PERFECTOID RINGS AND SPACES

JACOB SWENBERG

1. Introduction

We begin with a quote from Scholze:

> "...the theory of perfectoid spaces establishes a general framework relating geometric questions over local fields of mixed characteristic with geometric questions over local fields of equal characteristic."
> –Peter Scholze, *Perfectoid Spaces: A Survey* (2013)

For example, we have the following theorem of Fontaine–Wintenberger:

Theorem 1.1 (Fontaine–Wintenberger)**.** *The completions of the fields* $\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup_n \mathbb{Q}_p(p^{1/p^n})$ *and* $\mathbb{F}_p((t))(t^{1/p^\infty}) := \bigcup_n \mathbb{F}_p((t))(t^{1/p^n})$ *have isomorphic absolute Galois groups.*

In this talk, we will define perfectoid rings and spaces and their étale covers. For the sake of time, proofs are almost entirely omitted.

2. Perfectoid Rings

Fix a prime p. We always denote the p-power map $a \mapsto a^p$ by Φ. We denote by π a pseudo-uniformizer in the appropriate Tate ring.

Recall:

- A *Tate ring* R is a Huber ring with a topologically nilpotent unit. Let R be a Tate ring.

Date: Tuesday, January 24, 2023.

- A *uniform Huber ring* is a Huber ring R with R° bounded. Equivalently, R° is a ring of definition.
- A uniform Tate ring R with topologically nilpotent unit π, then R is of the form $R^\circ[\pi^{-1}]$, and R° has the π-adic topology.

Definition 2.1. A *perfectoid ring* is a complete uniform Tate ring that admits a topologically nilpotent unit π such that $p \in \pi^p R^\circ$ and such that the p-power map

$$R^\circ/\pi \to R^\circ/\pi^p$$

is an isomorphism.

Lemma 2.2. *Let R be a uniform Tate ring with topologically nilpotent unit π such that $p \in \pi^p R^\circ$. The following are equivalent:*

(1) $\Phi : R^\circ/\pi \to R^\circ/\pi^p$ *is an isomorphism.*

(2) $\Phi : R^\circ/\pi \to R^\circ/\pi^p$ *is surjective.*

(3) $\Phi : R^\circ/p \to R^\circ/p$ *is surjective.*

Proof. We first remark that $\Phi : R^\circ/\pi \to R^\circ/\pi^p$ is always injective. Indeed, let $a \in R^\circ$, and suppose $a^p = b\pi^p$ for some $b \in R^\circ$. then $(a/\pi)^p = b \in R^\circ$, so $a/\pi \in R^\circ$, and $a \in (\pi)$. Thus, (1) and (2) are equivalent. Furthermore, the commutative diagram

$$\begin{array}{ccc} R^\circ/p & \xrightarrow{\Phi} & R^\circ/p \\ \downarrow & & \downarrow \\ R^\circ/\pi & \xrightarrow{\Phi} & R^\circ/\pi^p \end{array}$$

show that (3) implies (2).

It remains to see (2) implies (3). Let $x \in R^\circ$. Then there exists $x_0 \in R^\circ$ such that $x - x_0^p \in \pi^p R^\circ$. Inductively, we can construct a sequence $x_0, x_1, \dots \in R^\circ$ such that

$$x - (x_0^p + x_1\pi^p + x_2^p\pi^{2p} + \cdots + x_n^p\pi^{np}) \in (\pi^{(n+1)p}).$$

Then $x = \sum_n x_n^p \pi^{np}$, so $x - (\sum_n x_n\pi^n)^p \in pR^\circ$. □

Definition 2.3. A *perfectoid field* is a nonarchimedean field (complete with respect to a non-archimedean $\mathbb{R}$-valued valuation) that is also a perfectoid ring.

Proposition 2.4. *A nonarchimedean field K is perfectoid if and only if the value group of K is not discrete, $|p| < 1$, and $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.*

Example 2.5. The completion $\mathbb{Q}_p^{cyc}$ of $\mathbb{Q}_p(\mu_{p^\infty})$ is a perfectoid field. Similarly, the completion $\mathbb{F}_p((t^{1/p^\infty}))$ of $\mathbb{F}_p((t))(t^{1/p^\infty})$ is a perfectoid field.

For an example of a perfectoid ring that is not a field, consider

$$\mathbb{Q}_p^{cyc}\langle T^{1/p^\infty}\rangle := \left(\varprojlim_n \mathbb{Z}_p^{cyc}[T^{1/p^\infty}]/p^n\right)[1/p]$$

Theorem 2.6 (Berkeley Lectures 6.1.10)**.** *Let (R, R^+) be a Huber pair with R perfectoid. Let $X = \mathrm{Spa}(R, R^+)$. Then for all rational subsets $U \subseteq X$, we have $\mathcal{O}_X(U)$ is perfectoid. It follows that (R, R^+) is sheafy.*

2.1. **Tilting.**

Proposition 2.7 (Berkeley Lectures 6.1.6)**.** *Let R be a complete Tate ring of characteristic p. Then R is perfectoid if and only if R is perfect, i.e. $\Phi : R \to R$ is an isomorphism.*

Proof. Omitted. □

Definition 2.8. Let R be a perfectoid ring. The *tilt* of R is the multiplicative monoid

$$R^\flat := \varprojlim_{\Phi} R$$

with the inverse limit topology.

Remark 2.9. Note that $\Phi : R \to R$ given by $x \mapsto x^p$ is not necessarily even a ring homomorphism, a priori. We give $R^\flat$ addition by

$$(x+y)_i = \lim_n (x_{i+n} + y_{i+n})^{p^n}.$$

(One checks this is well-defined.)

Remark 2.10. Note that if R has characteristic p, then $R^\flat = R$.

Proposition 2.11 (Berkeley Lectures 6.2.2)**.** *$R^\flat$ is a topological $\mathbb{F}_p$-algebra that is a perfect complete Tate ring. Equivalently, $R^\flat$ is a characteristic-p perfectoid ring. We have*

$$(R^\flat)^\circ = \varprojlim_{\Phi} R^\circ \cong \varprojlim_{\Phi} R^\circ/p \cong \varprojlim_{\Phi} R^\circ/\pi,$$

where $p \in \pi R^\circ$. We may also choose π such that $p \in \pi^p R^\circ$ with a compatible system

$$\pi^\flat := (\pi^{1/p^n})_n \in R^\flat$$

that is a pseudo-uniformizer in $R^\flat$. With this choice,

$$R^\flat = (R^\flat)^\circ[1/\pi^\flat].$$

Proof. Omitted. □

Remark 2.12. We denote by $(\cdot)^\sharp : R^\flat \to R$ the projection onto the first coordinate. If K is a perfectoid field, then $K^\flat$ is a complete nonarchimedean field with absolute value $|a| = |a^\sharp|$, as originally considered by Fontaine.

The map $\sharp$ defines an isomorphism

$$(R^\flat)^\circ/\pi^\flat \cong R^\circ/\pi.$$

Example 2.13. Let $(\zeta_{p^n})_n \subset \mathbb{Q}_p^{cyc}$ be a compatible system of p-power roots of unity. Then

$$t = (\zeta_{p^n} - 1)_n \in (\mathbb{Q}_p^{cyc})^\flat$$

is a pseudouniformizer. Furthermore,

$$(\mathbb{Q}_p^{cyc})^\flat = \mathbb{F}_p((t^{1/p^\infty})).$$

Theorem 2.14 (Kedlaya–Shahoseini 2022)**.** *Let K be a complete subfield of $\mathbb{C}_p$. Then K is perfectoid if and only if its tilt is not algebraic over $\mathbb{F}_p$.*

Proof. Dubious. A correction was literally posted today saying their proof is wrong. □

2.2. **Tilting Equivalence.** Let R be perfectoid.

Theorem 2.15 (Tilting Equivalence I (Berkeley Lectures 6.2.7))**.** *Tilting gives an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$-algebras.*

Theorem 2.16 (Tilting Equivalence II (Berkeley Lectures 6.2.6))**.** *Let (R, R^+) be a Huber pair with R perfectoid, let $X := \mathrm{Spa}(R, R^+)$, and let*

$$(R^\flat)^+ := \varprojlim_{\Phi} R^+ \subseteq R^\flat.$$

Then $(R^\flat, (R^\flat)^+)$ is a Huber pair. Furthermore, there is a homeomorphism

$$(\cdot)^\flat : X \to X^\flat := \mathrm{Spa}(R^\flat, (R^\flat)^+), \qquad |f(x^\flat)| = |f^\sharp(x)|.$$

preserving rational subsets. Moreover, given $U \subseteq X$ rational, we have $\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U^\flat)$.

3. Perfectoid Spaces

Definition 3.1. A *perfectoid space* is an adic space covered by adic spaces $\mathrm{Spa}(R, R^+)$ with R perfectoid.

Remark 3.2. If R is a perfectoid ring, we call $\mathrm{Spa}(R, R^+)$ a *affinoid perfectoid space*. Note that it is not clear that an affinoid space that is perfectoid must be an affinoid perfectoid space!

We can glue the tilting maps to obtain a functor $X \mapsto X^\flat$ for any perfectoid space.

Theorem 3.3 (Berkeley Lectures 7.1.4)**.** *Let X be a perfectoid space. There is an equivalence of categories between the category of perfectoid spaces over X and perfectoid spaces over $X^\flat$ given by the map $Y \mapsto Y^\flat$. We call this the* tilting functor*.*

We pause here with a quote from Scholze:

> 'The category of perfectoid spaces over $\mathbb{Q}_p$ is equivalent to the category of perfectoid spaces X of characteristic p together with a "structure morphism $X \to \mathbb{Q}_p$."'
> –Peter Scholze, Berkeley Lectures 7.2

It will turn out that tilting gives an equivalence of étale sites. We start with the case of fields

Theorem 3.4. *Let K be a perfectoid field. Every finite extension L/K is perfectoid. Moreover, the tilting functor gives an equivalence of categories*

$$\left\{\begin{matrix}\textit{finite extensions}\\ \textit{of } K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{finite extensions}\\ \textit{of } K^\flat\end{matrix}\right\}.$$

Corollary 3.5. *The absolute Galois groups of K and $K^\flat$ are isomorphic.*

4. Almost Mathematics

Throughout, R is a perfectoid ring.

Definition 4.1. An R°-module is *almost zero* if $\pi M = 0$ for all pseudo-uniformizers π of R.

Example 4.2. If K is a perfectoid field, then $\mathcal{O}_K/\mathfrak{m}_K$ is almost zero.

Lemma 4.3. *Let R^+ be a ring of integral elements in R. Then R°/R^+ is almost zero.*

Proof. Let $\pi \in R^\circ$ be a pseudo-uniformizer, and let $x \in R^\circ$. Then πx is topologically nilpotent, so $(\pi x)^n \in R^+$ by openness of R^+. Since R^+ is integrally closed, we have $\pi x \in R^+$. □

Definition 4.4. The category of almost R°-modules is the category of R°-modules quotiented by the subcategory of almost zero modules.

Question 4.5. Is this isomorphic to the category of R°-modules, but where morphisms are "morphisms mod almost zero morphisms?"

Theorem 4.6. *Let (R, R^+) be a Huber pair with R perfectoid. Let $X = \mathrm{Spa}(R, R^+)$. Then $H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$, and $H^0(X, \mathcal{O}_X^+) = R^+$.*

5. Étale morphisms

Definition 5.1. A morphism $f : X \to Y$ of perfectoid spaces is *finite étale* if for all $\mathrm{Spa}(B, B^+) \subset Y$ open, we have $X \times_Y \mathrm{Spa}(B, B^+) = \mathrm{Spa}(A, A^+)$ where A is a finite étale B-algebra, and A^+ is the integral closure of B^+ in A. A morphism $f : X \to Y$ is *étale* if X can be covered by open subsets U with open subset $V \supseteq f(U)$ of Y such that $U \to V$ factors through an open immersion $U \to W$ with $W \to V$ finite étale.

Theorem 5.2 (Tate)**.** *Let K be a perfectoid field and L/K a finite extension. Then $\mathcal{O}_L/\mathcal{O}_K$ is* almost finite étale.

Theorem 5.3. *Any finite étale R-algebra is perfectoid, and the category of such algebras is equivalent to the category of finite étale $R^\flat$-algebras under tilting.*

Theorem 5.4. *There exists an étale site $X_{ét}$ such that $X_{ét} \cong X^\flat_{ét}$. In particular, the tilt of a (finite) étale morphism is (finite) étale. Furthermore, if X is affinoid, then*

$$H^i(X_{ét}, \mathcal{O}_X^+)$$

is almost zero for $i > 0$.