UCLA NUMBER THEORY LEARNING SEMINAR 22F: HIDA THEORY

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The goal of this seminar is to introduce students of number theory to the basics of Hida theory. The main reference is Hida's *Elementary theory of L-functions and Eisenstein series* (the "Blue Book") [Hid93], in which Hida frequently follows the presentation of Wiles (in contrast to the presentation in papers of Hida [Hid86b][Hid86a]). Unpublished notes from Matt Lafferty on Hida Theory [Laf] give a streamlined presentation of the relevant content in [Hid93], but with less exposition or motivation. Many other useful references are given in the bibliography.

1. Modular Forms: Classical, p-adic, and Λ-adic

1.1. **Motivation and Background.** On Tuesday, September 27, Hida gave a talk at UCLA to begin this learning seminar: see his notes at `https://www.math.ucla.edu/~hida/UCLA22.pdf`.

1.2. **Introduction to Modular Forms.** Chapter 5 in the Blue Book [Hid93, Ch. 5] or Section 1 of Lafferty's notes [Laf, §1]. Defining modular forms, Eisenstein series, Hecke Operators. State the Hecke duality theorem [Hid93, §5.3, Theorem 1]. Briefly define p-adic modular forms ([Laf, §3], or read [Hid86b, §1]).

1.3. **Introduction to Λ-adic forms and ordinary forms.** Define Λ-adic forms; see [Hid93, §7.1-2] or [Laf, §4]. Define the corresponding Hecke algebras. Ordinary forms: introduce Hida's idempotent (the *ordinary projector*, see [Hid93, §7.2]), and define the space of ordinary forms [Laf, §3.2].

1.4. **Eichler-Shimura.** See Chapter 6 of Hida's Blue Book [Hid93, §6.1-3], or [Laf, §2] for Eichler–Shimura. Discuss the Λ-adic version of Eichler–Shimura.

1.5. **Structure Theorem of spaces of Λ-adic forms.** Finiteness and Freeness of the space of modular forms as a Λ-module. See [Hid93, §7.3, Theorem 1], or Propositions 4.5.1-3 in [Laf, §4.5].

1.6. **The Control Theorem.** Motto: "Specialization maps are isomorphisms." This is [Hid93, §7.3, Theorem 3] or [Laf, Theorem 4.5.1]. Perhaps begin with the discussion starting on page 211 of [Hid93, §7.3], including Proposition 1, Proposition 2, and Theorem 2.

2. Ordinary Galois representations, congruences of cusp forms, and further directions

The items labeled **(Optional)** give some possibilities for the last few talks, left to the discretion of the speaker.

2.1. **Galois representations from Hida families.** [Laf, §5], [Hid93, §7.5]. Alternatively, see [Hid86a].

2.2. **More on Congruences of Cusp Forms (Optional).** See the introduction of [Hid86b]. Also, see [Hid85]. Explain how congruences of modular forms are connected to the structure of certain Iwasawa modules and to the ring-theoretic properties of Hecke algebras.

2.3. **Modular Symbols and Hida Theory (Optional).** See [EPW06][DHH$^+$16] for the connection between modular symbols and Hida theory. See [Pol14] for a gentle introduction to modular symbols.

2.4. **Eigenvarieties (Optional).** See [Buz07][Bel21, §2.10]. There is possibly too much to discuss here, but one could briefly outline the connection between Hida theory and eigenvarieties (especially historically).

2.5. **Other Generalizations (Optional).** See [Hid88] for some generalizations to arbitrary totally real fields. Also see [Oht95].

References

[Bel21] Joël Bellaïche. *The eigenbook—eigenvarieties, families of Galois representations, p-adic L-functions*. Pathways in Mathematics. Birkhäuser/Springer, Cham, [2021] ©2021.

[Buz07] Kevin Buzzard. Eigenvarieties. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 59–120. Cambridge Univ. Press, Cambridge, 2007.

[DHH+16] Evan P. Dummit, Márton Hablicsek, Robert Harron, Lalit Jain, Robert Pollack, and Daniel Ross. Explicit computations of Hida families via overconvergent modular symbols. *Res. Number Theory*, 2:Paper No. 25, 54, 2016.

[EPW06] Matthew Emerton, Robert Pollack, and Tom Weston. Variation of Iwasawa invariants in Hida families. *Invent. Math.*, 163(3):523–580, 2006.

[Hid85] Haruzo Hida. Congruences of cusp forms and Hecke algebras. In *Séminaire de théorie des nombres, Paris 1983–84*, volume 59 of *Progr. Math.*, pages 133–146. Birkhäuser Boston, Boston, MA, 1985.

[Hid86a] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[Hid86b] Haruzo Hida. Iwasawa modules attached to congruences of cusp forms. *Ann. Sci. École Norm. Sup. (4)*, 19(2):231–273, 1986.

[Hid88] Haruzo Hida. On p-adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math. (2)*, 128(2):295–384, 1988.

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[Laf] Matthew Lafferty. Notes on hida theory.

[Oht95] Masami Ohta. On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms. *J. Reine Angew. Math.*, 463:49–98, 1995.

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