NOTES ON ÈTALE COHOMOLOGY

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Abstract. These are my notes for a 2022 UCLA Number Theory Learning Seminar on étale cohomology and the Weil Conjectures.

1. Introduction and History

In Number Theory, we care about solutions to polynomial equations. This can be reformulated as counting rational points on algebraic varieties. One could focus on finding solutions to systems of polynomial equations over finite fields. As is typical, we record the solutions in various extensions of a fixed finite field by using *zeta functions.*

Definition 1.1. Let X be a smooth geometrically irreducible projective variety of dimension n over the finite field $k = \mathbb{F}_q$. Recall that the *degree* of a closed point x of X is $[k(x) : k]$, where $k(x)$ is the residue field at x. The *zeta function* of X is

$$Z_X(T) = \prod_{x \in X \text{ closed}} (1 - T^{\deg x})^{-1}.$$

We sometimes write

$$\zeta_X(s) = Z_X(q^{-s}).$$

It turns out that this product converges for T small.

Weil made the following conjectures regarding this zeta function.

Conjecture 1.2. *For X and $Z_X(T)$ as above, the following are true:*

1. *(Rationality) $Z_X(T)$ is a rational function in T.*
2. *(Factorization) $Z_X(T)$ can be written as*
$$Z_X(T) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)},$$
where $n = \dim X$, $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^n T$, and each P_i is a polynomial in T with rational integer coefficients and constant term 1.
3. *(Riemann Hypothesis) For $0 < i < 2n$, the roots of P_i have absolute value $q^{-i/2}$.*
4. *(Betti numbers) If X is the good reduction of a smooth projective complex variety, then the degree of P_i is the ith (complex analytic) Betti number of this complex variety.*
5. *There is a functional equation for $\zeta_X(s)$.*

The following Lemma will be useful:

Lemma 1.3. *When T is small so that $Z_X(T)$ converges, we have*

$$\frac{d}{dT} \log Z_X(T) = \sum_{n=1}^{\infty} (\#X(\mathbb{F}_{q^n}))T^{n-1}.$$

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Proof. We note that a closed point of degree n corresponds to n distinct $\mathbb{F}_{q^n}$-points. Then

$$\begin{aligned}
\frac{d}{dT}\log Z_X(T) &= \frac{d}{dT}\sum_{x\in X \text{ closed}} -\log(1-T^{\deg x}) \\
&= \sum_{x\in X \text{ closed}} \deg x \frac{T^{\deg x-1}}{1-T^{\deg x}} \\
&= \sum_{x\in X \text{ closed}} \deg x \sum_{m=1}^{\infty} T^{m\deg x-1} \\
&= \sum_{n=1}^{\infty}\sum_{\{x\in X:\deg x=n\}}\sum_{m=1}^{\infty} nT^{mn-1} \\
&= \sum_{n=1}^{\infty}(\#X(\mathbb{F}_{q^n}))T^{n-1}.
\end{aligned}$$

□

Example 1.4. We consider $X=\mathbb{P}^1_{\mathbb{F}_p}$, where p is a rational prime. We have

$$\#X(\mathbb{F}_{p^n}) = p^n+1.$$

Note that a closed point of degree n corresponds to n points in $X(\mathbb{F}_{p^n})$. For example, we have a closed point (x^p-x-1) of $\operatorname{Spec}\mathbb{F}_p[x]\subset\mathbb{P}^1_{\mathbb{F}_p}$ that corresponds to p points in $X(\mathbb{F}_{p^p})$ (namely, the roots of x^p-x-1). So if N_n is the number of closed points of X of degree n, then

$$p^n+1 = \#X(\mathbb{F}_{p^n}) = \sum_{d|n} dN_d.$$

From this relation, we construct a table of values for N_n in Table 1.

By the previous Lemma,

$$\frac{d}{dT}\log Z_X(T) = \sum_{n=1}^{\infty}(p^n+1)T^{n-1} = \frac{1}{1-T}+\frac{p}{1-pT}.$$

So using that $Z_X(0)=1$, we have

$$\log Z_X(T) = -\log(1-T) + -\log(1-pT),$$

from which it follows that

$$Z_X(T) = \frac{1}{(1-T)(1-pT)}.$$

This agrees with the Weil conjectures.

In general, we consider $X=\mathbb{P}^N_{\mathbb{F}_q}$. Then

$$\#X(\mathbb{F}_{q^n}) = \frac{q^{n(N+1)}-1}{q^n-1} = \sum_{j=0}^{N} q^{jn}.$$

So

$$\frac{d}{dT}\log Z_X(T) = \sum_{j=0}^{N}\sum_{n=1}^{\infty} q^{jn}T^{n-1} = \sum_{j=0}^{N}\frac{q^j}{1-q^jT}.$$

n	N_n
1	$p+1$
2	$\frac{p^2-p}{2}$
3	$\frac{p^3-p}{3}$
4	$\frac{p^4-p^2}{4}$
5	$\frac{p^5-p}{5}$
6	$\frac{p^6-p^3-p^2+p}{6}$
n	$\frac{1}{n}\sum_{d\mid n}\mu(n/d)p^n$

TABLE 1. Points of given degree on $\mathbb{P}^1_{\mathbb{F}_p}$. Here, $\mu(d)$ denotes the Möbius function, and the final row is obtained by Möbius inversion.

A similar calculation to above shows that

$$Z_X(T) = \prod_{j=0}^{N} \frac{1}{1-q^jT}.$$

Remark 1.5 (History). For the special case of curves, the Weil conjectures were first formulated for curves by Emil Artin in 1924, and proved in this case by Weil [5]. At the end of Weil's paper, he posed his conjectures in what is essentially their current form. In 1960, Dwork proved the rationality conjecture using p-adic methods. In 1965, Grothendieck used the recently developed étale cohomology to prove almost all of the Weil conjectures, excluding the Riemann hypothesis. Grothendieck had hoped that this last conjecture could be proved using motivic methods (especially his *standard conjectures*). However, the Riemann hypothesis of the Weil conjectures was ultimately proved by Deligne in 1974.

2. Commutative Algebra Background

For this section, the main reference is Chapter I in the book of Freitag and Kiehl [2].

We would like to define a suitable notion of "covering" for schemes. The usual topological definition of "covering space" will not suffice.

Remark 2.1. Topological coverings are usually trivial for the Zariski topology. For instance, suppose $f : X \to Y$ is a surjective topological covering of irreducible topological spaces. Then for all $y \in Y$, there exists an open subset $V_y \subseteq Y$ containing y such that $f^{-1}(V_y) = \sqcup_\alpha U_\alpha$, where U_α are open subsets of X such that $f : U_\alpha \to V_y$ is an isomorphism. But X is irreducible, so there is only one sheet U_y in $f^{-1}(V_y)$. So $f : f^{-1}(V_y) \to V_y$ is an isomorphism, so f is an isomorphism.

In order to define the kinds of maps we want, we need to review some commutative algebra. In what follows, all rings are Noetherian (and commutative and unital). If R is a local ring, we denote by $\mathfrak{m}_R$ the maximal ideal of R and k_R the residue field.

Definition 2.2. Let A and B be local rings. A morphism $f : A \to B$ is called *local* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Equivalently, the unique closed point $\mathfrak{m}_A$ of $\operatorname{Spec} A$ is in the image of the morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. Equivalently, f induces an inclusion of fields $k_A \to k_B$.

Example 2.3. Let $\phi : R \to S$ be any ring map, and let $\mathfrak{p}$ be a prime of S. Then the induced map

$$R_{\phi^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$$

is a local homomorphism.

Definition 2.4. Let $f : A \to B$ be a local map of local rings. We say f is *unramified* if $f(\mathfrak{m}_A)B = \mathfrak{m}_B$ and the extension $k_A \to k_B$ is finite and separable. If $f : R \to S$ is any ring map, we say that f is unramified if the induced maps $R_{f^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$ are unramified for all $\mathfrak{p} \in \operatorname{Spec} S$.

Example 2.5. Let K be a field, and consider the map $K[x] \to K[x]$ given by $x \mapsto x^2$. This induces a local homomorphism $K[x]_{(x)} \to K[x]_{(x)}$. This map is ramified, since it extends the maximal ideal (x) to (x^2). (We have a non-reduced fiber).

Example 2.6. Consider $K \to K[x]$ where $\operatorname{char}(K) = 0$. This is unramified except at the ideal (0) of $K[x]$. Indeed, all of the other residue fields of $K[x]$ are finite separable extensions of K.

Remark 2.7. Note that even if $f : A \to B$ is of finite-type, an induced local map $A_{f^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ is not necessarily of finite type. For example, consider $K \to K[x]$, giving the induced map $K \to K[x]_{(x)}$. When we say "finite type local map," we mean that it is comes from a finite type ring map via localization. The Stacks project [4, Tag 024M] calls this "essentially finite type."

Definition 2.8. A local map $f : A \to B$ of local rings is *local étale* if it is flat, unramified, and a finite type local map.

Remark 2.9. See [2, bottom of p.9] for why flatness is good. Flatness is local, and finitely-generated flat modules over a local ring are free, for instance. Furthermore, if $A \to B$ is a flat finite type map of rings, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is open. An example of a morphism of schemes that is not flat is the normalization of a nodal curve [3, Example 9.7.1].

More geometrically, if $f : X \to Y$ is a flat morphism of schemes of finite type over a field k, then for all $x \in X$, we have

$$\dim_x(f^{-1}(f(x))) = \dim_x X - \dim_{f(x)} Y.$$

So for example, blowups are not flat.

We recall differentials briefly.

Definition 2.10. Let $f : A \to B$ be a ring map. The *module of relative differentials of B over A* is the B-module with generators db for each $b \in B$ and relations

$$d(b + b') = db + db', \qquad d(bb') = b\,db' + b'\,db, \qquad da = 0$$

for all $b, b' \in B$, $a \in A$.

Lemma 2.11. *A finite type local map $A \to B$ is unramified if and only if its module of differentials is trivial.*

Proof. Omitted. □

From here on, A and B are not necessarily local rings.

Definition 2.12. A morphism $f : A \to B$ of rings is *étale* if it is flat, unramified, and of finite type. (Similarly for schemes)

Remark 2.13. If $A \to B$ is étale, then the corresponding morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is quasi-finite (finite fibers).

Remark 2.14. According to Wikipedia, *étale* means "slack" as in a slack tide, e.g. calm and settled. According to Milne (`https://www.jmilne.org/math/CourseNotes/lec.html`), this is meaning that Grothendieck intended, which is different from the meaning of *étale* in *éspace étale*, where it means "spread out" or "on display."

Remark 2.15. The flat locus of a morphism is Zariski open, and the ramification locus is also Zariski open by the differential criterion.

Lemma 2.16. *Let K be a field. Then a map $K \to A$ is étale if and only if A is a finite product of finite separable extensions of K.*

Proof. The reverse implication is clear (the residue field of a field is itself).

Conversely, suppose A is a finitely-generated unramified K-algebra. Let $\mathfrak{p} \in \operatorname{Spec} A$, and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then $\operatorname{Frac}(A/\mathfrak{p})$ is a finite separable extension of K. In particular, every element of $A/\mathfrak{p}$ satisfies a separable monic polynomial over K. But then $A/\mathfrak{p}$ is an integral extension of K, so is a field. So $\mathfrak{p}$ is maximal. But A is also Noetherian, so A is Artinian. By a structure theorem, A is a product of local Artin rings. But all localizations must be fields since $K \to A$ is unramified. So A is a product of fields, each of which is a finite separable extension of K. □

Lemma 2.17. *A finitely-generated, flat map $\phi : A \to B$ is étale if and only if $B \otimes_A k_{A_\mathfrak{p}}$ is a finite separable extension of $k_{A_\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. Omitted. □

Example 2.18. We have a few more common examples:

1. Recall $AKLB$ setup: L/K is a finite unramified extension of number fields and B/A are their rings of integers. Then $A \to B$ is étale. Indeed, let P be a prime of B lying over a prime p of A. Then $pB = PP_1 \cdots P_n$ with the P, P_i distinct. So $pB_P = PB_P$, and B/P is a finite separable extension of A/p. So $A \to B$ is unramified. "Flat and finite type" follow from usual business.
2. Let L/K be a finite extension of number fields. Then this is a finite separable extension, so is *always* étale (even if ramified).
3. ([2, Prop 1.7]) If $X \to Y$ is a morphism of affine $\mathbb{C}$-varieties, then the map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is étale if and only if the map $X(\mathbb{C}) \to Y(\mathbb{C})$ is locally biholomorphic in the sense of complex analysis.
4. We saw above that $K[x] \to K[x]$ given by $x \mapsto x^2$ is ramified. However, the map $K[x, x^{-1}] \to K[x, x^{-1}]$ given by $x \mapsto x^2$ is étale.
5. What are étale morphisms $\mathbb{Z} \to A$?

We recall some more definitions.

Definition 2.19. A morphism of schemes is *locally of finite presentation if* [[replace "finite type" with "finitely presented" everywhere in the definition of "locally of finite type"]]

Definition 2.20. A morphism $f : X \to Y$ is smooth if it is flat, locally finitely presented, and has nonsingular geometric fibers. ((Recall that a geometric fiber of f is the fiber over a geometric point of Y, and a geometric point of Y is a k-point where k is algebraically closed.))

Proposition 2.21 (Basic properties of étale morphisms)**.** [4]
The following are true:

1. *A composition of étale maps is étale.*
2. *Étale morphisms are preserved under arbitrary base change.*
3. *Étale morphisms are local on the source and on the base. In other words, a morphism $f : X \to Y$ of schemes is étale if and only if "there is an open covering of X such that..." if and only if "there is an open covering of Y such that..."*
4. *(Cancellation) if $A \to B \to C$ is étale and $A \to B$ is étale, then $B \to C$ is étale. It follows that if $X \to Y$ is a morphism of S-schemes, where X and Y are both étale over S, then $X \to Y$ is étale.*
5. *Étale morphisms have relative dimension 0.*
6. *A morphism is étale if and only if it is flat and all fibers are étale.*
7. *Étale morphisms are open (see above).*
8. *Open immersions are étale.*
9. *A morphism is an open immersion if and only if it is étale and universally injective, i.e., injective on K-points for all fields K.*

Proof. See [4]. □

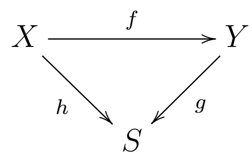
Theorem 2.22 (Étale via nilpotent thickenings)**.** *Let $f : X \to Y$ be a locally finitely presented scheme map. Then f is étale if and only if for all affine Y-schemes Z and all Y-subschemes Z_0 of Z with ideal sheaf squaring to 0, we have a natural bijection*

$$\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z_0, X).$$

3. Addendum: A "2 out of 3" Lemma

We will frequently see certain properties of morphisms of schemes that hold simultaneously for a morphism and the associated diagonal morphism, and that are preserved under arbitrary base change and composition. Recall that given a morphism $X \to S$ of schemes, the diagonal morphism $X \to X \times_S X$ is the morphism that post-composed with both projections $X \times_S X \to X$ gives the identity.

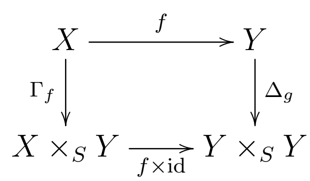
Lemma 3.1. *Let P be a class of morphisms of schemes that is closed under arbitrary base change and composition. Suppose we have a commutative diagram*



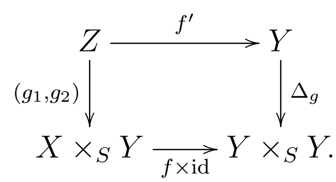
such that $h \in P$ and $\Delta_g \in P$. Then $f \in P$.

Proof. Consider the graph of f, which is the unique morphism $\Gamma_f : X \to X \times_S Y$ defined by identity $X \to X$ and $f : X \to Y$ (which are both morphisms over S). In particular, $f = \pi_2 \circ \Gamma_f$. Since P is closed under base change and $h \in P$, we have that $\pi_2 \in P$. Since P is closed under composition, it suffices to show that $\Gamma_f \in P$.

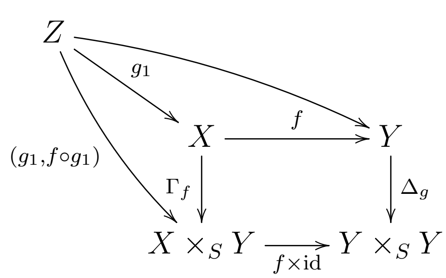
We claim that the commutative square



is cartesian. Suppose we are given a scheme Z fitting into the following commutative square:



By commutativity of the diagram, we have $(f', f') = (f \circ g_1, g_2)$. So the diagram can be written as



and g_1 is the unique morphism making the diagram commute. By the Yoneda Lemma, this implies that X is the pullback in the original square above. Since P is closed under base change and $\Delta_g \in P$, we have $\Gamma_f \in P$. This proves the claim. □

4. Commutative Algebra Background, Part 2

In the previous section, all rings were Noetherian (since Freitag and Kiehl assume this in Chapter 1). For this section, we remove any *a priori* finiteness assumptions. In particular, whenever we said “finite type” in the last section, it should be replaced with “locally of finite presentation” in general. Also, some authors (e.g. Wikipedia) seem to include lfp in the definition of unramified. For this section, this is the case.

The main references for this section are Stacks Project [4] and the paper of Bhatt and Scholze [1]. This time, I will attempt to be more consistent about notation!

In this talk, “formal” does not mean “formal scheme.” Rather, “formal” means something like “categorical.” We saw last time that étale maps are analogous to finite covering maps. However, we would like to remove the finiteness assumptions sometimes. This is analogous to the move from finite group cohomology to profinite group cohomology. There are a few ways to do this. One could attempt to replace “étale“ with any of the following:

(1) *formally étale*;
(2) *weakly étale*;
(3) *ind-étale*.

It turns out that “formal” is too weak. Slightly stronger is “weak,” but the definition of “weak étale” still seems too weak. We would ideally like to just take filtered colimits of rings: this is *ind-étale*. In 2013, Bhatt and Scholze proved that “weakly” and “ind-” give the same sites [1, Theorem 1.3].

We begin with some definitions.

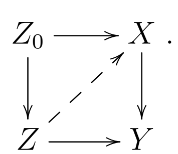
Definition 4.1. Let Z_0 be a scheme. A *thickening* of Z_0 is a scheme Z such that Z_0 is a closed subscheme of Z with the same underlying topological space. We say $Z_0 \to Z$ is a *first order thickening (FOT)* if $\mathcal{I}^2_{Z/Z_0} = 0$.

Example 4.2. Let R be a ring. Then $\operatorname{Spec} R/I$ is a closed subscheme of $\operatorname{Spec} R$, and has a first-order thickening $\operatorname{Spec} R/I^2$.

We now consider formal versions of the definitions made last time. We saw last time that if $f : X \to Y$ is an étale map of schemes, then for all Y-schemes Z_0 and first order Y-thickenings Z of Z_0, the morphism $Z_0 \to Z$ induces a bijection

$$\operatorname{Hom}_Y(Z, X) \cong \operatorname{Hom}_Y(Z_0, X).$$

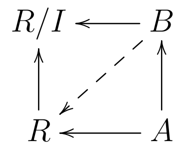
In a diagram, we have a unique induced dotted arrow making the following diagram commute:



This is analogous to a fibration. This motivates the following definition:

Definition 4.3.

(1) A ring map $\phi : A \to B$ is *formally étale* if for any ring R and any ideal $I \subseteq R$ with $I^2 = 0$, any solid diagram of the following form admits a compatible lift:



(2) A scheme map $f : X \to Y$ is *formally étale* if ... (similarly, but Z has to be affine!) for every **affine** Y-scheme Z_0 and every affine first order Y-thickening Z of Z_0, any Y-map $Z_0 \to X$ lifts uniquely to a first order thickening.

Remark 4.4. We can drop the requirement that Z and Z_0 be affine by [4, Lemma 04FD].

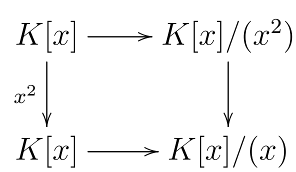
Recall that last time, we defined *étale* as flat, unramified, and finite type. We can make this "formal."

Definition 4.5. A ring map $\phi : A \to B$ is *formally unramified* if there is at most one lift to a nilpotent thickening in the above diagrams. Similarly, *formally smooth* if there is at least one lift.

Remark 4.6. There are similar definitions for schemes, using the same diagrams. However, for formally smooth, we cannot drop the requirement that Z and Z_0 are affine, since the proof that "affine" can be dropped relies on some kind of uniqueness to patch together maps on affine opens.

Remark 4.7. There is a strong analogy to classical covering maps with these definitions. For instance, recall that Ehresmann's theorem says that a proper submersion is a fibration. When one thinks of a fibration, we think of a lift existing, though not necessarily uniquely. Algebraically, we think of "smooth morphisms" as "submersions," so it makes sense that formally smooth should mean a lift exists. Formally étale, on the other hand, is stronger, saying that there is a unique such lift, just as covering spaces have unique path lifting.

Example 4.8. Let K be a field with $\mathrm{char}(K) \neq 2$. Consider them map $K[x] \to K[x]$ given by $x \mapsto x^2$ that we considered last time. We have a map $K[x] \to K[x]/(x^2)$ and $K[x] \to K[x]/(x)$ given by reductions, these making the following diagram commute:



But this does not lift! Indeed, suppose we had a map $\phi : K[x] \to K[x]/(x^2)$ such that $\phi(x^2) = \epsilon := \overline{x} \in K[x]/(x^2)$. Then $\phi(x)$ is a square root of $\overline{x}$ in $K[x]/(x^2)$. This is clearly impossible: $(a + b\epsilon)^2 = a^2 + 2ab\epsilon$. So this map is not formally smooth.

This map is also not formally ramified. If we let the top map be $x \mapsto 0$, then we are asking for a map $\phi : K[x] \to K[x]/(x^2)$ such that $\phi(x^2) = 0$. But we can set $\phi(x) = \epsilon$ and $\phi(x) = -\epsilon$, and these both work.

Example 4.9. We consider an inclusion $A \to B$ of number rings. Suppose $A \to B$ is tamely ramified in the number theoretic sense. Then there is some prime P of B lying over a prime p of A such that $pB \subset P^2$. Let e be the inertial degree. Then we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & B/P^2 \, . \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/P \end{array}$$

However, there is more than one lift $B \to B/pB$ that works here. Namely, precompose by a nontrivial element of tame inertia. (Additional assumptions are needed here in general)

Lemma 4.10. *A morphism $f : X \to Y$ of schemes is formally unramified if and only if $\Omega_{X/Y} = 0$.*

Proof. See [4, Lemma 02H9] □

Remark 4.11. In general, "formally blah" means satisfying some lifting property, "blah" or "G-blah" means requiring locally finitely presented (lfp) or locally of finite type (lft). We state the relevant facts below.

Proposition 4.12.

(1) [4, Lemma 02HE] *A morphism $f : X \to Y$ is unramified (resp. G-unramified) if and only if it is formally smooth and lft (resp. lfp).*
(2) [4, Lemma 02H6] *A morphism $f : X \to Y$ is smooth if and only if it is formally smooth and lfp.*
(3) [4, Lemma 02HM] *A morphism $f : X \to Y$ is étale if and only if it is formally étale and lfp.*

We now discuss two notions of "pro-étale" maps.

Definition 4.13.

(1) A ring map $A \to B$ is *weakly étale* (or *absolutely flat*) if it is flat and the diagonal map $B \otimes_A B \to B$ is flat.
(2) A ring map $A \to B$ is *ind-étale* if B is a filtered colimit of étale A-algebras.

Proposition 4.14. [1, Prop 2.3.3(a),(b)] *Ind-étale implies weakly étale implies formally étale.*

Example 4.15. (1) Let K be a field. Then $\prod_{n=1}^{\infty} K$ is not étale as a ring map, since it is not of finite type. Even though it is not finite type, it is locally of finite type (geometrically, a map of a bunch of points down to one point). So the map on Spec *is* étale, even though the map on rings is not... confused.
(2) Let K be a field, and let L/K be an infinite separable extension of K. Then $K \to L$ is not locally of finite type, so cannot be étale. However, L is ind-étale over K, since it is a union of finite separable extensions of K.

5. Étale fundamental groups

6. The Étale Site

Definition 6.1. A *site* is a category $\mathcal{C}$ equipped with a collection of coverings $\{U_i \to U\}_i$, where a covering is a collection of morphisms $U_i \to U$ with fixed target U, such that:

(1) For any isomorphism $U' \to U$, the singleton $\{U' \to U\}$ is a covering;
(2) Given a covering $\{U_i \to U\}_i$ and coverings $\{U_{ij} \to U_i\}_j$ for each i, the compositions $U_{ij} \to U_i \to U$ give a covering $\{U_{ij} \to U\}_{i,j}$;
(3) Given a covering $\{U_i \to U\}_i$ and any morphism $V \to U$ in $\mathcal{C}$, the pullbacks $U_i \times_U V \to V$ exist and give a covering $\{U_i \times_U V \to V\}_i$ of V.

A choice of coverings on a category is called a *Grothendieck topology* on the category.

Example 6.2. Let X be any topological space, regarded as a category of its open subsets with inclusion morphisms. Then the usual open coverings of open subsets make X into a site.

If X is a scheme, this site is called the *Zariski site*, denoted X_{Zar}.

Definition 6.3. The *(little) étale site* $X_{\acute{e}t}$ of a scheme X is the category of étale covers of X with morphisms given by morphisms of schemes over X (which will necessarily be étale morphisms by properties stated above). The coverings in $X_{\acute{e}t}$ are given by jointly surjective families of morphisms.

We can define presheaves on a site as contravariant functors on the site, and sheaves can be defined since we have a notion of "covering." Lots of the usual facts about sheaves on schemes hold. For example, we can consider the *étale structure sheaf* of a scheme X, whose sections at an etale cover U of X are just the global sections of the structure sheaf of U. Given a coherent $\mathcal{O}_X$-module, there is an associated sheaf on $X_{\acute{e}t}$ whose sections on $U \to X$ are global sections of the pullback to U of the original sheaf. There is also the usual notion of the *stalk* of a sheaf.

More remarkably, if K is a field, then sheaves on $\mathrm{Spec}(K)_{\acute{e}t}$ correspond to discrete G_K-modules.

It also turns out that the category of sheaves of abelian groups on $X_{\acute{e}t}$ is an abelian category that has enough injectives. One also checks that exact sequences of sheaves can be checked on stalks. Perhaps most importantly, the global sections functor is left exact. We can then define sheaf cohomology in the usual way, i.e., as derived functors of the global sections functor.

7. First Steps with Étale Cohomology

First, a clarification from last time. I claimed that all étale extensions of $\mathbb{Z}$ were just finite direct products of $\mathbb{Z}$. I asked if there was a way to prove that $\operatorname{Spec}\mathbb{Z}$ has no nontrivial étale covers by using lifting. There are a few things to note here.

- Any open subset of $\operatorname{Spec}\mathbb{Z}$ gives an étale extension of $\mathbb{Z}$. Namely, $\mathbb{Z}[1/n]$ for all nonzero integers n gives an étale extension of $\mathbb{Z}$ (however, this is not a *finite* étale extension).
- Even then, the lifting criterion does not fully characterize étale extensions of $\mathbb{Z}$. For instance, you can check that any number field is a formally étale extension of $\mathbb{Z}$. Indeed, $\mathbb{Z} \to \mathbb{Q}$ is formally étale since it is ind-étale.

Intuitively, this is like how an arbitrary intersection of open sets of a scheme does not have to be open, or even of the same dimension.

Of course, *finite* formally étale covers of $\operatorname{Spec}\mathbb{Z}$ are all trivial.

Now we continue discussing étale cohomology. For simplicity, we assume all schemes are Noetherian and separated. We will often ommit "ét" unless it is necessary to distinguish from classical sheaves.

Remark 7.1. Recall that a ring morphism $A \to B$ is faithfully flat if and only if it is flat and the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proposition 7.2. *Let M be a quasicoherent module on a scheme X.*

(a) *The presheaf $M_{ét}$ defined by*

$$M_{ét}(f : U \to X) = \Gamma(U, f^*M)$$

is a sheaf.

(b) *We have $H^i(X_{ét}, M_{ét}) = H^i(X, M)$.*

Proof. As usual, it suffices to check the sheaf property for Zariski open covers and singleton affine covers. For Zariski open covers, this follows from the fact that M is already a sheaf on X. For affine étale covers, it suffices to consider the case where M is an A-module, B is a (faithfully flat) étale A-algebra, and show that

$$0 \to M \to B \otimes_A M \rightrightarrows (B \otimes_A B) \otimes_B M$$

is exact. The proof is exactly the same as the one given last time: assume that $A \to B$ has a section, etc.

For the second part, we will need *Cartan's lemma* (basically, can we do Čech cohomology?) and a descent lemma (black-boxed for now, maybe another talk?). □

7.1. More Examples of Sheaves.

Example 7.3. Let F be a set (or finite abelian group). Then the *constant sheaf* F_X on X is given by

$$U \mapsto F^{\#\pi_0(U)}$$

where $\#\pi_0(U)$ is the number of connected components of U.

If F is a finite set, then the constant sheaf F_X on X is representable by the disjoint union $\bigsqcup_{i \in F} X$. Indeed, a morphism $U \to \bigsqcup_{i\in F} X$ over X is a choice of $i \in F$ on each connected component of U.

Example 7.4. We define an étale sheaf $U \to \Gamma(U, \mathcal{O}_U)^\times$ on $X_{ét}$, and call it $\mathcal{O}_{X,ét}^\times$ or just $\mathcal{O}_X^\times$. One checks that this is a sheaf. Indeed, for Zariski open covers, we can uniquely glue inverses into inverses, and for faithfully flat affine étale covers $A \to B$, we must show that $A^\times$ is the equalizer of the two maps $B^\times \to (B \otimes_A B)^\times$ induced by the usual maps $B \to B \otimes_A B$. But this follows from the equalizer diagram $0 \to A \to B \rightrightarrows B \otimes_A B$.

Definition 7.5. The étale sheaf $\mathcal{O}_X^\times$ is the *sheaf of units* on X.

Lemma 7.6. *The group scheme* $\mathbb{G}_m := \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$ *is such that* $\mathbb{G}_{m,X} := \mathbb{G}_m \times_{\mathbb{Z}} X$ *represents* $\mathcal{O}_X^\times$. *In other words, for all étale morphisms* $U \to X$, *we have a natural bijection (isomorphism of sheaves)*

$$\Gamma(U, \mathcal{O}_U)^\times \cong \operatorname{Hom}_X(U, \mathbb{G}_{m,X}).$$

Proof. By the sheaf property, it suffices to check this on affine étale morphisms. Indeed, let $U = \operatorname{Spec} B \to X$ be an étale morphism. Then an X-morphism $U \to \mathbb{G}_m \times X$ is the same as a morphism $U \to \mathbb{G}_m$. This is equivalent to a morphism $\mathbb{Z}[x, x^\times] \to B$, which is the same as a choice $b \in B^\times$. ☐

Because of this, it makes sense to denote the sheaf $\mathcal{O}_X^\times$ by $\mathbb{G}_{m,X}$, since we have shown that the functor of points of $\mathbb{G}_{m,X}$ on the étale site over X is the same as the sheaf $\mathcal{O}_X^\times$.

Example 7.7. Let n be a positive integer. We have a sheaf $U \mapsto \mu_n(\Gamma(U, \mathcal{O}_U))$ giving the nth roots of 1 in U. By a similar proof to above, this étale sheaf on X is represented by the scheme

$$\mu_{n,X} := \operatorname{Spec} \mathbb{Z}[x]/(x^n - 1) \times_{\mathbb{Z}} X.$$

7.2. **The Kummer Sequence.**

We would like to have an exact sequence looking like

$$\text{“}1 \to \mu_{n,X} \to \mathcal{O}_X^\times \xrightarrow{n} \mathcal{O}_X^\times \to 1.\text{”}$$

This is called the *Kummer sequence*. It is clear that the sequence is exact in the left and middle by definition of $\mu_{n,X}$. Exactness on the right often fails, on the other hand. We do have the following lemma:

Lemma 7.8. *If* $n \in \Gamma(X, \mathcal{O}_X)^\times$, *then the Kummer sequence is exact.*

Proof. It suffices to prove that the sequence is exact on affine étale covers of X. Since $n \in \Gamma(X, \mathcal{O}_X)^\times$, we know that any affine scheme $\operatorname{Spec} A$ over X has $n \in A^\times$. But then for all $a \in A^\times$, we have $A[x]/(x^n - a)$ is a finite, formally étale extension of A. By the definition of surjective sheaf map, we obtain the desired result. ☐

Remark 7.9. To see formally étale above, suppose we have a B algebra and an ideal $I \subseteq B$ with $I^2 = 0$, and a morphism $A[x]/(x^n - a) \to B/I$. Then we have $b \in B$ such that $b^n - a \in I$, and we must show that there is a unique $b' \in B$ such that $(b')^n - a = 0$. Let $c \in I$. Then

$$(b + c)^n - a = b^n - a + cnb^{n-1}.$$

Since a is a unit in A, we know that nx^{n-1} is a unit in $A[x]/(x^n - a)$. So $(nb^{n-1}) + I = B$. So

$$(nb^{n-1}) = (nb^{n-1}) + I^2 = B,$$

so nb^{n-1} is a unit in B. Then setting $c = (a - b^n)(nb^{n-1})^{-1}$ is the unique element of B above such that $(b + c)^n = a$. ☐

In general, this argument shows that $A[x]/(f)$ is an étale extension of A if f is a monic polynomial such that the derivative f' is a unit in $A[x]/(f)$ (this is related to *standard étale* maps of rings).

Remark 7.10. This Lemma is false if we only considered the Zariski site. For instance, the Zariski site of a field is trivial, but $\mathbb{Q}$ doesn't have a square root of 2.

7.3. A little more about other topologies.

Remark 7.11 (Other Topologies). See Part 2, Chapter 34 of the Stacks ("Topologies on Schemes").

- A *syntomic morphism* is a flat local complete intersection (so flat, locally finitely presented, and geometric fibers are local complete intersections). Etale implies syntomic.
- An *fppf* (French acronym for "faithfully flat finitely presented") cover is a cover by flat and lfp morphisms. So syntomic implies fppf.
- What is an *fpqc* cover? Again, "fp" means faithfully flat, and "qc" here means quasicompact. The Stacks uses a slightly different definition: they say a family of morphisms $\{U_i \to U\}_i$ is fpqc if each morphism is flat lfp, and every open affine of U is the union of the images of finitely many open affines of possibly multiple U_i. With this definition, etale implies fpqc (using that etale maps are open). NOTE: The fpqc topology has no "small basis." In other words, there is generally not a small category that contains a "cofinal system of coverings." So we will (and should) avoid talking about the category of fpqc sheaves.

The Kummer sequence is always exact in the big fppf and big syntomic sites.

Proposition 7.12. *The functor from the category of sheaves on the big etale site to the category of sheaves on the little etale site given by restriction is exact and maps injectives to injectives. Thus both sites give the same cohomology for sheaves on the big etale site.*

Let us discuss the fpqc site a little more. The fqpc site is nice for many reasons. For example, we can check the sheaf property by checking Zariski open covers and faithfully flat morphisms of affine schemes (similar to the etale case). On the other hand, more covers means presheaves are less likely to satisfy the sheaf property (see [4, Example 03O2], basically the sheaf of relative differentials is not an fpqc sheaf). However, we have the following Lemma:

Lemma 7.13. *Representable functors on the category of schemes over a fixed base S satisfy the sheaf condition for the (big) fpqc site over S.*

Proof. See [4, Lemma 03O3]. This is another consequence of descent theory. □

7.4. **Picard Groups.** Hopefully we can talk about descent at some point! Here's another reason we should discuss descent:

We can define an *etale $\mathcal{O}_X$-module*, or an *$\mathcal{O}_{X,et}$-module*, in the usual way: a module over the structure sheaf in each etale map in a compatible way. Similarly, an *etale line bundle* on a scheme X is a rank 1 locally trivial etale $\mathcal{O}_X$-module, where *locally trivial* means that there is an etale covering $\{U_i \to X\}_i$ such that the sheaf restricted to each U_i is isomorphic to $\mathcal{O}_{X,et}$ as a $\mathcal{O}_{X,et}$-module. As usual, the set $\mathrm{Pic}_{et}(X)$ of isomorphism classes of etale line bundles over X comes equipped with a group operation given by the tensor product. Using Čech cohomology, one shows that $\mathrm{Pic}_{et}(X)$ is the 1st Čech cohomology of $\mathcal{O}_{X,et}^\times$. By taking the etale sheaf induced by a quasicoherent sheaf, there is a natural map $\mathrm{Pic}(X) \to \mathrm{Pic}_{et}(X)$. By descent, this map is an isomorphism.

7.5. **Stalks.** One would like to also be able to prove exact sequences by looking at stalks. However, etale stalks of a quasicoherent sheaf are not the same as the usual stalks.

Recall that a *geometric point* of a scheme X is a morphism $\alpha : \operatorname{Spec}\Omega \to X$ where Ω is a separably closed field. An *etale neighborhood of* α is a factorization of α through an etale morphism $U \to X$.

Lemma 7.14. [4, Lemma 03PR] *If $\{U_i \to X\}_i$ is an etale cover of X and $\alpha : \operatorname{Spec}\Omega \to X$ is a geometric point, then α factors through some U_i. So there exists i such that $U_i \to X$ is an etale neighborhood of α.*

Proof. Consider the base change diagram:

$$\begin{array}{ccc} \operatorname{Spec}\Omega \times_X \coprod_i U_i & \longrightarrow & \coprod_i U_i \\ \downarrow & & \downarrow \\ \operatorname{Spec}\Omega & \xrightarrow{\alpha} & X \end{array}$$

Since the right map is etale, the left map is as well. But an etale cover of $\operatorname{Spec}\Omega$ is just a disjoint union of copies of $\operatorname{Spec}\Omega$. So the left map admits a section. □

Remark 7.15. Note that we are using etale here! This might not work in other topologies (see below for a general definition of a point of a site).

Recall that the *local ring* at α is

$$\mathcal{O}_{X,\alpha} := \varinjlim_{\alpha \to U} \mathcal{O}_X(U).$$

This ring is not the usual stalk of $\mathcal{O}_X$ at the image of α, but is rather its *Henselization.*

Remark 7.16. Given a point $x \in X$, all separable closures of the residue field at x are non-canonically isomorphic. So we can always choose a geometric point with image x, and the stalk F_x of a sheaf will be well-defined up to a non-canonical isomorphism. However, the support of a sheaf will not necessarily be closed anymore (see [2, p. 26]). I guess Henselizing can get rid of some points.

Definition 7.17. [4, Definition 00Y5] Let $\mathcal{C}$ be a site. A *point* of the site is a functor $\alpha : \mathcal{C} \to \mathbf{Set}$ such that

(1) For each covering $\{U_i \to U\}_i$, the function $\coprod_i \alpha(U_i) \to \alpha(U)$ is surjective. (This is analogous to how a geometric point should factor through any element of a cover.)
(2) For each cover $\{U_i \to U\}$ and any morphism $V \to U$, the maps

$$\alpha(U_i \times_U V) \to \alpha(U_i) \times_{\alpha(U)} \alpha(V)$$

are bijective. (This is like saying "points in the preimage of an open set are points that map to the open set.")
(3) The "stalk functor" is left exact. (See the stacks. It turns out that it will be exact in this case.)

8. Čech Cohomology and Descent

The following details were mentioned above, and we elaborate upon them below.

8.1. **Čech Cohomology.** Let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be an étale covering of a scheme X, and let F be a sheaf on X. The usual definition for Čech cohomology carries over: we define the *Čech complex* by

$$C^n(\mathcal{U}, F) := \prod_{i_0, \ldots, i_n} F(U_{i_0} \times_X \cdots \times_X U_{i_n}),$$

with boundary maps

$$C^n(\mathcal{U}, F) \longrightarrow C^{n+1}(\mathcal{U}, F),$$
$$(d(s_{i_0,\ldots,i_n})_{i_0,\ldots,i_n})_{j_0,\ldots,j_{n+1}} = \sum_{\ell} s_{j_0,\ldots,\hat{j_\ell},\ldots,j_{n+1}}.$$

On the right hand side, we are implicitly looking at the image of a section under the restriction map

$$F(U_{j_0} \times_X \cdots \times_X \widehat{U_{j_\ell}} \times_X \cdots \times_X U_{j_{n+1}}) \to F(U_{j_0} \times_X \cdots \times_X U_{j_\ell} \times_X \cdots \times_X U_{j_{n+1}}).$$

9. Finiteness and Constructible Sheaves

10. Computations for Curves

11. Proper Base Change

11.1. **Motivation.** In this subsection, we attempt to motivate the Proper Base Change Theorem as it is relevant to the Weil conjectures. Base change theorems have many uses, but here I focus on one important application.

We saw early on that results like Poincare duality are critical to proving the Weil conjectures. In order to discuss Poincare duality for non-complete varieties, we need to define cohomology with compact support. One could try to define these cohomology groups as higher derived functors of "sections with compact support." However, as discussed in FK (start of I.8), this will not be the cohomology theory we want.

In classical topology, there is another way to view cohomology with compact support, however. If X is a topological space, we can find a compactification $X \to \overline{X}$ such that $\overline{X}$ is compact and X is densely embedded in $\overline{X}$. If F is a sheaf of abelian groups on X, then let $\overline{F}$ be the usual extension by 0. Then

$$H^i_c(X, F) \cong H^i(\overline{X}, \overline{F}).$$

We would like to do something similar for etale cohomology. In this analogy, we will have to show that the above doesn't depend on the compactification $\overline{X}$ that is chosen. It turns out that this is a consequence of the Proper Base Change Theorem:

$$\text{base change theorems} \rightsquigarrow \text{cohomology with compact support} \rightsquigarrow \text{Poincare duality}$$

$$\rightsquigarrow \text{Weil conjectures.}$$

11.2. **The Proper Base Change Theorem in Etale Cohomology.** All sheaves considered in this section are etale.

We recall torsion sheaves.

Definition 11.1. An etale sheaf F of abelian groups on a scheme X is called a *torsion sheaf* if all of its stalks are torsion abelian groups.

Proposition 11.2. [2, Prop 4.8] *Suppose X is qcqs. A sheaf of abelian groups is constructible if and only if it is torsion and noetherian (every ascending chain of subsheaves terminates). Furthermore, all torsion sheaves are filtered colimits of constructible sheaves of abelian groups.*

The goal of this section is to prove the following theorem.

Theorem 11.3. *Let $f : X \to S$ be a proper scheme map, and let*

$$\begin{array}{ccc} X_T & \xrightarrow{h} & X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & S \end{array}$$

be a cartesian square (i.e. $X_T = X \times_S T$). Let F be a torsion sheaf on X. Then there is a natural isomorphism (called the ***base change map****)*

$$\boxed{g^*(R^i f_* F) \to R^i e_*(h^* F).}$$

Corollary 11.4. *Let $f : X \to S$ be a proper scheme map, and let s be a geometric point of S. Let $F \in \mathrm{Sh}(X_{et})$. Then*

$$(R^i f_* F)_s \cong H^i(X_s, F_s).$$

Proof. Take T to be the spectrum of a separably closed field above, then take stalks in the base change map. □

The remainder of this section is dedicated to the proof of this theorem.

First, we discuss how the base change map is defined. Recall that f^* is left adjoint to f_* for any morphism of schemes f. In this case, this gives us a unit $1 \Rightarrow h_* h^*$. So we have a map

$$F \to h_* h^* F.$$

We apply f_* to obtain

$$f_* F \to f_* h_* h^* F = g_* e_* h^* F.$$

By adjunction again, we get a natural morphism

$$g^* f_* F \to e_* h^* F.$$

Lemma 11.5. [4, Lemma 0A3U] *For f proper as above, the map $g^* f_* F \to e_* h^* F$ is an isomorphism*

Proof. See [4, Lemma 0A3U]. The proof is essentially checking on stalks. □

For higher direct images, we would like to replace F with an injective resolution. However, even though h^* is exact, it does not necessarily preserve injectives.

We recall derived categories. We embed the category of etale sheaves of abelian groups on X_{et} into its (bounded below) derived category

$$\mathrm{Sh}(X_{et}) \to \mathcal{D}^+(\mathrm{Sh}(X_{et}))$$

by sending a sheaf F to $F[0]$, the complex that is F at 0 and 0 elsewhere. We will frequently abuse notation and write F instead of $F[0]$. Recall that an injective resolution of a complex $C^\bullet \in \mathcal{K}(\mathrm{Sh}(X_{et}))$ of sheaves is a morphism $C^\bullet \to I^\bullet$, where:

- $I^n = 0$ for $n < 0$,
- I^n is injective for all n, and
- the morphism $C^\bullet \to I^\bullet$ is a quasi-isomorphism (inducing isomorphisms on all cohomology groups).

In particular, an injective resolution of $F[0]$ is an injective resolution of F in the usual sense. Since $\mathrm{Sh}(X_{et})$ has enough injectives, there is a functorial way to assign injective resolutions, which we will denote by $i_C : C^\bullet \to I(C^\bullet)$. Then $I : \mathcal{D}^+(\mathrm{Sh}(X_{et})) \to \mathcal{K}^+(\mathrm{Sh}(X_{et}))$ is a functor. Then the right-derived functor of f_* is

$$Rf_* : \mathcal{D}^+(\mathrm{Sh}(X_{et})) \to \mathcal{D}^+(\mathrm{Sh}(S_{et})), \qquad [C^\bullet] \mapsto f_*(I(C^\bullet)).$$

Then the homology groups $H^i(Rf_* F[0])$ are the usual higher direct images $R^i f_* F$.

In this language, we have a resolution $i_F : F \to IF$. Since h^* is exact, we get a resolution (not necessarily injective) $h^* i_F : h^* F \to h^* IF$. We then take another injective resolution $h^* IF \to I h^* IF$. The composition $h^* F \to I h^* IF$ is then an injective resolution of $h^* F$. From the above construction, we get a composition of maps in $\mathcal{K}(\mathrm{Sh}(T_{et}))$:

$$g^*(Rf_* F) = g^* f_* IF \to e_* h^* IF \to e_* I h^* IF = Re_*(h^* F).$$

In summary, we have a morphism in the derived category:

$$\boxed{g^*(Rf_*F) \to Re_*(h^*F)}$$

by exactness of the inverse image, this gives maps on homology:

$$g^*(R^if_*F) = g^*(H^i(Rf_*F)) = H^i(g_*(Rf_*F)) \to H^i(Re_*(h^*F)) = R^ie_*(h^*F)).$$

The goal is now to show that this is an isomorphism for all i.

11.3. **Proof of the Theorem.** For the sake of time, we will do the proof in the special case that F is a sheaf of $\mathbb{Z}/n$-modules, where $n \in \mathcal{O}_S^\times$. The general case can be proved by taking colimits. We follow the presentation in [4, Section 095S].

To prove Theorem 11.3, we will make a series of simplifications/reductions. First, we make the following observation.

Remark 11.6. Let $f : X \to S$ be proper. Let $\{S_i\}_i$ be an etale cover of S, and let $X_i = X \times_S S_i$. So $X_i \to S_i$ is proper for all i. Recall that R^if_*F is the sheafification of

$$(V \to S) \mapsto H^i(X \times_S V, F|_{X\times_S V}).$$

If we show the theorem for $X_i \to S_i$, then the base change map is locally an isomorphism, so is an isomorphism. So we can always look locally on S.

Lemma 11.7. [4, Lemma 0A4B] *Let $f : X \to S$ be proper, and $T \to S$ a morphism. Then Theorem 11.3 holds for f if and only if for all primes ℓ and all injective ℓ-torsion sheaves I on X,*

$$R^ie_*(h^*I) = 0 \qquad \text{for } i > 0.$$

Proof. The forward implication follows because higher direct images of an injective sheaf are zero.

Now assume the latter condition, and let F be an n-torsion sheaf on X. We induct on the number of prime factors of n. If n is prime, we choose an injective resolution $F \to I^\bullet$ of n-torsion sheaves. Then we must show that

$$g^*f_*I^\bullet \to Re_*(h^*F)$$

is a quasi-isomorphism. By assumption, we have that $h^*I^\bullet$ is an acyclic resolution of h^*F (using exactness of h^* and the hypothesis). So it suffices to show

$$g^*f_*I^\bullet \to e_*h^*I^\bullet$$

is an isomorphism. But this follows from Lemma 11.5.

Now suppose $n = \ell m$, where ℓ is prime. We have a short exact sequence of sheaves

$$0 \to F[\ell] \to F \to F/F[\ell] \to 0.$$

The base change maps give a commutative diagram

$$\begin{array}{ccccccccc} g_*(R^{i-1}f_*F/F[\ell]) & \longrightarrow & g_*(R^if_*F[\ell]) & \longrightarrow & g_*(R^if_*F) & \longrightarrow & g_*(R^if_*F/F[\ell]) & \longrightarrow & g_*(R^{i+1}f_*F) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R^{i-1}e_*(h^*F/F[\ell]) & \longrightarrow & R^ie_*(h^*F[\ell]) & \longrightarrow & R^ie_*(h^*F) & \longrightarrow & R^ie_*(h^*F/F[\ell]) & \longrightarrow & R^{i+1}e_*(h^*F[\ell]) \end{array}$$

By induction hypothesis, the outer four vertical maps are isomorphisms. So the middle map is an isomorphism. ∎

Lemma 11.8. *Theorem 11.3 holds for $f : X \to S$ a finite morphism.*

Proof. Recall that a finite morphism is proper. Furthermore, finiteness is preserved under base change. By [2, Cor 3.4], we have $R^i e_* G = 0$ for all sheaves of abelian groups G on X_T and all $i > 0$. By Lemma 11.7, the result follows. □

Lemma 11.9. [4, Lemmata 0A4B-0A4C] *Suppose $f_1 : X \to Y$ and $f_2 : Y \to S$ are proper morphisms such that f_1 satisfies Theorem 11.3. Then if f_2 satisfies the Theorem, so does $f_2 \circ f_1$. Conversely, if $f_2 \circ f_1$ satisfies the theorem and f_1 is surjective, then f_2 satisfies the theorem.*

Proof. See [4, Lemmata 0A4B-0A4C]. Essentially, we use Lemma 11.7 and the Leray spectral sequence. □

We now reduce to showing the theorem for $\mathbb{P}^1_S \to S$.

Lemma 11.10. [4, Lemma 0A4F] *If proper base change holds for $\mathbb{P}^1_S \to S$ for all schemes S, then it is true for all proper morphisms.*

Proof. Suppose that the theorem is true for $\mathbb{P}^1_S \to S$. Let $f : X \to S$ be proper. By the above remark, we can look locally and assume that S is affine.

We recall Chow's Lemma, which gives the following commutative diagram for some N:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{P}^N_S & \longrightarrow & S \end{array}$$

Here, we get that $X' \to \mathbb{P}^N_S$ is a closed immersion and $X' \to X$ is proper and surjective.

Remark 11.11. By [4, Lemma 0203], Chow's Lemma hold for $X \to S$ separated and of finite type, where S is qcqs (unless X has finitely many irreducible components, we lose $X' \to X$ being an isomorphism over a dense open subset of X).

Since $X' \to X$ is surjective and proper, Lemma 11.9 shows that it suffices to prove the theorem for $X' \to S$. The map $X' \to \mathbb{P}^N_S$ is a closed immersion, so finite, so the theorem holds for this as explained above. Then we have reduced to showing the theorem for $\mathbb{P}^N_S \to S$.

There is a finite surjective morphism

$$\mathbb{P}^1_S \times_S \cdots \times_S \mathbb{P}^1_S \to \mathbb{P}^N_S$$

given by looking at the coefficients of $\prod_i (x_i t + y_i)$. The map

$$\mathbb{P}^1_S \times_S \cdots \times_S \mathbb{P}^1_S \to S$$

is a composition of morphisms of the form $\mathbb{P}^1_Z \to Z$, so the theorem holds by assumption and Lemma 11.9. But then the theorem holds for $\mathbb{P}^N_S \to S$ by the same Lemma, completing the proof. □

So we have reduced to the situation where $f : \mathbb{P}^1_S \to S$. Again, we can assume that S is affine. Let t be a geometric point of T. Then considering stalks (and using facts about Henselian local rings), we are reduced to showing that

$$H^i(\mathbb{P}^1_L, F) \to H^i(\mathbb{P}^1_K, F)$$

is an isomorphism, where L/K is a purely inseparable extension of (separably closed) fields. But this follows from the fact that a purely inseparable morphism induces an equivalence of abelian categories between the categories of sheaves on each space.

12. Smooth Base Change

13. ℓ-ADIC COHOMOLOGY

In this section, we primarily follow [2], with references to Conrad's notes.

Recall that the primary strategy for proving the Weil Conjectures is to find a "Weil Cohomology Theory" satisfying some properties. In particular, we need to have our cohomology groups be vector spaces over a field of characteristic 0. Unfortunately, everything we have been doing so far is for torsion sheaves.

Example 13.1. [2, §I.12, p. 118] Let X be a smooth projective curve of genus g over an algebraically closed field k, and let ℓ be a prime not dividing char(k). Then if we use the definition of etale cohomology we've used previously, $H^i_{et}(X, \mathbb{Q}_\ell) = 0$ for $i > 0$. On the other hand,

$$\varprojlim_n H^i(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ \mathbb{Q}_\ell^{2g} & i = 1 \\ \mathbb{Q}_\ell & i = 2 \\ 0 & \text{else.} \end{cases}$$

This example also shows that while etale cohomology commutes with colimits (namely, direct limits), this is not so with projective limits of sheaves.

The above example shows that it is not enough to just consider the category of etale sheaves. In this talk, we will define ℓ-adic sheaves and see some first properties. The following subtleties are noteworthy:

- We will see that ℓ-adic sheaves are not literally sheaves, but rather certain projective systems of sheaves.
- Morphisms of ℓ-adic sheaves are more complicated than literal morphisms of projective systems.
- In order to look at things like "trace of Frobenius" later, we want to be sure that the resulting cohomology theory gives finite-dimensional cohomology groups (otherwise, how will we look at trace?).
- We will also need to make sense of how pushforward and pullback work in this category.

13.1. **Review of Artin–Rees for Modules.**

We fix a prime ℓ. Let $R_n := \mathbb{Z}/\ell^{n+1}\mathbb{Z}$ and $R = \varprojlim_n R_n \cong \mathbb{Z}_\ell$.

We could just consider a category of projective systems

$$\cdots \longrightarrow M_n \xrightarrow{p_n} M_{n-1} \xrightarrow{p_{n-1}} \cdots \longrightarrow 0$$

that eventually terminate (without loss of generality, we will assume $M_n = 0$ for $n < 0$), where each M_n is a torsion R-module. However, making sense of Hom in this category is not always easy. For instance, consider the map of projective systems $f : (R_n) \to (R_n)$ given by multiplication by ℓ. Note that $\ker(f_n) = \ell^n R_n$ and the maps between kernels at each step are 0. On the other hand, $\ker(f : R \to R) = 0$, and the projective systems $(\ker(f_n))_n$ and $(0)_n$ are not isomorphic as projective systems. We can resolve this apparent contradiction in the spirit of the *Artin–Rees lemma*:

Lemma 13.2 (Artin–Rees)**.** *Let R be any Noetherian ring, and $I \subseteq R$ an ideal. Let M be a finitely generated R-module and $N \subseteq M$. Then there exists an integer $n_0 \geq 0$ such that for $n \geq n_0$,*

$$I^n M \cap N = I^{n-n_0}((I^{n_0} M \cap N).$$

Corollary 13.3. *Assume the notation of the Artin–Rees Lemma. Then in the kernel complex of $N \to M$ (i.e., the projective system*

$$\cdots \to (I^{n+1} M \cap N)/I^{n+1} N \to (I^n M \cap N)/I^n N \to (I^{n-1} M \cap N)/I^{n-1} N \to \cdots)$$

the composition of n_0 consecutive maps is 0.

Proof. This Corollary says that for $n \geq n_0$,

$$I^n M \cap N \subseteq I^{n-n_0} N.$$

This follows immediately from the Artin–Rees Lemma. □

If $M = (M_n)$ is a projective system, we denote by $M[d]$ the projective system given by

$$M[d]_n = M_{n+d}.$$

Note that we have natural maps $M[d] \to M$ given by the d-fold composition of the maps in the system.

Definition 13.4. Let $M = (M_n)$ be a projective system.

1. We say M is *null* if there exists $d \geq 0$ such that $M[d] \to M$ is the zero map.
2. We say M satisfies the *Mittag-Leffler condition (ML)* if for all n, there exists $m \geq n$ such that for all $k \geq m$, the image of $M_k \to M_n$ is the same as the image of $M_m \to M_n$.

We can now define the category we want to work with.

Definition 13.5. The *AR category* (of projective systems over (R_n)) has objects given by projective systems $M = (M_n)$, and for systems M, N, we define

$$\mathrm{Hom}_{\mathrm{AR}}(M, N) = \varinjlim_{d \geq 0} \mathrm{Hom}(M[d], N).$$

If two systems are isomorphic in the AR category, we will say they are "*AR equivalent*." If a system is AR equivalent to a system with property P, we say that the system is "*AR P*."

Example 13.6. If M is a null system, then M is AR isomorphic to 0. Indeed, the zero map $M \to M$ and the identity map are both identified with the natural map $M[d] \to M$ in $\mathrm{Hom}_{\mathrm{AR}}(M, M)$ when d is large enough.

Remark 13.7. One checks that the AR category is abelian. Furthermore, a morphism of projective systems is an isomorphism if and only if its kernel and cokernel are null.

Definition 13.8. An *ℓ-adic system* is a projective system (F_n) such that:

1. Each F_n is of finite length;
2. $\ell^{n+1} F_n = 0$ for all n;
3. For $n > 0$, the map $F_{n+1} \to F_n$ induces an isomorphism

$$F_{n+1}/\ell^{n+1} F_{n+1} \cong F_n.$$

A projective system is called *AR ℓ-adic* if it is AR isomorphic to an ℓ-adic one.

Lemma 13.9. [2, p. 120] *Let F be an ℓ-adic system, and let G be a system such that $\ell^{n+1}G_n = 0$ for all n. Then* $\mathrm{Hom}(F, G) = \mathrm{Hom}_{\mathrm{AR}}(F, G)$.

Proof. Let $f : F \to G$ be an morphism that is AR equivalent to 0. Then there exists $d > 0$ such that $F[d] \to F \to G$ is zero. This means $F_{n+d} \to G_n$ is zero for all n. So the natural map $F[d] \to F$ has image contained in the kernel of f. But by condition 3 above, the map $F_{n+d} \to F_n$ is surjective for all n. So the kernel is all of F, and $f = 0$.

On the other hand, suppose we have an AR morphism $f : F[d] \to G$ for some $d > 0$. By induction, it suffices to show that this map factors through $F[d] \to F[d-1]$. But this follows from the fact that $\ell^{n+d}F_{n+d}$ is in the kernel of $f_n : F_{n+d} \to G_n$, so each of these maps factors through $F_{n+d}/\ell^{n+d}F_{n+d} \cong F_{n+d-1}$. □

Theorem 13.10. [2, Prop 12.4] *The full subcategory of the A–R category consisting of AR ℓ-adic objects is an exact subcategory, and the functor $\varprojlim$ gives an equivalence of categories between the AR ℓ-adic category and the category of finitely generated R-modules. Furthermore, suppose*

$$0 \to F \to G \to H \to 0$$

is an exact sequence in the AR category with F and H AR ℓ-adic, and suppose that for some d, we have $\ell^{n+d}G_n = 0$ for all n. Then G is AR ℓ-adic.

Proof. See Conrad's notes, Theorem 1.4.2.5. We use the fact that ℓ-adic systems will satisfy the ML condition. □

13.2. **Back to Etale Sheaves.** For the remainder of this section, X is a Noetherian separated scheme such that $\ell \in \mathcal{O}_X^\times$. We follow the model above to get ℓ-adic sheaves. In what follows, "ℓ-torsion" means "ℓ-power torsion."

Definition 13.11. The *AR category of ℓ-torsion sheaves on X* is the category whose objects are projective systems $\mathcal{F} = (\mathcal{F}_n)$ of ℓ-torsion (etale) sheaves on X, and such that for systems $\mathcal{F}$ and $\mathcal{G}$,

$$\mathrm{Hom}_{\mathrm{AR}}(\mathcal{F}, \mathcal{G}) = \varinjlim_d \mathrm{Hom}(\mathcal{F}[d], \mathcal{G}).$$

Definition 13.12. A system $(\mathcal{F}_n)$ in the AR category of ℓ-torsion sheaves on X is called *ℓ-adic* if:

(1) All $\mathcal{F}_n$ are constructible;
(2) $\ell^{n+1}\mathcal{F}_n = 0$ for all n;
(3) For all n, the map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism $\mathcal{F}_{n+1}/\ell^{n+1} \to \mathcal{F}_n$.

Example 13.13.

(1) If M is a finitely generated $\mathbb{Z}_\ell$-module, then the projective system $(M/\ell^{n+1}M)$ of constant sheaves on X is an ℓ-adic sheaf.
(2) The system $(\mu_{\ell^n,X})$ with morphisms $\mu_{\ell^{n+1}} \to \mu_{\ell^n}$ given by $\zeta \mapsto \zeta^\ell$ is an ℓ-adic sheaf.
(3) If $\mathcal{F}$ is a constructible sheaf with $\ell^m\mathcal{F} = 0$, then the system $(\mathcal{F}/\ell^{n+1}\mathcal{F})$ is eventually constant, but is still an ℓ-adic sheaf. In fact, this construction embeds the category of constructible ℓ-torsion sheaves as a full subcategory of ℓ-adic sheaves.

Lemma 13.14 (Useful properties)**.** *If X is connected and $(\mathcal{F}_n)$ is a system of LCC sheaves, then $\mathcal{F}$ is AR ℓ-adic if and only if the stalks give AR ℓ-adic systems of R-modules. In* *general, given a geometric point of X, taking stalks is an exact functor from the AR category of sheaves to the AR category of R-modules.*

Definition 13.15. We say an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ is *locally constant* if each $\mathcal{F}_n$ is locally constant.

Proposition 13.16. [2, Prop 12.10] *Let $\mathcal{F}$ be an ℓ-adic sheaf on X. Then there is a dense open subscheme U of X such that $\mathcal{F}|_U$ is locally constant.*

Proof. See Freitag and Kiehl [2, Prop 12.10] □

Proposition 13.17. [2, Prop 12.12] *The category of AR ℓ-adic sheaves is noetherian.*

Proof. Use noetherian induction. □

Definition 13.18. The *category of sheaves of $\mathbb{Q}_\ell$-vector spaces on X* has objects same as the AR category of ℓ-torsion sheaves, and hom sets are given by tensoring the AR hom sets with $\mathbb{Q}_\ell$.

14. Poincare Duality

14.1. **Cohomology with Compact Support.** Recall how extension of a sheaf by 0 works in the topological setting. Given an open set $U \subseteq X$ and a sheaf F of abelian groups on U, we define $j_!F$ as the sheafification of the presheaf

$$V \mapsto \begin{cases} F(V), & V \subseteq U, \\ 0 & \text{else.} \end{cases}$$

We only consider open immersions here so that the stalks are zero outside U and the same inside U. In other words, there is a natural isomorphism $F \cong j^{-1}j_!F$. We generalize this as follows.

Definition 14.1. Let $f : U \to X$ be an etale map and $\mathcal{F}$ a sheaf on U. We define $f_!\mathcal{F}$ to be the sheafification of

$$V \mapsto \bigoplus_{\phi \in \mathrm{Hom}_X(V,U)} \mathcal{F}(\phi : V \to U).$$

Here are the key facts about this construction:

- $f_!$ is left adjoint to f^* (recall that f^* is left adjoint to f_*).
- The stalk of $f_!\mathcal{F}$ at a point is the direct sum of stalks of the fiber. Namely, if α is a geometric point of X and $\alpha_1, \ldots, \alpha_r$ are the geometric points of U over α, then

$$(f_!\mathcal{F})_\alpha = \prod_{i=1}^{r} \mathcal{F}_{\alpha_i}.$$

- The functor $f_!$ is exact.
- Extension by 0 is compatible with base change: if we have a cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{h} & X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

  where g is etale and $\mathcal{F}$ is a sheaf of abelian groups on T, then

$$f^*(g_!\mathcal{F}) \cong h_!(e^*\mathcal{F}).$$

References

[1] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[2] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[3] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[4] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2022.

[5] André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.