290B: Week 5

Today we will talk about the construction teased last time, namely a functor

$$\left\{\begin{smallmatrix}\text{analytic pre-adic}\\ \text{spaces/Spa }\mathbb{Z}_p\end{smallmatrix}\right\} \to \{\text{diamonds}\}$$
$$X \mapsto X^\diamond$$

which is similar to the tilting construction of week 3: it changes the characteristic that we're looking at, but retains some topological data. A key ingredient will be untilting and the untilting correspondence, which is described in sections 6 and 7 of the Berkeley notes.

§1 : Recall diamonds

Recall that last time we saw two equivalent definitions of diamonds:

Proposition-definition 1.

1. *A* diamond *is a pro-étale sheaf* $\mathcal{D}$ *on* Perf *of the form*

$$\mathcal{D} = X/R \in \mathrm{Shv}_{\text{proét}}(\mathrm{Perf}),\ \exists R \subset X \times X \text{ an equivalence relation}$$

   for X *a perfectoid space of characteristic* p *and* R *a perfectoid space such that the two projections* $R \rightrightarrows X$ *are pro-étale.*

2. *Equivalently,*[1] *a* diamond *is a pro-étale sheaf* $\mathcal{D}$ *on* Perf *with a surjective* quasi-pro-étale *morphism*

$$X \twoheadrightarrow \mathcal{D} \ \text{ in } \mathrm{Shv}_{\text{proét}}(\mathrm{Perf})$$

   for some perfectoid space X *of characteristic* p*.*

In proving that a particular X/R yields a diamond, proving that $R \to X \times X$ is actually an injection of perfectoid spaces will be an important step.

We won't define the term quasi-pro-étale, as we'll mainly be using the original definition. If you want to learn more about this second definition, see subsection 9.2. The subsection also contains a cool illustration of a diamond.

We will also use the following proposition in the proof of the last theorem:

[1] Prop 9.2.3

Proposition 2 (Proposition 8.3.3)**.** *Let $f : Y \to X$ be a map of perfectoid spaces. Then the following are equivalent (and if any holds, we say f is injective):*

1. *For all perfectoid spaces T, $\mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$ is injective.*
2. *For all algebraically closed affanoid fields (C, C^+), the map $Y(K, K^+) \to X(K, K^+)$ is injective.*

§2 : G-torsors

For a finite group G, a *G-torsor* (in any topos) is a morphism $\mathcal{F}' \to \mathcal{F}$ with a G-action $G \times \mathcal{F}' \to \mathcal{F}'$ over $\mathcal{F}$ such that, locally on $\mathcal{F}$, there is a G-equivariant isomorphism $\mathcal{F}' \cong \mathcal{F} \times G$.

We now modify the definition to suit a topological group G.

- For a topological space T, the presheaf

$$\begin{aligned} \underline{T} : \mathrm{Perf}^{\mathrm{op}} &\to \mathrm{Set} \\ X &\mapsto C^0(|X|, T) \end{aligned}$$

  forms a pro-étale sheaf.

- If G is a topological group, then $\underline{G}$ forms a pro-étale sheaf of groups. Moreover, if $G = \lim_i G_i$ is profinite, then $\underline{G} = \lim_i \underline{G_i}$.

Remark 1. $\underline{G}$ is not representable, as Perf has no final object. However, if X is a perfectoid space and G is profinite, then $X \times \underline{G}$ becomes representable (in perfectoid spaces over X?) by

$$X \times \underline{G} = \lim_{H \trianglelefteq G:\ \mathrm{open}} X \times G/H.$$

//

We now define a torsor in our context:

Definition 3. *Let G be a topological group. A $\underline{G}$-*torsor *is a morphism*

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \in \mathrm{Mor}\big(\mathrm{Shv}(\mathrm{Perf})\big), \quad \text{together with an action } \underline{G} \times \mathcal{F}' \to \mathcal{F}' \text{ over } \mathcal{F},$$

such that locally on $\mathcal{F}$ there is a $\underline{G}$-equivariant isomorphism $\mathcal{F}' \cong \mathcal{F} \times \underline{G}$.

Proposition 4 (Proposition 9.3.1)**.** *Let $f : \mathcal{F}' \to \mathcal{F}$ be a $\underline{G}$-torsor, with G profinite. Then for any affinoid $X = \mathrm{Spa}(B, B^+)$ and any $X \to \mathcal{F}$, the pullback $\mathcal{F}' \times_{\mathcal{F}} X$ is representable by a perfectoid affinoid $X' = \mathrm{Spa}(A, A^+)$. Moreover, A is the completion of $\mathrm{colim}_H A_H$ where for each normal open subgroup $H \subset G$, A_H/B is a finite étale G/H-torsor in the algebraic sense.*

§3 : The diamond Spd $\mathbb{Q}_p$ and the pro-étale sheaf Untilt

Consider the perfectoid space $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ as a pro-étale sheaf on Perf.

Let's recall some of the notation from week 3. $\mathbb{Q}_p^{\mathrm{cycl}}$ is the completion of the maximal cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$ of $\mathbb{Q}_p$. It's a perfectoid ring (of characteristic zero).

For any perfectoid ring R, we have its *tilt*

$$R^\flat := \lim_{\Phi} R, \quad \Phi : R \xrightarrow{x \mapsto x^p} R,$$

which forms a perfectoid ring[2] *of characteristic* p; if R was already of characteristic p, then $R^\flat = R$. If (R, R^+) is a Huber pair with R perfectoid, then $\left(R^\flat, (R^+)^\flat\right)$ is a Huber pair, and there's a homeomorphism

$$\mathrm{Spa}(R, R^+) \to \mathrm{Spa}\left(R^\flat, \left(R^+\right)^\flat\right).$$

For $R = \mathbb{Q}_p^{\mathrm{cycl}}$, we have

$$(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \cong \mathbb{F}_p((t^{1/p^\infty})).$$

Lastly, here is a definition we'll need in a moment:

Definition 5. *For a perfectoid Tate ring (R, R^+), an* untilt *is a perfectoid Tate ring $(R^\sharp, {R^\sharp}^+)$ together with an isomorphism ${R^\sharp}^\flat \to R$ such that ${R^\sharp}^+$ and R^+ are identified (under lemma 6.2.5).*

We can describe untilting in terms of the Witt construction:

Theorem 6 (Lemma 6.2.8)**.** *Let $(R^\sharp, R^{\sharp+})$ be an untilt of (R, R^+).*

1. *There is a canonical surjection of rings $\theta : W(R^+) \twoheadrightarrow R^{\sharp+}$*

2. *The kernel of θ is generated by a nonzero-divisor ζ of the form $\zeta = p + [\pi]\alpha$, where $\pi \in R^+$ is a pseudo-uniformizer and $\alpha \in W(R^+)$.*

[2] With addition defined as in week 3.

See also theorem 6.2.11 for a more precise correspondence.

Proposition 7 (Theorem 7.1.4)**.** *If $f : X \to Y$ is in* Perf *and $Y^\sharp$ is an untilt of Y, then there's a unique morphism of perfectoid spaces $X^\sharp \to Y^\sharp$ whose tilt is f.*

For any perfectoid ring (R, R^+), the (R, R^+)-valued points of $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ are

$$\left\{\mathbb{F}_p((t^{1/p^\infty})) \cong (\mathbb{Q}_p^{\mathrm{cycl}})^\flat \xrightarrow{f} R; f \text{ cont.}\right\} \cong \left\{x \in R^\times; x \text{ top. nilpotent}\right\}.$$

We now define the diamond $\mathrm{Spd}\ \mathbb{Q}_p$:

Definition 8. *We consider $\underline{\mathbb{Z}_p^\times}$ as inducing a equivalence relation (by its action[3]) on $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$. Then*

$$\mathrm{Spd}\ \mathbb{Q}_p := \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat / \underline{\mathbb{Z}_p^\times} = \mathrm{CoEq}\left(\underline{\mathbb{Z}_p^\times} \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \underset{\text{action}}{\overset{\mathrm{pr}_2}{\rightrightarrows}} \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat\right).$$

Remark 2. To be more consistent with the earlier notation for diamonds, this should maybe instead be written

$$\mathrm{Spd}\ \mathbb{Q}_p := \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \Big/ \left(\underline{\mathbb{Z}_p^\times} \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat\right), \tag{1}$$

$$\text{with} \quad g : \underline{\mathbb{Z}_p^\times} \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \xrightarrow{(\mathrm{pr}_2, \text{action})} \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat. \tag{2}$$

//

For (1) to be a diamond, we need (2) to be an injection. This is shown in lemma 9.4.2.

Proposition 9 (Corollary of lemma 9.4.2)**.** $\mathrm{Spa}\ \mathbb{Q}_p^{\mathrm{cycl}} \to \mathrm{Spd}\ \mathbb{Q}_p$ *is a $\underline{\mathbb{Z}_p^\times}$-torsor.*

Additionally, if $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space of characteristic p, then $\mathrm{Spd}\ \mathbb{Q}_p(X)$ *is the set of isomorphism classes of data of the following type:*

1. *A $\mathbb{Z}_p^\times$-torsor $R \to \tilde{R} = (\mathrm{colim}\, R_n)^\wedge$, where $R \to R_n$ is finite étale with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$.*
2. *A top. nilpotent $t \in \tilde{R}^\times$ such that for all $\gamma \in \mathbb{Z}_p^\times$, $\gamma(t) = (1+t)^\gamma - 1$.*

[3] $\mathbb{Z}_p^\times$ acts on $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \cong \mathbb{F}_p((t^{1/p^\infty}))$ by $\gamma(t) = (1+t)^\gamma - 1$ for $\gamma \in \mathbb{Z}_p^\times$.

The idea of the proof is as follows:

We first show that $\underline{\mathbb{Z}_p^\times} \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \to \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \times \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ is actually an injection, using proposition 8.3.3 and the fact that for any perfectoid affinoid field (K, K^+), $\mathbb{Z}_p^\times$ acts freely on $\mathrm{Hom}(\mathbb{F}_p((t^{1/p^\infty})), K)$.

Then we use proposition 9.3.1 and the fact that for any $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}\ \mathbb{Q}_p$, we get via pullback a $\underline{\mathbb{Z}_p^\times}$-torsor over $\mathrm{Spa}(R, R^+)$.

The following is a special case of the theorem which was teased last week as the motivation for diamonds:

Theorem 10. *The following categories are equivalent:*

- *Perfectoid spaces over* $\mathbb{Q}_p$.
- *Perfectoid spaces* X *of characteristic* p *equipped with a "structure morphism"* $X \to \mathrm{Spd}\ \mathbb{Q}_p$.

These are both fibered over Perf. Respectively, the morphisms to Perf are

- $X \mapsto X^\flat$
- $X \mapsto X$.

Recall that for a (Grothendieck) fibration of categories, we need to be able to pull back morphisms in the base category or, more precisely, to have at least one inverse image for every morphism of Perf whose codomain is in the image of the projection. This follows (for the first category) from proposition 7. These fibered categories correspond, respectively, to two presheaves of groupoids on Perf:

- $X \mapsto \mathrm{Untilt}_{\mathbb{Q}_p}(X) = \{(X^\sharp, \iota); X^\sharp\text{: perfectoid space over } \mathbb{Q}_p, \iota : X^{\sharp^\flat} \cong X\}$
- $\mathrm{Spd}\ \mathbb{Q}_p$.

Definition 11. *Let* Untilt *be the presheaf on* Perf *taking*

$$S \mapsto \left\{(S^\sharp, \iota); {}^{S^\sharp \text{ a perfectoid space,}}_{\iota : S^{\sharp^\flat} \cong S}\right\} /\text{isomorphism}$$

We state the following important lemma without proof:

Lemma 12 (Lemma 9.4.5)**.** Untilt *is a pro-étale sheaf.*

Note that the uniqueness of proposition 7 means that we don't have the obvious obstruction.

In particular, this implies $\mathrm{Untilt}_{\mathbb{Q}_p}$ is a pro-étale sheaf.

Proof of theorem 10. We need give an isomorphism between the sheaves $\mathrm{Untilt}_{\mathbb{Q}_p}$ and $\mathrm{Spd}\ \mathbb{Q}_p$.

Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space of characteristic p. For an untilt $X^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$, let

$$\tilde{R}^\sharp := R^\sharp \hat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{cycl}},$$

and let $\tilde{R}^{\sharp +}$ be the completion of the integral closure of $R^{\sharp +}$ in $\tilde{R}^\sharp$, and let

$$\tilde{X}^\sharp := \mathrm{Spa}(\tilde{R}^\sharp, \tilde{R}^{\sharp +}).$$

Then $\tilde{X}^\sharp \to X^\sharp$ is a pro-étale $\underline{Z_p^\times}$-torsor, whose tilt $\tilde{X} \to X$ is a pro-étale $\underline{Z_p^\times}$-torsor with a $\underline{Z_p^\times}$-equivariant map $\tilde{X} \to \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$, so we've produced $X \to \mathrm{Spd}\ \mathbb{Q}_p$.

Going the other way, suppose $\tilde{X} \to X$ is a pro-étale $\underline{Z_p^\times}$-torsor with a $\underline{Z_p^\times}$-equivariant map $\tilde{X} \to \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$. By proposition 7, there is a unique $\tilde{X}^\sharp \to \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ which is also $\underline{Z_p^\times}$-equivariant. This equivarance means exactly that $\tilde{X}^\sharp$ comes with a descent datum along $\tilde{X} \to X$, and so by lemma 12 we get an untilt $X^\sharp$ of X over $\mathbb{Q}_p$. $\square$

§4 : Pre-adic spaces and the functor $X \mapsto X^\diamond$

In this section, we make good on the motivation for diamonds from week 4.

Remember that there was an issue with Huber pairs in general: the structure presheaf on $\mathrm{Spa}(A, A^+)$ can fail to be a sheaf. One way to deal with this is—similar to the theory of algebraic spaces—to enlarge the category of adic spaces, by looking at a larger subcategory of $\mathrm{PSh}(\mathrm{CAff}^{\mathrm{op}})$, where CAff is the category of complete Huber pairs. This enlargement will, in particular, include more fiber products than the category of adic spaces alone.

We turn $\mathrm{CAff}^{\mathrm{op}}$ into a site by giving it the coarsest topology for which every rational cover is a cover, i.e. the topology where for every rational cover $X = \mathrm{Spa}(A, A^+) = \bigcup_i U_i$, the set of maps $(\mathcal{O}_X(U_i), \mathcal{O}_X^+(U_i))^{\mathrm{op}} \to (A, A^+)^{\mathrm{op}}$ is a cover, and these generate all covers. For $(A, A^+)^{\mathrm{op}} \in \mathrm{CAff}^{\mathrm{op}}$, we denote by

$$\mathrm{Spa}^Y(A, A^+) \colon \text{the sheafification of } (B, B^+) \mapsto \mathrm{Hom}_{\mathrm{CAff}}((B, B^+), (A, A^+)).$$

Definition 13. *Let $\mathcal{F}$ be a sheaf on* $\mathrm{CAff}^{\mathrm{op}}$*, and let* $X = \mathrm{Spa}(A, A^+)$ *for a complete Huber pair* (A, A^+)*. A map* $\mathcal{F} \to \mathrm{Spa}^Y(A, A^+)$ *is an open immersion if there's an open* $U \subset X$ *such that*

$$\mathcal{F} = \operatorname*{colim}_{V \subset U\text{: rational}} \mathrm{Spa}^Y(\mathcal{O}_X(V), \mathcal{O}_X^+(V)).$$

A map of sheaves $\mathcal{F} \to \mathcal{G}$ *is an open immersion if for all* $(A, A^+) \in \mathrm{CAff}$ *with* $\mathrm{Spa}^Y(A, A^+) \to \mathcal{G}$*, the fiber product*

$$\mathcal{F} \times_{\mathcal{G}} \mathrm{Spa}^Y(A, A^+) \to \mathrm{Spa}^Y(A, A^+)$$

is an open immersion.

We call a sheaf $\mathcal{F}$ *on* $\mathrm{CAff}^{\mathrm{op}}$ *a* pre-adic space *if*

$$\mathcal{F} = \operatorname*{colim}_{\mathrm{Spa}^Y(A, A^+) \subset \mathcal{F}\text{:open}} \mathrm{Spa}^Y(A, A^+).$$

Now recall (from week 2):

Proposition-definition 14. *A Huber ring A is* analytic *if the ideal generated by topologically nilpotent elements is the unit ideal.*

If (A, A^+) *is a complete Huber pair, then A is analytic if and only if all points of* $\mathrm{Spa}(A, A^+)$ *are analytic, in the sense of definition 4.2.1, and a point* $x \in \mathrm{Spa}(A, A^+)$ *is analytic if and only if there's a rational neighborhood* $x \in U \subset \mathrm{Spa}(A, A^+)$ *such that* $\mathcal{O}_X(U)$ *is Tate.*

We define a functor

$$\left\{\begin{matrix}\text{analytic pre-adic}\\ \text{spaces/Spa } \mathbb{Z}_p\end{matrix}\right\} \to \{\text{diamonds}\}$$
$$X \mapsto X^\diamondsuit$$

as follows:

Definition 15. *Let X be an analytic pre-adic space over* $\mathrm{Spa}\,\mathbb{Z}_p$*. We define a presheaf $X^\diamondsuit$ on* Perf *by*

$$X^\diamondsuit(T) = \left\{(T^\sharp, T^\sharp \to X)\right\}/\text{isomorphism}, \text{ for } T \text{ perfectoid of characteristic } p,$$

where $T^\sharp$ stands for an untilt of T and $T^\sharp \to X$ is a map of pre-adic spaces.

If $X = \mathrm{Spa}(R, R^+)$*, write* $\mathrm{Spd}(R, R^+) := \mathrm{Spa}(R, R^+)^\diamondsuit$*.*

Remark 3. If X is perfectoid, then $X^\diamond = X^\flat$. Note also that the pairs $(T^\sharp, T^\sharp \to X)$ have no nontrivial automorphisms, so we don't have this potential obstruction to $X^\diamond$ forming a sheaf. //

Definition 16. *Let* $\operatorname{Spd} \mathbb{Z}_p := \mathrm{Untilt}$ *be the sheaf on* Perf *taking*

$$S \mapsto \left\{ (S^\sharp, \iota);\, {}^{S^\sharp \text{ a perfectoid space (fibered uniquely over } \operatorname{Spa} \mathbb{Z}_p),}_{\iota: S^{\sharp\flat} \cong S} \right\} / \text{isomorphism}$$

This is a pro-étale sheaf by lemma 12.

Theorem 17. $X^\diamond$ *is a diamond.*

Sketch. $X^\diamond$ being a sheaf follows from lemma 12 and the argument of proposition 8.2.8, which shows that maps from perfectoid spaces $T^\sharp$ to a fixed pre-adic X form a pro-étale sheaf.

Since X is analytic, we may thus assume (by restriction to a rational cover) that $X = \operatorname{Spa}(R, R^+)$ is affinoid, with R a Tate ring. Since $X \to \operatorname{Spa} \mathbb{Z}_p$, we must have $p \in R$ topologically nilpotent.

It remains to prove (by proposition 4):

Lemma 18. *Let* $\operatorname{colim} R_i$ *be a filtered direct limit of* $\{R \to R_i\}$*, all finite étale algebras, which admits no nonsplit finite étale covers. Endow* $\operatorname{colim} R_i$ *with the topology making* $\operatorname{colim} R_i^\circ$ *open and bounded, and let* $\tilde{R}$ *be the completion. We state without proof that*

Claim 19. $\tilde{R}$ *is perfectoid.*

Assume further that each R_i *is a* G_i*-torsor over* R*, compatibly with change in* i *for an inverse system of finite groups* $\{G_i\}$*. Let* $G = \lim_i G_i$*. Then*

$$\operatorname{Spd}(R, R^+) = \operatorname{Spd}(\tilde{R}, \tilde{R}^+)/\underline{G},$$

and $\operatorname{Spd}(R, R^+) \to \operatorname{Spd}(\tilde{R}, \tilde{R}^+)$ *is a* $\underline{G}$*-torsor. In particular,* $\operatorname{Spd}(R, R^+)$ *is a diamond.*

Proof of lemma 18. By proposition 8.3.3 and proposition 4, we need to show that for any algebraically closed nonarchimedian field C of characteristic p, the group G acts freely on $\operatorname{Hom}(\tilde{R}^\flat, C)$.

Fix $f : \tilde{R}^\flat \to C$. By the tilting equivalence, this corresponds to a map $f^\sharp : \tilde{R} \to C^\sharp$, or more precisely $\tilde{R}^\circ = W(\tilde{R}^{\flat\circ})/I$, where I is G-stable. We get $W(f^\circ) : W(\tilde{R}^{\flat\circ}) \to W(\mathcal{O}_C)$, which descends mod I to $\tilde{R}^\circ \to \mathcal{O}_{C^\sharp}$.

Assume there exists $\gamma \in G$ such that

$$\begin{tikzcd} \tilde{R}^\flat \arrow[rr, "f"] & & C \\ & \tilde{R}^\flat \arrow[ul, "\gamma"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. Applying W and reducing mod I, we get

$$\begin{tikzcd} \tilde{R}^\circ \arrow[rr, "f^{\sharp\circ}"] & & \mathcal{O}_{C^\sharp} \\ & \tilde{R}^\circ \arrow[ul, "\gamma"] \arrow[ur, "f^{\sharp\circ}"'] & \end{tikzcd}$$

Inverting π, we have for all i that

$$\begin{tikzcd} R_i \arrow[rr, "f^\sharp"] & & C^\sharp \\ & R_i \arrow[ul, "\gamma"] \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes, so γ must be 1. By definition of $\tilde{R}$, G acts freely on $\mathrm{Hom}(\tilde{R}^\flat, C)$. $\square$

$\square$