Homework 9

15.6

5. $f_x(x,y) = \dfrac{5}{2\sqrt{5x-4y}} = \dfrac{5}{8}$ at $(4,1)$.

$f_y(x,y) = -\dfrac{4}{2\sqrt{5x-4y}} = -\dfrac{1}{2}$ at $(4,1)$.

$$D_u f = f_x \cos\left(-\tfrac{\pi}{6}\right) + f_y \sin\left(-\tfrac{\pi}{6}\right)$$
$$= \frac{5}{8}\cdot\frac{\sqrt{3}}{2} + -\frac{1}{2}\cdot -\frac{1}{2} = \frac{5\sqrt{3}}{16} + \frac{1}{4} = \frac{5\sqrt{3}+4}{16}.$$

8. (a) $f_x = \frac{y}{x}$, $f_y = \ln x \Rightarrow \nabla f = \left(\frac{y}{x}, \ln x\right)$.

(b) $\nabla f(1,-3) = (-3, 0)$.

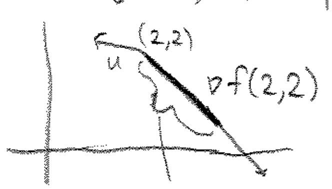
(c) $D_u f = (-3,0)\cdot\left(-\frac{4}{5}, \frac{3}{5}\right) = \frac{12}{5}$.

14. $g_r = -e^{-r}\sin\theta$, $g_\theta = e^{-r}\cos\theta$. At $(0, \frac{\pi}{3})$, $g_r = -\frac{\sqrt{3}}{2}$, $g_\theta = \frac{1}{2}$.

$$\frac{v}{\|v\|} = \frac{1}{\sqrt{3^2+2^2}}(3,-2) = \frac{1}{\sqrt{14}}(3,-2)$$
$$D_{v/\|v\|}g = \frac{3}{\sqrt{14}}\cdot -\frac{\sqrt{3}}{2} + -\frac{2}{\sqrt{14}}\cdot\frac{1}{2} = \frac{-3\sqrt{3}-2}{2\sqrt{14}}.$$

18. The angle looks to be about $\frac{3\pi}{4}$.

Then $D_u f(2,2) = |\nabla f|\cos\theta = |\nabla f|\left(-\frac{\sqrt{2}}{2}\right)$.



length $= \frac{\sqrt{2}}{2}|\nabla f|$.

27. (a) $D_u f = |\nabla f|\cos\theta \geq -|\nabla f|$ with equality when $\theta = \pi$.

This corresponds to the direction of $-\nabla f$.

(b) $\nabla f = (4x^3y - 2xy^3,\ x^4 - 3x^2y^2) = (-96+108,\ 16-96) = (12,-80)$

So $-\nabla f = (-12, 80)$, the direction of fastest descent.

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39. $x^2+2y^2+3z^2=21 \quad (4,-1,1)$

(a) $F(x,y,z) = x^2+2y^2+3z^2$.

$F_x(x,y,z) = 2x, \; F_y = 4y, \; F_z = 6z$.

At $(4,-1,1)$, $F_x = 8, F_y = -4, F_z = 6$.

So the tangent plane is $8(x-4)-4(y+1)+6(z-1)=0$.

(b) The symmetric equations of the normal line are

$$\frac{x-4}{8} = \frac{y+1}{-4} = \frac{z-1}{6}.$$

32. $T(x,y,z) = 200e^{-x^2-3y^2-9z^2}$.

(a) $\frac{\partial T}{\partial x} = -400xe^{-x^2-3y^2-9z^2} = -800e^{-43}$ at $(2,-1,2)$

$\frac{\partial T}{\partial y} = -600ye^{-x^2-3y^2-9z^2} = 600e^{-43}$ at $(2,-1,2)$

$\frac{\partial T}{\partial z} = -1800ze^{-x^2-3y^2-9z^2} = -3600e^{-43}$ at $(2,-1,2)$.

$D_uT = (-800e^{-43}, 600e^{-43}, -3600e^{-43}) \cdot \frac{(3,-3,3)}{\sqrt{27}}$

$= -\frac{13800}{3\sqrt{3}}e^{-43} = -\frac{4600}{\sqrt{3}}e^{-43}$.

(b) $\nabla T = 200e^{-43}(-4,3,-18)$, so the temperature increases fastest in the direction of $(-4,3,-18)$.

(c) The rate is given by $|\nabla T| = 200e^{-43}\sqrt{4^2+3^2+18^2}$

$= 200e^{-43}\sqrt{349}$

40. (a) $F(x,y,z) = -x+y^2+z^2$. $\quad F_x=-1, \; F_y=-2y=-2, \; F_z=2z=0$ at $(-1,1,0)$

So the tangent plane is $-(x+1)-2(y-1)=0$.

(b) The symmetric equations of the normal line are

$$\frac{x+1}{-1} = \frac{y-1}{-2}, \quad z=0$$

41. (a) $F(x,y,z) = x^2-2y^2+z^2+yz$. $\quad F_x = 2x = 4, \; F_y = -4y+z = -5, \; F_z = 2z+y = -1$ at $(2,1,-1)$.

The tangent plane is $4(x-2)-5(y-1)-(z+1)=0$.

(b) The symmetric equations of the normal line are

$$\frac{x-2}{4} = \frac{y-1}{-5} = \frac{z+1}{-1}.$$

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43. (a) $F(x,y,z) = xe^y\cos z - z$. $(1,0,0)$

$F_x = e^y\cos z = 1$ at $(1,0,0)$.

$F_y = xe^y\cos z = 1$ at $(1,0,0)$.

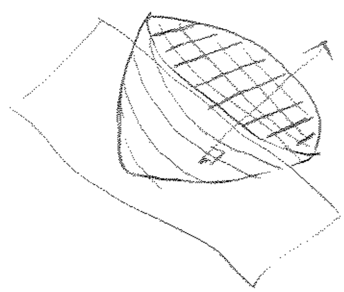
$F_z = -xe^y\sin z - 1 = -1$ at $(1,0,0)$.

Tangent plane: $(x-1)+(y-0)-(z-0)=0$.

(b) Normal line:

$$\frac{x-1}{1} = \frac{y-0}{1} = \frac{z-0}{-1}$$

46.



Sorry, not a very good picture. But basically it's a hyperboloid (it has four sheets, but only one is displayed here).

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4. It looks like $(-1,1)$, $(-1,-1)$, $(1,0)$ are critical points since the circular level curves get smaller and smaller around these points. $(-1,1)$ is likely a minimum since the level curves are decreasing and $(1,0)$ is a maximum since the level curves are increasing. $(-1,-1)$ is then probably a saddle point since no values are specified.

5. $f_x = -2-2x = 0 \Rightarrow x=-1$. $f_y = 4-8y=0 \Rightarrow y=\frac{1}{2}$.

$f_{xx} = -2 < 0$. $f_{yy} = -8$. $f_{xy} = 0$. $D = -2\cdot -8 = 16 > 0$.

So $(-1,\frac{1}{2})$ is a local maximum.

6. $f_x = 3x^2y+24x = 0 \Rightarrow x=0$ or $3xy+24=0$.

$f_y = x^3-8=0 \Rightarrow x=2$ or $x^2+2x+4=0$, which has no solutions.

So $x=2$, $3xy+24=0 \Rightarrow (2,-4)$ is a critical point.

$f_{xx} = 6xy+24 = -24 < 0$. $f_{yy} = 0$. $f_{xy} = 3x^2 = 12$.

$D<0$, so $(2,-4)$ is a saddle point.

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12. $f_x = y - 2xy - y^2 = 0$. $f_y = x - x^2 - 2xy = 0$

$\Rightarrow y = 0$ or $1 - 2x - y = 0$, $x = 0$ or $1 - 2y - x = 0$.

$x = 0 \Rightarrow y = 1$. $y = 0 \Rightarrow x = 1$.

$1 - 2x - y = 0$, $1 - 2y - x = 0 \Rightarrow y = 1 - 2x$, $1 - 2y - x = 0$

$\Rightarrow 1 - 2(1 - 2x) - x = 0 \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$, $y = \frac{1}{3}$.

The critical points are $(0,1), (1,0), (\frac{1}{3}, \frac{1}{3})$.

$f_{xx} = -2y$, $f_{xy} = -1 - 2x - 2y$, $f_{yy} = -2x$

$D = (-2x)(-2y) - (-1 - 2x - 2y)^2 = 4xy - (1 + 2x + 2y)^2$.

For $(0,1)$, $D = -9 < 0 \Rightarrow$ saddle point.

For $(1,0)$, $D = -9 < 0 \Rightarrow$ saddle point.

For $(\frac{1}{3}, \frac{1}{3})$, $D = \frac{4}{9} - \frac{25}{9} = \frac{-21}{9} < 0 \Rightarrow$ saddle point.

14. $f_x = 2x - 2x^{-3}y^{-2} = 0$. $f_y = 2y - 2x^{-2}y^{-3} = 0$

$\Rightarrow x^4y^2 = 1$, $x^2y^4 = 1 \Rightarrow \frac{1}{x^6} = 1 \Rightarrow x = 1, -1$. Also, $y = 1, -1$.

So $(1,1), (1,-1), (-1,1), (-1,-1)$ are the critical points.

$f_{xx} = 2 + 6x^{-4}y^{-2}$. $f_{xy} = 4x^{-3}y^{-3}$. $f_{yy} = 2 + 6x^{-2}y^{-4}$

$D = [2 + 6(\pm 1)^{-4}(\pm 1)^{-2}][2 + 6(\pm 1)^{-2}(\pm 1)^{-4}] - [4(\pm 1)^{3}(\pm 1)^{3}]^2$

$= 50 > 0$.

$f_{xx} = 2 + 6(\pm 1)^{-4}(\pm 1)^{-2} = 8 > 0$,

so we have local minimums for each of $(1,1), (1,-1), (-1,1), (-1,-1)$

18. $f_x = 2xye^{-x^2-y^2} - 2x^3ye^{-x^2-y^2} = 0$, $f_y = x^2e^{-x^2-y^2} - 2x^2y^2e^{-x^2-y^2} = 0$

$\Rightarrow x = 0$ or $y = 0$, $x = \pm 1$; $x = 0$ or $\frac{1}{2} = y^2$.

$x = 0 \Rightarrow y$ is arbitrary. $y = 0 \Rightarrow x = 0$. $x = \pm 1 \Rightarrow y = \pm\sqrt{\frac{1}{2}}$.

So $(0, y)$, $(\pm 1, \pm\sqrt{\frac{1}{2}})$ are the critical points.

$f_{xx} = 2ye^{-x^2-y^2} - 4x^2ye^{-x^2y^2} - 6x^2ye^{-x^2-y^2} + 4x^3ye^{-x^2-y^2}$

$= (2y - 10x^2y + 4x^3y)e^{-x^2-y^2}$

$f_{xy} = 2xe^{-x^2-y^2} - 4xy^2e^{-x^2-y^2} - 2x^3e^{-x^2-y^2} + 4x^3y^2e^{-x^2-y^2}$

$= (2x - 4xy^2 - 2x^3 + 4x^3y^2)e^{-x^2-y^2}$

$f_{yy} = -2x^2ye^{-x^2-y^2} - 4x^2ye^{-x^2-y^2} + 4x^2y^3e^{-x^2-y^2}$

$= (-6x^2y + 4x^2y^3)e^{-x^2-y^2}$

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18. (continued)

$x=0 \Rightarrow f_{xx} = 2ye^{-y^2},\ f_{xy} = 0,\ f_{yy} = 0 \Rightarrow D=0.$

$x=1, y=\sqrt{\frac{1}{2}} \Rightarrow f_{xx} = -4\sqrt{\frac{1}{2}}\,e^{-3/2},\ f_{xy} = 0,\ f_{yy} = -4\sqrt{\frac{1}{2}}\,e^{-3/2}$

$\Rightarrow D = 8e^{-3} > 0,\ f_{xx} < 0 \Rightarrow$ local max.

$x=1, y=-\sqrt{\frac{1}{2}} \Rightarrow f_{xx} = 4\sqrt{\frac{1}{2}}\,e^{-3/2},\ f_{xy} = 0,\ f_{yy} = 4\sqrt{\frac{1}{2}}\,e^{-3/2}$

$\Rightarrow D = 8e^{-3} > 0,\ f_{xx} > 0 \Rightarrow$ local min.

$x=-1, y=\sqrt{\frac{1}{2}} \Rightarrow f_{xx} = -12\sqrt{\frac{1}{2}}\,e^{-3/2},\ f_{xy} = 0,\ f_{yy} = -4\sqrt{\frac{1}{2}}\,e^{-3/2}$

$\Rightarrow D = 24e^{-3} > 0,\ f_{xx} < 0 \Rightarrow$ local max.

$x=-1, y=-\sqrt{\frac{1}{2}} \Rightarrow f_{xx} = 12\sqrt{\frac{1}{2}}\,e^{-3/2},\ f_{xy} = 0,\ f_{yy} = 4\sqrt{\frac{1}{2}}\,e^{-3/2}$

$\Rightarrow D = 24e^{-3} > 0,\ f_{xx} > 0 \Rightarrow$ local min.

So we've taken care of all the critical points except for $x=0$, y arbitrary. In this case $f(0,y) = 0$.

For $x=0$, $y>0$, $x^2ye^{-x^2-y^2} \geq 0$ around a neighborhood of $(0,y)$, so $x=0$, $y>0$ is a local minimum.

For $x=0$, $y<0$, $x^2ye^{-x^2-y^2} \leq 0$ around a neighborhood of $(0,y)$, so $x=0$, $y<0$ is a local maximum.

For $x=0$, $y=0$, directions of increasing y, increasing (or decreasing) x give an increase in f, while decreasing y, increasing (or decreasing x) give a decrease in f, so $x=0, y=0$ is a saddle point.