to tell the rank of a matrix just by eyeballing it. In those cases, you’ll have to buckle down and actually calculate the rank. But don’t worry, it’s not too hard!

First we’ll explain the **Problem**, then we’ll establish a good **Way of Thinking**, and then finally we’ll tackle the **Solution**.

**Problem**

Calculate the rank of the following 2×4 matrix:

\[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix}
\]

**Way of Thinking**

Before we can solve this problem, we need to learn a little bit about elementary matrices. An *elementary matrix* is created by starting with an identity matrix and performing exactly one of the elementary row operations used for Gaussian elimination (see Chapter 4). The resulting matrices can then be multiplied with any arbitrary matrix in such a way that the number of linearly independent columns becomes obvious.
With this information under our belts, we can state the following four useful facts about an arbitrary matrix \( A \):

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

**Fact 1**

Multiplying the elementary matrix

\[
\begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 1
\end{pmatrix}
\]

Column \( i \) Column \( j \)

Row \( i \) Row \( j \)

...to the left of an arbitrary matrix \( A \) will switch rows \( i \) and \( j \) in \( A \).

If we multiply the matrix to the right of \( A \), then the columns will switch places in \( A \) instead.

- **Example 1** (Rows 1 and 4 are switched.)

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43}
\end{pmatrix}
\begin{pmatrix}
0 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41} \\
0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} \\
0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} + 0 \cdot a_{41} \\
1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41}
\end{pmatrix}
\begin{pmatrix}
a_{12} & 0 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 1 \cdot a_{42} \\
0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} \\
0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} + 0 \cdot a_{42} \\
1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42}
\end{pmatrix}
\begin{pmatrix}
a_{13} & 0 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 1 \cdot a_{43} \\
0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\
0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} + 0 \cdot a_{43} \\
1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43}
\end{pmatrix}
\]

\[
\begin{pmatrix}
a_{41} & a_{42} & a_{43} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{11} & a_{12} & a_{13}
\end{pmatrix}
\]
Example 2 (Columns 1 and 3 are switched.)

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43}
\end{bmatrix}
\begin{bmatrix}
 0 & 0 & 1 \\
 0 & 1 & 0 \\
 1 & 0 & 0
\end{bmatrix}
\]

= \[
\begin{bmatrix}
 a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 & a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot 0 & a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot 0 \\
 a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 1 & a_{21} \cdot 0 + a_{22} \cdot 1 + a_{23} \cdot 0 & a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot 0 \\
 a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 1 & a_{31} \cdot 0 + a_{32} \cdot 1 + a_{33} \cdot 0 & a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot 0 \\
 a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 1 & a_{41} \cdot 0 + a_{42} \cdot 1 + a_{43} \cdot 0 & a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43}
\end{bmatrix}
\]

**FACT 2**

Multiplying the elementary matrix

\[
\begin{bmatrix}
 1 & \cdots & 0 & \cdots & 0 \\
 \vdots & \ddots & \vdots & \ddots & \vdots \\
 0 & \cdots & k & \cdots & 0 \\
 \vdots & \ddots & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
\]

to the left of an arbitrary matrix \( A \) will multiply the \( i \)th row in \( A \) by \( k \).

Multiplying the matrix to the right side of \( A \) will multiply the \( i \)th column in \( A \) by \( k \) instead.
Example 1 (Row 3 is multiplied by \(k\).)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & k & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43}
\end{bmatrix}
= \begin{bmatrix}
1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} \\
0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} \\
0 \cdot a_{11} + 0 \cdot a_{21} + k \cdot a_{31} + 0 \cdot a_{41} \\
0 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
ka_{31} & ka_{32} & ka_{33} \\
a_{41} & a_{42} & a_{43}
\end{bmatrix}
\]

Example 2 (Column 2 is multiplied by \(k\).)

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot 0 \\
a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot 0 \\
a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot 0 \\
a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot 0
\end{bmatrix}
\begin{bmatrix}
a_{11} & ka_{12} & a_{13} \\
a_{21} & ka_{22} & a_{23} \\
a_{31} & ka_{32} & a_{33} \\
a_{41} & ka_{42} & a_{43}
\end{bmatrix}
\]
**Fact 3**

Multiplying the elementary matrix

\[
\begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & k & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}
\]

Row \(i\) to the left of an arbitrary matrix \(A\) will add \(k\) times row \(i\) to row \(j\) in \(A\).

Multiplying the matrix to the right side of \(A\) will add \(k\) times column \(j\) to column \(i\) instead.

- **Example 1** (\(k\) times row 2 is added to row 4.)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & k & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43}
\end{pmatrix}
= \begin{pmatrix}
1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\
0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\
0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} + 0 \cdot a_{43} \\
0 \cdot a_{11} + k \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41} & 0 \cdot a_{12} + k \cdot a_{22} + 0 \cdot a_{32} + 1 \cdot a_{42} & 0 \cdot a_{13} + k \cdot a_{23} + 0 \cdot a_{33} + 1 \cdot a_{43}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} + ka_{21} & a_{42} + ka_{22} & a_{43} + ka_{23}
\end{pmatrix}
\]
Example 2 (\(k\) times column 3 is added to column 1.)

\[
\begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33} \\
    a_{41} & a_{42} & a_{43}
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    k & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
   a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot k \\
   a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot k \\
   a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot k \\
   a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot k
\end{pmatrix}
\begin{pmatrix}
   a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot k \\
   a_{21} \cdot 0 + a_{22} \cdot 1 + a_{23} \cdot k \\
   a_{31} \cdot 0 + a_{32} \cdot 1 + a_{33} \cdot k \\
   a_{41} \cdot 0 + a_{42} \cdot 1 + a_{43} \cdot k
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    a_{11} + ka_{13} \\
    a_{21} + ka_{23} \\
    a_{31} + ka_{33} \\
    a_{41} + ka_{43}
\end{pmatrix}
\begin{pmatrix}
    a_{12} \\
    a_{22} \\
    a_{32} \\
    a_{42}
\end{pmatrix}
\]

**FACT 4**

The following three \(m\times n\) matrices all have the same rank:

1. The matrix:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

2. The left product using an invertible \(m\times m\) matrix:

\[
\begin{pmatrix}
    b_{11} & b_{12} & \ldots & b_{1m} \\
    b_{21} & b_{22} & \ldots & b_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{m1} & b_{m2} & \ldots & b_{mm}
\end{pmatrix}
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

3. The right product using an invertible \(n\times n\) matrix:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    c_{11} & c_{12} & \ldots & c_{1n} \\
    c_{21} & c_{22} & \ldots & c_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{n1} & c_{n2} & \ldots & c_{nn}
\end{pmatrix}
\]

In other words, multiplying \(A\) by any elementary matrix—on either side—will not change \(A\)'s rank, since elementary matrices are invertible.
The following table depicts calculating the rank of the 2×4 matrix:

\[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix}
\]

Begin with

\[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix}
\]

Add \((-1 \cdot \text{column 2})\) to column 3

\[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

Add \((-1 \cdot \text{column 1})\) to column 4

\[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 0 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

Add \((-3 \cdot \text{column 1})\) to column 3

\[
\begin{pmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -3 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\]

Add \((-2 \cdot \text{column 2})\) to column 4

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[\text{SOLUTION}\]

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Because of Fact 4, we know that both \[
\begin{pmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
have the same rank.

One look at the simplified matrix is enough to see that only \[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\]
are linearly independent among its columns.
This means it has a rank of 2, and so does our initial matrix.

**THE RELATIONSHIP BETWEEN LINEAR TRANSFORMATIONS AND MATRICES**

We talked a bit about the relationship between linear transformations and matrices on page 168. We said that a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) could be written as an \( m \times n \) matrix:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

As you probably noticed, this explanation is a bit vague. The more exact relationship is as follows:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}

\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

for some matrix \( A \).
8
EIGENVALUES AND EIGENVECTORS
And Jumonji from Nanhou University!

Yurino from Hanamichi University!

He looks tough. I'll have to give it my all.

Begin!

I'll have to give it my all.
I've got to show them how strong I can be!

I have to win this!
Thank you... very much...

Damn...

Good match.
I'm sorry about the match...

Yeah...

My brother said you fought well, though.

Really?

don't worry about it.

You'll do better next time.

I know it!

I'm sorry! You're completely right!

Sulking won't accomplish anything.
Anyway... Today's our last lesson.

And I thought we'd work on eigenvalues and eigenvectors.

Okay. I'm ready for anything!

Studying eigenvalues and eigenvectors comes in handy when doing physics and statistics, for example.

They also make these kinds of problems much easier.

Finding the $p^{th}$ power of an $n \times n$ matrix.

It's a pretty abstract topic, but I'll try to be as concrete as I can.

I appreciate it!
What Are eigenvalues and eigenvectors?  

What do you say we start off with a few problems? Sure.

Hmm... Like this?

So close!

Oh, like this?

Exactly!

So the answer can be expressed using multiples of the original two vectors?

Okay, first problem. Find the image of \( \mathbb{R}^2 \) using the linear transformation determined by the 2x2 matrix:

\[
\begin{pmatrix}
3 & 1 \\
1 & 2
\end{pmatrix}
\]

(where \( c_1 \) and \( c_2 \) are real numbers).

\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}
\]
THAT'S RIGHT! SO YOU COULD SAY THAT THE LINEAR TRANSFORMATION EQUAL TO THE MATRIX

\[
\begin{pmatrix}
8 & -3 \\
2 & 1 \\
\end{pmatrix}
\]

TRANSFORMS ALL POINTS ON THE $x_1 x_2$ PLANE...

LIKE SO.

OH...
Let's move on to another problem.

Find the image of $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ using the linear transformation determined by the $3 \times 3$ matrix

\[
\begin{pmatrix}
4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

(where $c_1$, $c_2$, and $c_3$ are real numbers).

\[
\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}
\]

\[
\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}
\]

So the solution can be expressed with multiples as well...
...transforms every point in the $x_1 x_2 x_3$ space...

So you could say that the linear transformation equal to the matrix

\[
\begin{pmatrix}
4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

Like this.

I get it!
If the image of a vector \( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \) through the linear transformation determined by the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

is equal to \( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \), \( \lambda \) is said to be an eigenvalue to the matrix, and

\[
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

is said to be an eigenvector corresponding to the eigenvalue \( \lambda \).

The zero vector can never be an eigenvector.

You can generally never find more than \( n \) different eigenvalues and eigenvectors for any \( n \times n \) matrix.

\[
\begin{array}{c|c}
R^n & R^n \\
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} & \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{array}
\]

So the two examples could be summarized like this?

\[
\begin{array}{c|c|c}
\text{MATRIX} & \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\text{EIGENVALUE} & \lambda = 7, 2 & \lambda = 4, 2, -1 \\
\text{EIGENVECTOR} & \begin{pmatrix} 3 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{array}
\]
Let's have a look at calculating these vectors and values.

The relationship between the determinant and eigenvalues of a matrix.

\[ \lambda \text{ is an eigenvalue of the matrix} \]
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

if and only if
\[
\text{det}
\begin{pmatrix}
a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{pmatrix} = 0
\]

The 2×2 matrix
\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}
\]
will do fine as an example.

Okay.

Let's start off with the relationship between the determinant and eigenvalues of a matrix.

The relationship between the determinant and eigenvalues of a matrix.

\[ \lambda \text{ is an eigenvalue of the matrix} \]
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

if and only if
\[
\text{det}
\begin{pmatrix}
a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{pmatrix} = 0
\]
This means that solving this characteristic equation gives us all eigenvalues corresponding to the underlying matrix.

It's pretty cool.

Go ahead, give it a shot.

The values are seven and two?

Correct!

\[
\begin{vmatrix}
8 - \lambda & -3 \\
2 & 1 - \lambda
\end{vmatrix} = (8 - \lambda) \cdot (1 - \lambda) - (-3) \cdot 2
\]

\[
= (\lambda - 8) \cdot (\lambda - 1) - (-3) \cdot 2
\]

\[
= \lambda^2 - 9\lambda + 8 + 6
\]

\[
= \lambda^2 - 9\lambda + 14
\]

\[
= (\lambda - 7)(\lambda - 2) = 0
\]

\[
\lambda = 7, 2
\]
FINDING EIGENVECTORS IS ALSO PRETTY EASY.

For example, we can use our previous values in this formula:

\[
\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ that is } \begin{pmatrix} 8 - \lambda & -3 \\ 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

**PROBLEM 1**

Find an eigenvector corresponding to \( \lambda = 7 \).

Let's plug our value into the formula:

\[
\begin{pmatrix} 8 - 7 & -3 \\ 2 & 1 - 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 - 6x_2 \end{pmatrix} = [x_1 - 3x_2] \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

This means that \( x_1 = 3x_2 \), which leads us to our eigenvector

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\]

where \( c_1 \) is an arbitrary nonzero real number.

**PROBLEM 2**

Find an eigenvector corresponding to \( \lambda = 2 \).

Let's plug our value into the formula:

\[
\begin{pmatrix} 8 - 2 & -3 \\ 2 & 1 - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6x_1 - 3x_2 \\ 2x_1 - x_2 \end{pmatrix} = [2x_1 - x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

This means that \( x_2 = 2x_1 \), which leads us to our eigenvector

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ 2c_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

where \( c_2 \) is an arbitrary nonzero real number.
Calculating the $p$th Power of an $n \times n$ Matrix

It's finally time to tackle today's real problem! Finding the $p$th power of an $n \times n$ matrix.

We've already found the eigenvalues and eigenvectors of the matrix

\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}
\]

So let's just build on that example.

For simplicity's sake, let's choose $c_1 = c_2 = 1$.

Using the two calculations above...

Let's multiply $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$ to the right of both sides of the equation. Refer to page 91 to see why $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$ exists.

Makes sense.
TRY USING THE FORMULA TO CALCULATE
\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}^2
\]

Hmm... Okay.

\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}^2
= \begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix} \begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix}
= \begin{pmatrix}
31 & 70 & 31 \\
12 & 2 & 12
\end{pmatrix}
\]

Yep!

Yay!

Is...this it?

Looking at your calculations, would you say this relationship might be true?

\[
\begin{pmatrix}
8 & -3 \\
2 & 1
\end{pmatrix} = \begin{pmatrix}
31 & 7^p & 0 \\
12 & 0 & 2^p
\end{pmatrix} \begin{pmatrix}
31 \\
12
\end{pmatrix}^{-1}
\]

Uhhh...
It actually is! This formula is very useful for calculating any power of an \( n \times n \) matrix that can be written in this form.

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}^p
= \begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}
\begin{pmatrix}
  \lambda_1^p & 0 & \cdots & 0 \\
  0 & \lambda_2^p & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \lambda_n^p
\end{pmatrix}
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}^{-1}
\]

GOT IT!

Oh, and by the way...

When \( p = 1 \), we say that the formula diagonalizes the \( n \times n \) matrix

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

The right side of the equation is the diagonalized form of the middle matrix on the left side.

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix} \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} = \begin{pmatrix}
  \lambda_1 & 0 & \cdots & 0 \\
  0 & \lambda_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\]

And that's it!

Nice!
That was the last lesson!

How do you feel? Did you get the gist of it?

Yeah, thanks to you.

Awesome!

Really, though, thanks for helping me out.

I know you're busy, and you've been awfully tired because of your karate practice.

Not at all! How could I possibly have been tired after all that wonderful food you gave me?

I should be thanking you!

I'll miss these sessions, you know! My afternoons will be so lonely from now on...
Well...we could go out sometime...

Hmm?

Yeah...to look for math books, or something... you know...

If you don't have anything else to do...

Sure, sounds like fun!

So when would you like to go?
We said on page 221 that any $n \times n$ matrix could be expressed in this form:

$$
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
= 
\begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{1n} \\
    x_{21} & x_{22} & \cdots & x_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
\lambda \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
= 
\begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{1n} \\
    x_{21} & x_{22} & \cdots & x_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
\lambda \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
= 
\begin{bmatrix}
    \lambda_1 & 0 & \cdots & 0 \\
    0 & \lambda_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \lambda_n
\end{bmatrix}
\begin{bmatrix}
    x_{11} \\
    x_{21} \\
    \vdots \\
    x_{n1}
\end{bmatrix}
\begin{bmatrix}
    x_{12} \\
    x_{22} \\
    \vdots \\
    x_{n2}
\end{bmatrix}
\begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
\end{bmatrix}
\begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
\end{bmatrix}
\begin{bmatrix}
    x_{1n} \\
    x_{2n} \\
    \vdots \\
    x_{nn}
\end{bmatrix}
\end{equation}

This isn't totally true, as the concept of multiplicity\(^1\) plays a large role in whether a matrix can be diagonalized or not. For example, if all $n$ solutions of the following equation

$$
\det \begin{bmatrix}
    a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{bmatrix} = 0
$$

are real and have multiplicity 1, then diagonalization is possible. The situation becomes more complicated when we have to deal with eigenvalues that have multiplicity greater than 1. We will therefore look at a few examples involving:

- Matrices with eigenvalues having multiplicity greater than 1 that can be diagonalized
- Matrices with eigenvalues having multiplicity greater than 1 that cannot be diagonalized

---

1. The multiplicity of any polynomial root reveals how many identical copies of that same root exist in the polynomial. For instance, in the polynomial $f(x) = (x - 1)^4(x + 2)^3x$, the factor $(x - 1)$ has multiplicity 4, $(x + 2)$ has 2, and $x$ has 1.
A Diagonalizable Matrix with an Eigenvalue Having Multiplicity 2

**Problem**

Use the following matrix in both problems:

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
\]

1. Find all eigenvalues and eigenvectors of the matrix.
2. Express the matrix in the following form:

\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}^{-1}
\]

**Solution**

1. The eigenvalues \( \lambda \) of the 3x3 matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
\]

are the roots of the characteristic equation: \( \det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & -1 \\ -2 & 0 & 3 - \lambda \end{pmatrix} = 0 \).

\[
\det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & -1 \\ -2 & 0 & 3 - \lambda \end{pmatrix}
= (1 - \lambda)(1 - \lambda)(3 - \lambda) + 0 \cdot (-1) \cdot (-2) + 0 \cdot 1 \cdot 0
- 0 \cdot (1 - \lambda) \cdot (-2) - 0 \cdot 1 \cdot (3 - \lambda) - (1 - \lambda) \cdot (-1) \cdot 0
= (1 - \lambda)^2(3 - \lambda) = 0
\]

\[\lambda = 3, 1\]

Note that the eigenvalue 1 has multiplicity 2.
A. The eigenvectors corresponding to $\lambda = 3$

Let's insert our eigenvalue into the following formula:

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \lambda
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix},
\]
that is

\[
\begin{pmatrix}
1 - \lambda & 0 & 0 \\
1 & 1 - \lambda & -1 \\
-2 & 0 & 3 - \lambda
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 0
\]

This gives us:

\[
\begin{pmatrix}
1 - 3 & 0 & 0 \\
1 & 1 - 3 & -1 \\
-2 & 0 & 3 - 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
-2 & 0 & 0 \\
1 & -2 & -1 \\
-2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 0
\]

The solutions are as follows:

\[
\begin{pmatrix}
x_1 = 0 \\
x_3 = -2x_2
\end{pmatrix}
\quad \text{and the eigenvector}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=egin{pmatrix}
0 \\
c_1 \\
-2c_1
\end{pmatrix}
= c_1 \begin{pmatrix}
0 \\
1 \\
-2
\end{pmatrix}
\]

where $c_1$ is a real nonzero number.

B. The eigenvectors corresponding to $\lambda = 1$

Repeating the steps above, we get

\[
\begin{pmatrix}
1 - 1 & 0 & 0 \\
1 & 1 - 1 & -1 \\
-2 & 0 & 3 - 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & -1 \\
-2 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 0
\]

and see that $x_3 = x_1$ and $x_2$ can be any real number. The eigenvector consequently becomes

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
c_1 \\
c_2 \\
c_1
\end{pmatrix}
= c_1 \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\]

where $c_1$ and $c_2$ are arbitrary real numbers that cannot both be zero.
3. We then apply the formula from page 221:

The eigenvector corresponding to 3

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-2 & 0 & 3
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 3 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
-2 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
-2 & 1 & 0
\end{pmatrix}
\]

The linearly independent eigenvectors corresponding to 1

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}
= \begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}^{-1}
\begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}
\]

**A NON-DIAGONALIZABLE MATRIX WITH A REAL EIGENVALUE HAVING MULTIPLICITY 2**

? **PROBLEM**

Use the following matrix in both problems:

\[
\begin{pmatrix}
1 & 0 & 0 \\
-7 & 1 & -1 \\
4 & 0 & 3
\end{pmatrix}
\]

1. Find all eigenvalues and eigenvectors of the matrix.

2. Express the matrix in the following form:

\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}
\lambda
\begin{pmatrix}
x_{21} & x_{22} & x_{23}
\end{pmatrix}
= \begin{pmatrix}
x_{31} & x_{32} & x_{33}
\end{pmatrix}^{-1}
\]

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}
= \begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}^{-1}
\begin{pmatrix}
x_{11} & x_{12} & x_{13}
\end{pmatrix}
\]

? **SOLUTION**

1. The eigenvalues \( \lambda \) of the 3×3 matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
-7 & 1 & -1 \\
4 & 0 & 3
\end{pmatrix}
\]

are the roots of the characteristic equation: \( \det \begin{pmatrix}
1 - \lambda & 0 & 0 \\
-7 & 1 - \lambda & -1 \\
4 & 0 & 3 - \lambda
\end{pmatrix} = 0. \)
\[ \det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ -7 & 1 - \lambda & -1 \\ 4 & 0 & 3 - \lambda \end{pmatrix} = (1 - \lambda)(1 - \lambda)(3 - \lambda) + 0 \cdot (-1) \cdot 4 + 0 \cdot (-7) \cdot 0 \]
\[ - 0 \cdot (1 - \lambda) \cdot 4 - 0 \cdot (-7) \cdot (3 - \lambda) - (1 - \lambda) \cdot (-1) \cdot 0 \]
\[ = (1 - \lambda)^2(3 - \lambda) = 0 \]

\[ \lambda = 3, 1 \]

Again, note that the eigenvalue 1 has multiplicity 2.

A. The eigenvectors corresponding to \( \lambda = 3 \)

Let's insert our eigenvalue into the following formula:

\[ \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 - \lambda & -1 \\ 4 & 0 & 3 - \lambda \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \]

This gives us

\[ \begin{pmatrix} 1 - 3 & 0 & 0 \\ -7 & 1 - 3 & -1 \\ 4 & 0 & 3 - 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ -7 & -2 & -1 \\ 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -7x_1 - 2x_2 - x_3 \end{pmatrix} = 0 \]

The solutions are as follows:

\[ \begin{cases} x_1 = 0 \\ x_3 = -2x_2 \end{cases} \]

and the eigenvector

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ -2c_1 \end{pmatrix} \]

where \( c_1 \) is a real nonzero number.
B. The eigenvectors corresponding to $\lambda = 1$

We get

$$
\begin{pmatrix}
1 & -1 & 0 \\
-7 & 1 & -1 \\
4 & 0 & 3 - 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 \\
-7 & 0 & -1 \\
4 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
0 \\
-7x_1 - x_3 \\
4x_1 + 2x_3
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

and see that

$$
\begin{cases}
\begin{aligned}
x_3 &= -7x_1 \\
x_3 &= -2x_1
\end{aligned}
\end{cases}
$$

But this could only be true if $x_1 = x_3 = 0$. So the eigenvector has to be

$$
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
0 \\
c_2 \\
0
\end{pmatrix}
=c_2
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
$$

where $c_2$ is an arbitrary real nonzero number.

3. Since there were no eigenvectors in the form

$$
c_2
\begin{pmatrix}
x_{12} \\
x_{22} \\
x_{32}
\end{pmatrix}
+c_3
\begin{pmatrix}
x_{13} \\
x_{23} \\
x_{33}
\end{pmatrix}
$$

for $\lambda = 1$, there are not enough linearly independent eigenvectors to express

$$
\begin{pmatrix}
1 & 0 & 0 \\
-7 & 1 & -1 \\
4 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}^{-1}
$$

It is important to note that all diagonalizable $n \times n$ matrices always have $n$ linearly independent eigenvectors. In other words, there is always a basis in $\mathbb{R}^n$ consisting solely of eigenvectors, called an eigenbasis.
Hey there, been waiting long?

Reij—

Kyaa!

That voice!

Aww, don't be like that!

Stop it! Let me go!

We just want to get to know you better...

Misa!
I HAVE TO DO SOMETHING...

AND FAST!

THOSE JERKS...

WHAT IF IT HAPPENS ALL OVER AGAIN?

BUT...

ON MY FIRST DATE WITH YUKI, MY GIRLFRIEND IN MIDDLE SCHOOL...

HEY, LET ME GO!

WE JUST WANT TO HANG OUT...

I ALREADY HAVE A DATE.

YUKI!
THOSE JERKS...

I HAVE TO DO SOMETHING!

YOU... STOP IT!

UH... WHO ARE YOU?

KICK

GUH?

HAHA

SERIOUSLY? ONE KICK? WHAT A WUSS...

LEMME GO!

TIME TO GO.

YOU THERE!
Stop. Can't you see she doesn't want to go with you?

Jeez, who is it now?

Oh no!

You're... The legendary leader of the Hanamichi karate club.

"The Hanamichi Hammer!"

"The Hanamichi Hammer!"

In the flesh.

Unng...

Hanamichi Hammer?

Messing with that guy is suicide!

T-Thank you!

Don't worry about it... but I think your boyfriend needs medical attention...

Let's get outta here!

MOAN...
Listen... I'm glad you stood up for me, but...

I...

Don't think I can see you anymore.

I'm so sorry... I couldn't do anything...

Not this time.

I'll show them—

I'm so much stronger now—

Come on, sweetie...

I'm so sorry... I couldn't do anything...

This time will be different!

Help!

Let her go!
WHA--?

COME ON, LET'S GO.
REIJI!

HEH! STOP RIGHT THERE.

JUST WHO DO YOU THINK YOU ARE?!

STAY AWAY FROM HER, ALL OF YOU!

HAHA, LOOK! HE THINKS HE'S A HERO!

LET'S GET HIM!
RUN!

BE CAREFUL, REIJI!

OOPH

YOU LITTLE...

THUMP

STOP IT!

PLEASE!

LET ME GO!

STUBBORN, HUH?

ENOUGH!

GRAB
Attacking my little sister, are we?

Tetsuo!

I don’t like excessive violence... but in attacking Misa, you have given me little choice...

It’s Ichinose!

The Hanamichi Hammer!

Mommy!

Run!

Sensei?

He’s out cold.
REIJI?

REIJI!

REIJI, WAKE UP.

YOU'RE OKAY!

WHOA!

YURINO...MISA TOLD ME WHAT HAPPENED.

THANK YOU.

UM...NO PROBLEM.

BUT I DON'T DESERVE YOUR THANKS...
I couldn't help Misa...I couldn't even help myself...

I haven't changed at all! I'm still a weakling!

Well, you may not be a black belt yet...

But you're definitely no weakling.

Putting Misa's safety before your own shows great courage. That kind of courage is admirable, even though the fight itself was unnecessary.

He's right.

I don't know what to say... thank you.

You should be proud!

But--

Reiji!

Misa...
Thank you for everything!

Ah...

I thought I was pretty clear about the rules...

Huh?!

I, uh...

Heh...

Well, I guess it's okay... Misa's not a kid anymore.

Thanks, Sensei...

By the way, would you consider doing me another favor?

S-sure.
I'd like you to teach me, too.

Math, I mean.

What?

He could really use the help, being in his sixth year and all.

If he doesn't graduate soon...

So, what do you say?

Sure! Of course!

Great! Let's start off with plus and minus, then!

Um...

Plus and minus?

It'd mean a lot to me, too.

Sounds like you'll need more lunches!
ONLINE RESOURCES

THE APPENDIXES

The appendixes for *The Manga Guide to Linear Algebra* can be found online at [http://www.nostarch.com/linearalgebra](http://www.nostarch.com/linearalgebra). They include:

- Appendix A: Workbook
- Appendix B: Vector Spaces
- Appendix C: Dot Product
- Appendix D: Cross Product
- Appendix E: Useful Properties of Determinants

UPDATES

Visit [http://www.nostarch.com/linearalgebra](http://www.nostarch.com/linearalgebra) for updates, errata, and other information.
I still don't get it!

No need to get violent...

You can do it, bro!

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