Homework 7

15.3

14. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4$

$\frac{\partial f}{\partial x} = 5x^4 + 9x^2y^2 + 3y^4$. $\frac{\partial f}{\partial y} = 6x^3y + 12xy^3$.

18. $f(x,y) = x^y$

$\frac{\partial f}{\partial x} = yx^{y-1}$. $\frac{\partial f}{\partial y} = (\ln x)x^y$.

26. $f(x,y,z) = x^2e^{yz}$.

$\frac{\partial f}{\partial x} = 2xe^{yz}$. $\frac{\partial f}{\partial y} = x^2ze^{yz}$. $\frac{\partial f}{\partial z} = x^2ye^{yz}$.

37. $f(x,y,z) = \frac{x}{y+z}$.

$f_z(x,y,z) = -\frac{x}{(y+z)^2}$. $f_z(3,2,1) = -\frac{3}{3^2} = -\frac{1}{3}$.

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44. $\sin(xyz) = x + 2y + 3z$.

Keep y constant: $yz\cos(xyz) + xy\cos(xyz)\frac{\partial z}{\partial x} = 1 + 3\frac{\partial z}{\partial x}$

$$\Rightarrow \left[xy\cos(xyz) - 3\right]\frac{\partial z}{\partial x} = 1 - yz\cos(xyz)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}.$$

Keep x constant: $xz\cos(xyz) + xy\cos(xyz)\frac{\partial z}{\partial y} = 2 + 3\frac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial z}{\partial y}\left[xy\cos(xyz) - 3\right] = 2 - xz\cos(xyz)$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}.$$

52. $\frac{\partial v}{\partial x} = \frac{1}{2\sqrt{x+y^2}}$. $\frac{\partial v}{\partial y} = \frac{2y}{2\sqrt{x+y^2}} = \frac{y}{\sqrt{x+y^2}}$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{2}[x+y^2]^{-1/2}\right) = -\frac{1}{4}(x+y^2)^{-3/2}$$

$$\frac{\partial^2 v}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{1}{2}(x+y^2)^{-1/2}\right) = -\frac{1}{2}y(x+y^2)^{-3/2}.$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x+y^2}}\right) = \frac{\sqrt{x+y^2}\cdot 1 - y\cdot\frac{y}{\sqrt{x+y^2}}}{x+y^2} = \frac{\frac{x+y^2-y^2}{\sqrt{x+y^2}}}{x+y^2}$$

$$= \frac{x}{(x+y^2)^{3/2}}.$$

53. $u_x = \sin(x+2y) + x\cos(x+2y)$. $u_y = 2x\cos(x+2y)$.

$u_{xy} = (u_x)_y = 2\cos(x+2y) - 2x\sin(x+2y)$.

$u_{yx} = (u_y)_x = 2\cos(x+2y) - 2x\sin(x+2y)$.

68. (a) $u_x = 2x$, $u_y = 2y \Rightarrow u_{xx} = 2$, $u_{yy} = 2 \Rightarrow u_{xx} + u_{yy} = 4$. No.

(b) $u_x = 2x$, $u_y = -2y \Rightarrow u_{xx} = 2$, $u_{yy} = -2 \Rightarrow u_{xx} + u_{yy} = 0$. Yes.

(c) $u_x = 3x^2 + 3y^2$, $u_y = 6xy \Rightarrow u_{xx} = 6x$, $u_{yy} = 6x \Rightarrow u_{xx} + u_{yy} = 12x$. No.

(d) $u = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$. $u_x = \frac{x}{x^2+y^2}$, $u_y = \frac{y}{x^2+y^2}$

$$\Rightarrow u_{xx} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}. \quad u_{yy} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\Rightarrow u_{xx} + u_{yy} = 0$. Yes.

(e) $u_x = \cos x\cosh y - \sin x\sinh y$, $u_y = \sin x\sinh y + \cos x\cosh y$

$\Rightarrow u_{xx} = -\sin x\cosh y - \cos x\sinh y$. $u_{yy} = \sin x\cosh y + \cos x\sinh y$

$\Rightarrow u_{xx} + u_{yy} = 0$. Yes.

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68. (f) $u_x = -e^{-x}\cos y + e^{-y}\sin x$, $u_y = -e^{-x}\sin y + e^{-y}\cos x$

$u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, $u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

$u_{xx} + u_{yy} = 0$. Yes.

80. $W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$

We need to evaluate $\frac{\partial W}{\partial T}$ and $\frac{\partial W}{\partial v}$ at $T = -15$, $v = 30$.

$\frac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16} \approx 1.3048$ for $v = 30$.

$\frac{\partial W}{\partial v} = (-11.37 + 0.3965T)\cdot 0.16v^{-0.84} \approx -0.1592$.

So if the actual temperature drops by 1, the apparent temperature should drop by $\frac{\partial W}{\partial T} \approx 1.3048°C$.

If the wind speed increases by 1 km/h, the apparent temperature should drop by $\frac{\partial W}{\partial v} \approx 0.1592°C$.

82. Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$.

$2a = -2bc(-\sin A)\frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc\sin A}$.

$0 = 2b - 2c\cos A + 2bc\sin A\frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{2c\cos A - 2b}{2bc\sin A}$

$\Rightarrow \frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$.

$0 = 2c - 2b\cos A + 2bc\sin A\frac{\partial A}{\partial c} \Rightarrow \frac{\partial A}{\partial c} = \frac{b\cos A - c}{bc\sin A}$.

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4. $\frac{\partial z}{\partial x} = \frac{y}{x}$. At $(1,4,0)$, $= 4$. $\frac{\partial z}{\partial y} = \ln x$. At $(1,4,0)$, $= 0$.

So $z - 0 = 4(x-1) + 0(y-4) \Rightarrow z = 4x - 4$.

5. $\frac{\partial z}{\partial x} = -y\sin(x-y)$. At $(2,2,2)$, $= 0$. $\frac{\partial z}{\partial y} = \cos(x-y) - y\sin(x-y)$. At $(2,2,2)$, $= 1$.

So $z - 2 = 0(x-2) + 1(y-2) \Rightarrow z - 2 = y - 2 \Rightarrow y = z$.

13. $f_x = e^x\cos xy - ye^x\sin xy$, which exists everywhere and is continuous as the sum and product of continuous functions.

Similarly, $f_y = -xe^y\sin xy$ exists and is continuous.

Theorem 8 $\Rightarrow$ f is differentiable.

$$
\begin{aligned}
L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\
&= 1 + x + 0 \\
&= 1 + x.
\end{aligned}
$$

Homework 8

15.4

14. $f_x = \dfrac{1}{2\sqrt{x+e^{4y}}}$, which exists and is continuous at $(3,0)$.

$f_y = \dfrac{4e^{4y}}{2\sqrt{x+e^{4y}}} = \dfrac{2e^{4y}}{\sqrt{x+e^{4y}}}$, which exists and is continuous at $(3,0)$.

By Theorem 8, f is differentiable.

$$L(x,y) = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$$
$$= 2 + \tfrac{1}{4}(x-3) + y$$
$$= \tfrac{1}{4}x + y + \tfrac{5}{4}.$$

17. $f_x = -\dfrac{x}{\sqrt{20-x^2-7y^2}} = -\dfrac{2}{3}$ at $(2,1)$.

$f_y = -\dfrac{7y}{\sqrt{20-x^2-7y^2}} = -\dfrac{7}{3}$ at $(2,1)$

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(y-1)$$
$$= 3 + -\tfrac{2}{3}(x-2) + -\tfrac{7}{3}(y-1)$$
$$= -\tfrac{2}{3}x - \tfrac{7}{3}y + \tfrac{20}{3}.$$
$$L(1.95, 1.08) = -\tfrac{2}{3}\cdot 1.95 - \tfrac{7}{3}\cdot 1.08 + \tfrac{20}{3} = \tfrac{427}{150}.$$

28. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz$

$$= (ye^{xz} + xyze^{xz})dx + xe^{xz}dy + x^2ye^{xz}dz$$

32. $A(x,y,z) = 2xy + 2xz + 2yz$. $\quad \dfrac{\partial A}{\partial x} = 2y+2z$. $\quad \dfrac{\partial A}{\partial y} = 2x+2z$.

$dA = \dfrac{\partial A}{\partial x}dx + \dfrac{\partial A}{\partial y}dy + \dfrac{\partial A}{\partial z}dz$. $\quad \dfrac{\partial A}{\partial z} = 2x+2y$.

$$= 2(60+50)\cdot 0.2 + 2(80+50)\cdot 0.2 + 2(80+60)\cdot 0.2$$
$$= 44 + 52 + 56$$
$$= 152 \text{ cm}^2.$$

33. $V(r,h) = \pi r^2 h$. $\quad \dfrac{\partial V}{\partial r} = 2\pi r h$. $\quad \dfrac{\partial V}{\partial h} = \pi r^2$.

$dV = \dfrac{\partial V}{\partial r}dr + \dfrac{\partial V}{\partial h}dh = 2\pi\cdot 4\cdot 12\cdot 0.04 + \pi\cdot 4^2\cdot 0$

$$= 3.84\pi \text{ cm}^3.$$

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34. As in #33, $V(r,h) = \pi r^2 h$, $\frac{\partial V}{\partial r} = 2\pi r h$, $\frac{\partial V}{\partial h} = \pi r^2$.

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = 2\pi \cdot 2 \cdot 10 \cdot 0.05 + \pi \cdot 2^2 \cdot 0.2$$
$$= 2.8\pi \text{ cm}^3.$$

36. $P = \frac{8.31T}{V}$. $\frac{\partial P}{\partial T} = \frac{8.31}{V}$, $\frac{\partial P}{\partial V} = -\frac{8.31T}{V^2}$.

$$dP = \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV = \frac{8.31}{12}(-5) + -\frac{8.31(310)}{12^2}(0.3)$$
$$= -\frac{8.31 \cdot 152}{144} = -8\frac{463}{600} \text{ kiloPascals.}$$

38. $P = a_1 a_2 a_3 a_4$. $\frac{\partial P}{\partial a_1} = a_2 a_3 a_4, \ldots, \frac{\partial P}{\partial a_4} = a_1 a_2 a_3$.

$$dP = \frac{\partial P}{\partial a_1}da_1 + \frac{\partial P}{\partial a_2}da_2 + \frac{\partial P}{\partial a_3}da_3 + \frac{\partial P}{\partial a_4}da_4$$
$$\leq 50 \cdot 50 \cdot 50 \cdot 0.05 + 50 \cdot 50 \cdot 50 \cdot 0.05 + 50 \cdot 50 \cdot 50 \cdot 0.05 + 50 \cdot 50 \cdot 50 \cdot 0.05$$

since the maximum error will occur for $a_1 = a_2 = a_3 = a_4 = 50$ and the largest error in rounding to the first decimal place is 0.05

$$= 6250.$$

15.5

2. $z = \sqrt{x^2+y^2}$. $x = e^{2t}$. $y = e^{-2t}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y^2}} \cdot 2e^{2t} + \frac{y}{\sqrt{x^2+y^2}} \cdot -2e^{-2t}$$
$$= \frac{2}{\sqrt{x^2+y^2}}\left(xe^{2t} - ye^{-2t}\right).$$

8. $z = \frac{x}{y}$. $x = se^t$, $y = 1 + se^{-t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{y}e^t + -\frac{x}{y^2} \cdot e^{-t}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}se^t + -\frac{x}{y^2} \cdot -se^{-t} = \frac{1}{y}se^t + \frac{x}{y^2}se^{-t}.$$

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24. $M = xe^{y-z^2}$, $x = 2uv$, $y = u-v$, $z = u+v$.

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u}$$
$$= e^{y-z^2}\cdot 2v + xe^{y-z^2}\cdot 1 + -2xze^{y-z^2}\cdot 1$$
$$\frac{\partial M}{\partial v} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v}$$
$$= e^{y-z^2}\cdot 2u + xe^{y-z^2}\cdot -1 + -2xze^{y-z^2}\cdot 1.$$

When $u=3$, $v=-1$, we have $x=-6$, $y=4$, $z=2$.

$$\frac{\partial M}{\partial u} = -2 + -6 + 24 = 16. \quad \frac{\partial M}{\partial v} = 6+6+24 = 36.$$

28. $F(x,y) = y^5 + x^2y^3 - 1 - ye^{x^2} = 0$.

$F_x = 2xy^3 - 2xye^{x^2}$. $\quad F_y = 5y^4 + 3x^2y^2 - e^{x^2}$.

$$\frac{dy}{dx} = -\frac{2xy^3 - 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

32. $F(x,y,z) = xyz - \cos(x+y+z) = 0$.

$\frac{\partial F}{\partial x} = yz + \sin(x+y+z)$. $\quad \frac{\partial F}{\partial y} = xz + \sin(x+y+z)$. $\quad \frac{\partial F}{\partial z} = xy + \sin(x+y+z)$

$$\frac{\partial z}{\partial x} = -\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}. \quad \frac{\partial z}{\partial y} = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}.$$

39. (a) $V(l,w,h) = lwh$. $\quad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$

$$= 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot -3 = 6 \text{ m}^3/\text{s}.$$

(b) $A(l,w,h) = 2lw + 2lh + 2wh$.

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial l}\frac{dl}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}$$
$$= 2(2+2)\cdot 2 + 2(1+2)\cdot 2 + 2(1+2)\cdot -3$$
$$= -26 \text{ m}^2/\text{s}.$$

$\frac{\partial A}{\partial l} = 2w+2h$

$\frac{\partial A}{\partial w} = 2l+2h$.

$\frac{\partial A}{\partial h} = 2l+2w$

(c) $L(l,w,h) = \sqrt{l^2+w^2+h^2}$.

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial l}\frac{dl}{dt} + \frac{\partial L}{\partial w}\frac{dw}{dt} + \frac{\partial L}{\partial h}\cdot\frac{dh}{dt}$$
$$= \frac{1}{3}\cdot 2 + \frac{2}{3}\cdot 2 + \frac{2}{3}\cdot -3$$
$$= 0 \text{ m/s}$$

$\frac{\partial L}{\partial l} = \frac{l}{\sqrt{l^2+w^2+h^2}}$

$\frac{\partial L}{\partial w} = \frac{w}{\sqrt{l^2+w^2+h^2}}$.

$\frac{\partial L}{\partial h} = \frac{h}{\sqrt{l^2+w^2+h^2}}$

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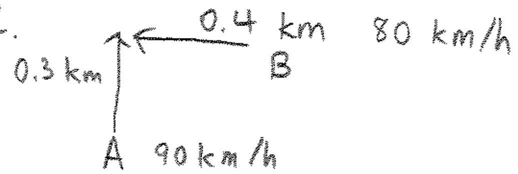
42.



$D = \sqrt{x^2+y^2}$, where

$x =$ distance of A to intersection

$y =$ distance of B to intersection.

$$\frac{\partial D}{\partial t} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt}$$

$$= \frac{3}{5}\cdot 90 + \frac{4}{5}\cdot 80$$

$$= 118 \text{ km/h}.$$

$$\frac{\partial D}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial D}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

45. $z = f(x-y)$. $\quad \frac{\partial z}{\partial x} = f'(x-y)$. $\quad \frac{\partial z}{\partial y} = -f'(x-y)$.

So $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(x-y) - f'(x-y) = 0$

50. $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta. \quad \frac{\partial x}{\partial \theta} = -r\sin\theta. \quad \frac{\partial y}{\partial r} = \sin\theta. \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

(a) $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= f_x\cos\theta + f_y\sin\theta.$$

(b) $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= f_x(-r\sin\theta) + f_y\cdot r\cos\theta.$$

(c) $$\frac{\partial^2 z}{\partial r\,\partial\theta} = \frac{\partial}{\partial r}\left[-f_x r\sin\theta + f_y r\cos\theta\right]$$

$$= -\left[\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right] r\sin\theta - f_x\sin\theta$$

$$+ \left[\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right] r\cos\theta + f_y\cos\theta$$

$$= -\left[f_{xx}\cos\theta + f_{xy}\sin\theta\right] r\sin\theta - f_x\sin\theta$$

$$+ \left[f_{xy}\cos\theta + f_{yy}\sin\theta\right] r\cos\theta + f_y\cos\theta$$