Garnett MATH 32A - PRACTICE FINAL EXAM March 8, 2006.

Name:(Last, First)____________________________________

Signature ____________________________________

Quiz Section (or Time and TA's Name) ____________________________________

Answers on exam. No books or notes. You may remove the scratch paper at the end of your exam. (n) means the problem is worth n points.

Solutions

(15) 1. Let

$$\vec{a} = \vec{j} - \vec{k}$$

and

$$\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}.$$

Find a vector $\vec{u}$ such that

(i) $\vec{u} = c\vec{b}$ for some scalar c, and

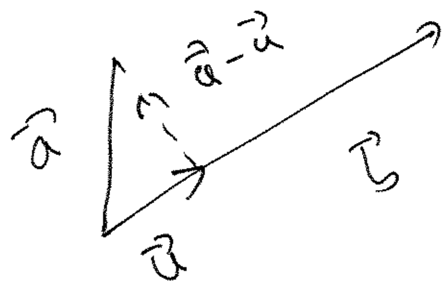
(ii) $\vec{a} - \vec{u} \perp \vec{b}$,

where $\perp$ means orthogonal or perpendicular.

Draw a picture illustrating the relations between $\vec{a}, \vec{b}$ and $\vec{u}$.

$$\vec{u} = \mathrm{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}$$

$$= \frac{-1}{9}\left(2\vec{i} + \vec{j} + 2\vec{k}\right)$$



(15) 2. Suppose that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. Find the following:

(a) $(\vec{u} \times \vec{v}) \cdot \vec{w}$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 2.$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w}) = 2$$

(2 interchanges of rows)

(b) $\vec{u} \cdot (\vec{w} \times \vec{v})$.

$$\vec{u} \cdot (\vec{w} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{w}) = -2.$$

(c) $\vec{v} \cdot (\vec{u} \times \vec{w})$. $=$

$$\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = -2.$$

(interchange 2 rows)

(d) $(\vec{u} \times \vec{v}) \cdot \vec{v}$. $= 0$

$\vec{u} \times \vec{v} \perp \vec{v}$

(15) 3. Find the length of the curve

$$\vec{r}(t) = (\sin 2t, 2t^{\frac{3}{2}}, \cos 2t)$$

for $0 \le t \le 1$.

$$\vec{r}\,'(t) = \langle 2\cos 2t, \; t^{1/2}, \; -2\sin 2t \rangle.$$

$$\text{length} = \int_0^1 |\vec{r}\,'(t)| \, dt$$

$$= \int_0^1 \sqrt{4 + t} \; dt$$

$$= \frac{2}{3}(4+t)^{3/2} \Big|_0^1 = \frac{2}{3} \cdot 5^{3/2}.$$

$$= \frac{20}{3}\sqrt{5}.$$

(10) 4. Suppose a particle in 3-space moves with motion $\vec{r}(t)$, velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$. Assume that the speed $|\vec{v}(t)|$ is constant. Prove that

$$\vec{v}(t) \cdot \vec{a}(t) = 0.$$

$\vec{v}(t) \cdot \vec{v}(t) = |\vec{v}(t)|^2$ is constant.

$\therefore \quad \frac{d}{dt} \vec{v}(t) \cdot \vec{v}(t) = 0$

But by the product rule

$$\frac{d}{dt} \vec{v}(t) \cdot \vec{v}(t)$$

$$= \vec{v}(t) \cdot \vec{a}(t) + \vec{a}(t) \cdot \vec{v}(t)$$

$$= 2 \vec{v}(t) \cdot \vec{a}(t) = 0$$

(35) 5. Let $\vec{r}(t) = t\vec{i} + \frac{t^2}{2}\vec{j} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\vec{k}$, for $1 \le t \le 4$.

(a) Find the velocity vector $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ and the speed $\frac{ds}{dt} = |\vec{v}(t)|$.

$$\vec{v}(t) = \vec{i} + t\vec{j} + \sqrt{2}\, t^{1/2}\,\vec{k}$$

$$\frac{ds}{dt} = \sqrt{1 + t^2 + 2t} = 1+t.$$

(b) Find the the unit tangent vector $T(t)$.

$$T(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{1+t}\left(\vec{i} + t\vec{j} + \sqrt{2}\,t^{1/2}\vec{k}\right)$$

(c) Find the derivative $T'(t)$ and its length $|T'(t)|$. Calculate carefully here, you will need these results below.

$$T'(t) = \frac{-1}{(1+t)^2}\vec{i} + \frac{1}{(1+t)^2}\vec{j} + \frac{1}{\sqrt{2}}\frac{1}{(1+t)^2}\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)\vec{k}$$

$$|T'(t)| = \frac{1}{(1+t)^2}\sqrt{1 + \cancel{t^2} + \frac{1}{2}\left(\frac{1}{t} - 2 + t\right)}$$

(d) Find the curvature $\kappa(t)$.

$$|T'(t)| = \frac{1}{\sqrt{2}}\frac{1}{(1+t)^2}\sqrt{t+2+\frac{1}{t}}$$

$$= \frac{\sqrt{t} + \frac{1}{\sqrt{t}}}{\sqrt{2}(1+t)^2}.$$

$$\kappa(t) = \frac{T'(t)}{ds/dt} = \frac{\sqrt{t} + 1/\sqrt{t}}{\sqrt{2}(1+t)^3}.$$

(e) Find the normal vector $N(t)$. $= \dfrac{T'(t)}{|T'(t)|} =$

$$\frac{\sqrt{2}}{\sqrt{t}+\sqrt{\frac{1}{t}}}\left(-\vec{i}+\vec{j}+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{t}}-\sqrt{t}\right)\vec{k}\right)$$

(f) For $t = 1$ find the binormal vector $B(1)$.

$$B(1) = T(1) \times N(1) = \left(\frac{\vec{i}}{2}+\frac{\vec{j}}{2}+\frac{1}{\sqrt{2}}\vec{k}\right) \times$$

$$\times \sqrt{2}\left(-\vec{i}+\vec{j}+0\vec{k}\right) = -\vec{i}+\vec{j}+\sqrt{2}\,\vec{k}$$

(g) For $t = 1$ find the equation of the osculating plane for the curve $\vec{r}(t)$.

$$\vec{r}(1) = \vec{i}+\frac{\vec{j}}{2}+\frac{2\sqrt{2}}{3}\vec{k}$$

osculating plane is $\perp$ $\vec{B}(1)$ and through $\vec{r}(t)$.

plane is $(-1)(x-1)+1(y-1/2)$

$$+\sqrt{2}\left(z-\frac{2\sqrt{2}}{3}\right)=0.$$

(20) 6. Let

$$f(x,y) = \sqrt{x^2+y^2+z^2}.$$

(a) Find the partial derivatives $f_x(x,y,z)$, $f_y(x,y,z)$ and $f_z(x,y,z)$.

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2+z^2}},$$

$$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad \vec{\nabla} f(x,y,z) = \frac{(x,y,z)}{|(x,y,z)|}.$$

(b) Find the linear approximation of $f(x,y,z)$ at the point $(3,2,6)$.

$$L(x,y,z) = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6)$$

(c) Use your answer to (b) to approximate $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}$.

$$7 + \frac{3}{7}(.02) + \frac{2}{7}(-(.03)) + \frac{6}{7}(.01)$$

$$= 7 + \frac{.06}{7}.$$

(25) 7. Find the equation of the tangent plane and normal line to the surface

$$xy = \ln(x + z)$$

at the point $(1, 0, 0)$.

$$f(x,y,z) = xy - \ln(x+z)$$

$$\nabla f = \left\langle y - \frac{1}{x+z}, x, -\frac{1}{x+z} \right\rangle.$$

$$\nabla f(1,0,0) = \langle -1, 1, -1 \rangle.$$

Tangent plane.

$$(-1)(x-1) + 1(y-0) - 1(z-0) = 0$$

Normal line:

$$\frac{x-1}{-1} = \frac{y}{1} = \frac{z}{-1}.$$

(25) 7'. (a) Give the $\epsilon - \delta$ defintion of

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

For all $\varepsilon > 0$ there is $\delta > 0$ such that if $\sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$.

(b) Let $f(x,y) = x^2 + y^2 - 1$. Prove using the definition in (a) that

$$\lim_{(x,y)\to(1,0)} f(x,y) = 0.$$

Fix $\varepsilon > 0$. Let $|x^2+y^2| < 1$. Then $|x| < 1$ and $|y| < 1$

$$|f(x,y)| \le |x^2-1| + |y|^2$$

$$\le |x-1||x+1| + |y|^2$$

Hence (because $|x| < 1$ and $|y| < 1$)

$$|f(x,y)| < |x-1| + |y|$$

Now let $\sqrt{(x-1)^2+y^2} <$

$$\min\left(\frac{\varepsilon}{2}, 1\right) = \delta$$

Then $|f(x,y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

(20) 8. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by

$$(x(t), y(t)) = (\sqrt{1+t}, 2 + \frac{t}{3})$$

where x and y are measured in centimeters. The temperature function $T(x, y)$ satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after $t = 3$ seconds?

By the chain rule,

$$T'(x(t), y(t)) = \frac{d}{dt} T(x(t), y(t))$$

$$= \frac{1}{2\sqrt{1+t}} T_x(x(t), y(t)) + \frac{1}{3} T_y(x(t), y(t)).$$

When $t = 3$, $(x(t), y(t)) = (2, 3)$

$$\cancel{T'(t)} \quad \left. \frac{d}{dt} T(x(t), y(t)) \right|_{t=3} =$$

$$\frac{1}{2 \cdot 2} \cdot T_x(2, 3) + \frac{1}{3} T_y(2, 3)$$

$$= \frac{4}{4} + \frac{3}{3} = 2 \text{ deg/sec}$$

(20) 8'. Near a buoy, the depth of a lake at the point (x, y) is $z = 200 - 0.02x^2 - 0.001y^3$, where x, y and z are measured in meters. A boat starts at $(80, 60)$ and moves in a straight line toward the buoy, which is at $(0, 0)$. Is the water under the boat getting deeper or shallower as the boat starts off? Calculate an explicit directional derivative to justify your answer.

$$\vec{u} = -\frac{\langle 80, 60\rangle}{|\langle 80, 60\rangle|} = -\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle.$$

(= ~~ve~~ unit vector in direction that boat is moving)

$$\nabla \vec{z} = -.04x\,\vec{i} - .003y^2\,\vec{j}.$$

$$\nabla \vec{z}\,(80,60) = -3.2\,\vec{i} + 10.8\,\vec{j}.$$

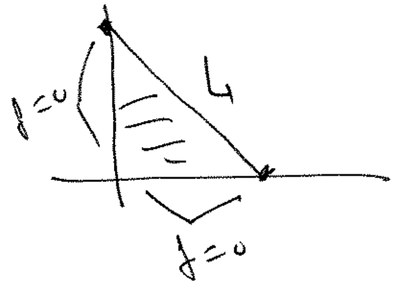
$$D_{\vec{u}}\vec{z}(80,60) = \left(\frac{4}{5}\right)(3.2) - \frac{3}{5}(10.8)$$

$$= \frac{13.8}{5} - \frac{32.4}{5} < 0.$$

Hence depth is <u>decreasing</u>.

(25) 9. On the closed triangle in the xy-plane with vertices $(0,0)$, $(6,0)$ and $(0,6)$ find the absolute maximum and minimum values of the function

$$f(x,y) = x^3y + x^2y^2 - 4x^2y.$$



$$\nabla f = \left(3x^2y + 2xy^2 - 8xy,\ x^3 + 2x^2y - 4x^2\right).$$

At a critical point

$$3x^2y + 2xy^2 - 8xy = 0 \quad \text{and}$$
$$x^3 + 2x^2y - 4x^2 = 0$$

Thus $(xy)(3x + 2y - 8) = 0$

and $x^2(x + 2y - 4) = 0$

All points $(0,y)$ are critical points, but not inside our triangle. If $x \neq 0$, then critical points at $y=0, x=4$, and $x=2,\ y=1$. At $(2,1)$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 4 & 8 \end{vmatrix} = 32. \quad \left(\begin{array}{l} D > 0 \\ f_{xx} > 0 \end{array} \Rightarrow \text{local } \cancel{\text{max}} \text{ min} \right)$$

$f(2,1) = -4.$

On L, $x+y=6$ and $f(x,y) = 2x^2(6-x)$ is largest at $x=4, y=2$.

Max $= f(4,2) = 64$ Min $= -4$.

(20) 10. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

$$g(x,y,z) = x + y + 2z$$

$$\nabla g = \langle 1, 1, 2 \rangle.$$

$$h(x,y,z) = x^2 + y^2 - z$$

$$\nabla h = \langle 2x, 2y, -1 \rangle.$$

$$f(x,y,z) = (\text{distance}((x,y,z),(0,0,0)))^2$$

$$= x^2 + y^2 + z^2$$

$$\nabla f = \langle 2x, 2y, 2z \rangle.$$

Max and min occur when

$\nabla f = \lambda \nabla g + \mu \nabla h$, λ, μ scalars

That gives

$$\left.\begin{array}{l} 2x = \lambda + 2\mu x \\ 2y = \lambda + 2\mu y. \\ 2z = 2\lambda - \mu \\ x + y + 2z = 2 \\ x^2 + y^2 = z \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = y = \dfrac{\lambda}{2(1-\mu)}. \\ z = 2x^2 \\ 2x + 4x^2 = 2 \end{array}\right\} \Rightarrow \begin{array}{c} 2x^2 + x - 1 = 0 \\ \Downarrow \\ x = -1 \text{ or } x = 1/2. \end{array}$$

$(x,y,z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \Rightarrow f = \frac{3}{4}$ Min

$(x,y,z) = (-1, -1, 2) \Rightarrow f = 6$ Max