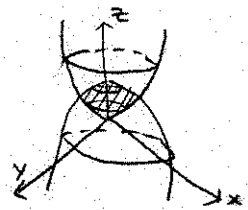
Homework 6

13.6

42.



45. The distance from (x,y,z) to $(-1,0,0)$ is $\sqrt{(x+1)^2+y^2+z^2}$ and the distance from (x,y,z) to the plane $x=1$ is just the distance from (x,y,z) to $(1,y,z)$ (the line connecting these two points is perpendicular to the plane).

So the distance is $x-1$.

Now if the two distances are equal,

$$x-1=\sqrt{(x+1)^2+y^2+z^2} \Rightarrow (x-1)^2=(x+1)^2+y^2+z^2$$

$$\Rightarrow 0 = 4x+y^2+z^2 \Rightarrow x = -\frac{y^2+z^2}{4},$$

an elliptic paraboloid.

15.2

7. Let $y=mx$. Then $\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+m^2x^2} = \frac{1}{1+m^2}$, and for different values of m, we get different values, so the limit does not exist.

8. Let $y=mx$. Then $\frac{x^2+\sin^2 y}{2x^2+y^2} = \frac{x^2+\sin^2(mx)}{2x^2+m^2x^2} = \frac{1+\frac{\sin^2(mx)}{x^2}}{2+m^2}$.

If we let $m=0$, as $x\to 0$, we get $\frac{1}{2}$. Letting $m=1$, we get $\lim_{x\to 0} \frac{1+\frac{\sin^2 x}{x^2}}{2+1} = \frac{1+1}{3} = \frac{2}{3}$ $\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$.

$\frac{1}{2}\neq\frac{2}{3} \Rightarrow$ the limit does not exist.

12. $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2}(x^2-y^2) = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0$.

13. Let $y=mx^2$. Then $\frac{2x^2y}{x^4+y^2} = \frac{2mx^4}{x^4+(mx^2)^2} = \frac{2m}{1+m^2}$, and taking $m=0$ gives 0 while $m=1$ gives 1. So the limit does not exist.

15.2

14. Let $x = my \Rightarrow \dfrac{x^2\sin^2 y}{x^2+2y^2} = \dfrac{m^2y^2\sin^2 y}{m^2y^2+2y^2} = \dfrac{m^2\sin^2 y}{m^2+2} \to 0$ as $y \to 0$.

Then we might suspect that the limit is 0.

We want $\delta > 0$ so that $\left|\dfrac{x^2\sin^2 y}{x^2+2y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

That is, $\dfrac{x^2\sin^2 y}{x^2+2y^2} < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

Now $x^2+2y^2 \geq x^2$, so $\dfrac{x^2\sin^2 y}{x^2+2y^2} \leq \dfrac{x^2\sin^2 y}{x^2} = \sin^2 y$.

If we choose $\delta \leq (\sin^{-1}\sqrt{\varepsilon})$, then $x^2+y^2 \leq (\sin^{-1}\sqrt{\varepsilon})^2$, so $y^2 \leq (\sin^{-1}\sqrt{\varepsilon})^2$

$\Rightarrow y \leq \sin^{-1}\sqrt{\varepsilon} \Rightarrow \sin y \leq \sqrt{\varepsilon} \Rightarrow \sin^2 y < \varepsilon$.

Therefore, $\dfrac{x^2\sin^2 y}{x^2+2y^2} < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta \leq \sin^{-1}\sqrt{\varepsilon}$.

15. We also show that the limit here is 0.

Make the substitution $x = r\cos\theta$, $y = r\sin\theta$. Then proving that the limit is 0 is equivalent to showing that the limit is 0 as $r \to 0$ (this covers all directions).

$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{r^2}{\sqrt{r^2+1}-1} = \frac{r}{\sqrt{1+1/r^2}-1/r}$$

and as $r \to 0$, $\frac{1}{r^2}, \frac{1}{r} \to 0$, so $\sqrt{1+\frac{1}{r^2}} - \frac{1}{r} \to 1$, so $\dfrac{r}{\sqrt{1+1/r^2}-1/r} \to 0$.

29. arctan is a continuous function defined everywhere, so we only need to consider when $x+\sqrt{y}$ is continuous. This is true for $x \in (-\infty, \infty)$, $y \in [0, \infty)$.

35. The function $\dfrac{x^2y^3}{2x^2+y^2}$ is defined everywhere and continuous except at $(0,0)$.

To determine whether f is continuous at $(0,0)$, we only need to check whether $\dfrac{x^2y^3}{2x^2+y^2} \to 0$ as $(x,y) \to 0$.

If we let $x = r\cos\theta$, $y = r\sqrt{2}\sin\theta$,

$$\frac{x^2y^3}{2x^2+y^2} = \frac{r^2\cos^2\theta \cdot 2\sqrt{2}r^3\sin^3\theta}{2r^2\cos^2\theta+2r^2\sin^2\theta} = \sqrt{2}r^3\cos^2\theta\sin^3\theta \to 0 \text{ as } r \to 0.$$

So the limit exists, but it is equal to $0 \neq 1$.

Therefore, $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$.