ON OHTA'S ORDINARY p-ADIC EICHLER–SHIMURA ISOMORPHISM

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1. Introduction

In this short note, we given an outline of Ohta's proof of a p-adic Eichler–Shimura isomorphism. We mainly follows Ohta's original proof given in [5] which excludes certain eigenspaces.

Throughout, let $p \geq 5$ be a prime number, and let $N \geq 1$ be an integer not divisible by p. Recall that for each $r \geq 0$, we have modular curves $X_r := X_1(Np^r)$ defined over $\mathbb{Q}$. Following Ohta, we choose our models of X_r such that the cusp corresponding to $i\infty$ is rational. Let C_r be the set of cusps on X_r, and let $Y_r := X_r - C_r$ be the corresponding open modular curve.

We begin by considering the étale cohomology groups

$$H_r := H^1_{\text{ét}}(X_{r,\overline{\mathbb{Q}}}, \mathbb{Z}_p).$$

These groups admit an action of certain (adjoint) Hecke algebras $\mathfrak{h}_r$ over $\mathbb{Z}_p$, and in particular an action of the Hecke operator $T(p)^*$. We will consider a certain idempotent e acting on H_r which is the composition of Hida's ordinary projector for $T(p)^*$ and projection away from the $1, \omega^{-1}$ eigenspaces for the action of $(\mathbb{Z}/p\mathbb{Z})^\times$. There are natural trace mappings $H_{r+1} \to H_r$ that commute with $T(p)^*$, and so we may define

$$H := e \cdot \varprojlim_r H_r.$$

This group admits an action of the algebra $\mathfrak{h} := \varprojlim_r e\mathfrak{h}_r$. This group also admits an action of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, though for this note we will be focused on the action of $G_{\mathbb{Q}_p}$.

We fix embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \mathbb{C}_\ell$ for all primes ℓ. Let $I_p \subseteq G_{\mathbb{Q}_p}$ be the inertia subgroup. Let $\chi_{cyc} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be the mod-p cyclotomic character. Let K_∞ be a complete subfield of $\mathbb{C}_p$ containing all roots of unity, and let $\mathcal{O}_\infty$ be its ring of integers. Let $\Lambda := \mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ be the Iwasawa algebra. Then $\mathfrak{h}$ is an algebra over Λ by sending group elements to adjoint diamond operators. Let $\Lambda_{\mathcal{O}_\infty} := \Lambda \widehat{\otimes} \mathcal{O}_\infty \cong \mathcal{O}_\infty[[T]]$. We may consider the space $S(N, \Lambda_{\mathcal{O}_\infty})$ of $\Lambda_{\mathcal{O}_\infty}$-adic cusp forms (in the sense of Hida).

Theorem 1.1 (Ohta)**.** *There is a canonical short exact sequence of $\mathfrak{h}[G_{\mathbb{Q}_p}]$-modules*

$$0 \to H_{sub} \to H \to H_{quo} \to 0$$

such that:

1. *For all $\sigma \in I_p$, we have that σ acts on H_{sub} trivially and on H_{quo} as $\chi_{cyc}(\sigma)^{-1}\langle \chi_{cyc}(\sigma)^{-1}\rangle^*$;*
2. *The sequence is non-canonically split as $\mathfrak{h}$-modules;*
3. *There is a Λ-bilinear pairing $H \times H \to \Lambda$ that induces a perfect pairing*

$$H_{sub} \times H_{quo} \to \Lambda.$$

.

(4) *There is a canonical isomorphism of* $\mathfrak{h}[G_{\mathbb{Q}_p}]$*-modules*

$$H_{quo}\widehat{\otimes}\mathcal{O}_\infty \cong eS(N;\Lambda_{\mathcal{O}_\infty})(-1).$$

This Theorem has numerous implications. In particular, it is a key ingredient in the construction of the map Υ in Sharifi's Conjectures.

2. Good quotients of Jacobians

In this section, we review the construction of "good quotients" of Jacobians of modular curves used by Ohta. Ohta references Tilouine [7] and Mazur–Wiles [4] for this construction.

Let J_r be the Jacobian of X_r. Let X'_r be the modular curve corresponding to the congruence subgroup $\Gamma_1(Np^r)\cap\Gamma_0(p^{r+1})$, and let J'_r be its Jacobian. We have natural projections

$$X_{r+1} \xrightarrow{\pi_r} X'_r \xrightarrow{\rho_r} X_r$$

inducing maps on Jacobians

$$(\rho_r)_* : J'_r \to J_r, \qquad \pi_r^* : J'_r \to J_{r+1}.$$

Ohta defines the "good quotients" of the J_r inductively for $r \geq 1$ as follows: we set $B_1 := J_1/\pi_0^*(J'_0)$, and we let $\alpha_1 : J_1 \to B_1$ be the quotient map. Now suppose we have constructed $\alpha_r : J_r \to B_r$ for some $r \geq 1$. We let

$$K_r := \ker(\alpha_r \circ (\rho_r)_*), \qquad B_{r+1} := J_{r+1}/\pi_r^*(K_r)^\circ,$$

and $\alpha_{r+1} : J_{r+1} \to B_{r+1}$ is the quotient map. Note that the kernel of α_r is always connected by construction.

Lemma 2.1 (Tilouine)**.** *For all* $r \geq 1$*, we have that* $\ker(\alpha_r)$ *is* $\mathbb{Q}$*-rational and stable under all (usual) Hecke operators. Thus, we have a well-defined* $\mathbb{Q}$*-rational action of Hecke operators on* B_r*.*

Let w_r be the Atkin–Lehner involution of X_r. In what follows, we replace K_r with $w_r(K_r)$, and correspondingly replace B_r with $J_r/w_r(\ker(\alpha_r))$. Then $\mathfrak{h}_r$ acts on B_r, compatibly in r. Let $\mathcal{O}_r := \mathbb{Z}_p[\mu_{p^r}]$.

Proposition 2.2 (c.f. [5], bottom of page 76)**.** *The abelian scheme* $B_{r/\mathbb{Q}_p(\mu_{p^r})}$ *has good reduction. In other words, there is an abelian scheme* $B_{r/\mathcal{O}_r}$ *over* $\mathcal{O}_r$ *with generic fiber* $B_{r/\mathbb{Q}_p(\mu_{p^r})}$*.*

For each $r \geq 1$, we let

$$G_r := e(B_{r/\mathcal{O}_r}[p^\infty]),$$

which is a p-divisible group over $\mathcal{O}_r$.

Proposition 2.3. [5, Theorem 3.2.5] *The following are true:*

(1) *The map* α_r *induces an isomorphism*

$$e(J_r[p^\infty]) \cong e(B_r[p^\infty]).$$

Consequently, the p*-adic Tate module* T_pG_r *is isomorphic to*

$$e(T_pJ_r) \cong eH_r(1).$$

(2) G_r *is ordinary, in the sense that* G_r° *has étale Cartier dual.*

(3) *Let* $(\cdot,\cdot) : T_pJ_r \times T_pJ_r \to \mathbb{Z}_p(1)$ *be the usual Weil pairing. Define a "twisted Weil pairing" by*

$$[\cdot,\cdot]_r : T_pJ_r \times T_pJ_r \to \mathbb{Z}_p(1), \qquad [x,y]_r := (x, w_r y)_r.$$

Then $T_p(G_r^\circ)$ *is self-annihilating under this pairing, and this pairing gives a perfect duality between* $T_p(G_r^\circ)$ *and* $T_p(G_r^{ét})$*.*

(4) *The action of the inertia subgroup* $I_p \subset G_{\mathbb{Q}_p}$ *leaves* $T_p(G_r^\circ)$ *(and hence* $T_p(G_r^{ét})$*) stable. For* $\sigma \in I_p$*, we have that* σ *acts trivially on* $T_p(G_r^{ét})$*, and acts on* $T_p(G_r^\circ)$ *as* $\chi_{cyc}(\sigma)\langle\chi_{cyc}(\sigma)\rangle^*$*.*

(5) *The following exact sequence of* $e\mathfrak{h}_r$*-modules (non-canonically) splits:*

$$0 \to T_p(G_r^\circ) \to T_pG_r \to T_p(G_r^{ét}) \to 0.$$

Corollary 2.4. *Let*

$$H_{r,sub} := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G_r^{ét}), \mathbb{Z}_p), \qquad H_{r,quo} := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G_r^\circ), \mathbb{Z}_p).$$

Then we have a short exact sequence

$$0 \to H_{r,sub} \to eH_r \to H_{r,quo} \to 0$$

satisfying conditions (1)-(3) of Theorem 1.1. These exact sequences are compatible in r*, so taking a limit, we obtain a short exact sequence*

$$0 \to H_{sub} \to H \to H_{quo} \to 0$$

satisfying conditions (1)-(3) of Theorem 1.1, where

$$H_{sub} := \varprojlim_r H_{r,sub}, \qquad H_{quo} := \varprojlim_r H_{r,quo}.$$

3. From cohomology to cusp forms

For the following argument, Ohta refers to Tate [6].

We saw above that the sub and quo of H come from the connected-étale sequences for the p-divisible groups G_r (after taking Tate modules and $\mathbb{Z}_p$-duals). In particular, we saw that

$$H_{r,quo} = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G_r^\circ), \mathbb{Z}_p).$$

We now relate this group to a space of weight 2 cusp forms. As in the Introduction, let K_∞ be a complete subfield of $\mathbb{C}_p$ containing all roots of unity, and let $\mathcal{O}_\infty$ be its ring of integers.

Proposition 3.1. [5, c.f. 3.3.2] *We have an isomorphism*

$$H_{r,quo} \otimes \mathcal{O}_\infty \cong \mathrm{Cot}(G_{r,\mathcal{O}_\infty})(-1).$$

Here, $\mathrm{Cot}(G_{r,\mathcal{O}_\infty})$ *denotes the cotangent space of* $G_{r,\mathcal{O}_\infty}$ *along the unit section over* $\mathcal{O}_\infty$*.*

Remark 3.2. In [5, 3.3.2], Ohta claims that this follows from [6, §4]. However, no explanation is given for how this follows from Tate's argument.

Lemma 3.3. *We have an injection*

$$\mathrm{Cot}(G_{r,\mathcal{O}_\infty}) \to eS_2(\Gamma_1(Np^r), K_\infty).$$

Proof. We have an isomorphism

$$\operatorname{Cot}(G_{r,\mathcal{O}_\infty}) \cong e\operatorname{Cot}(B_{r,\mathcal{O}_r}[p^\infty]) \cong e\operatorname{Cot}(B_{r,\mathcal{O}_r}).$$

Futhermore, we have $\mathfrak{h}_r$-equivariant injections

$$\operatorname{Cot}(B_{r,\mathcal{O}_r}) \to \operatorname{Cot}(J_{r,K_\infty}) \to H^0(X_r,\Omega^1)\otimes_{\mathbb{Q}} K \cong S_2(\Gamma_1(Np^r),K_\infty).$$

Combining the above facts gives the claim. $\square$

Proposition 3.4. [5, Prop 3.3.6] *If f is in the image of the injection $\operatorname{Cot}(G_{r,\mathcal{O}_\infty}) \to eS_2(\Gamma_1(Np^r),K_\infty)$, then both f and $w_r f$ are contained in $\mathcal{O}_\infty[[q]]$.*

Proof sketch. We denote by $\mathfrak{X}_r$ the integral closure of the integral model of the level 1 modular curve $X(1)_{/\mathbb{Z}}$ in $X_{r,\mathbb{Q}_p(\mu_{Np^r})}$. Then the smooth locus $\mathfrak{X}_r^{sm}$ of $\mathfrak{X}_r$ over $\mathbb{Z}_p[\mu_{Np^r}]$ contains the sections corresponding to cusps. By universal property of the Neron model $\mathfrak{J}_r$ of J_r over $\mathbb{Z}_p[\mu_{Np^r}]$, we have a morphism $\mathfrak{X}_r^{sm} \to \mathfrak{J}_r$ whose base change to $\mathbb{Q}_p(\mu_{Np^r})$ factors through the morphism $X_{r,\mathbb{Q}(\mu_{Np^r})} \to J_{r,\mathbb{Q}(\mu_{Np^r})}$. There is a formal neighborhood $\operatorname{Spec}(\mathbb{Z}_p[\mu_{Np^R}][[q]] \to \mathfrak{X}_r^{sm}$ of the cusp corresponding to ∞ such that the q-expansion of $f \in S_2(\Gamma_1(Np^r),\mathbb{Q}_p(\mu_{Np^r})) \cong H^0(X_{r,\mathbb{Q}(\mu_{Np^r})},\Omega^1)$ coming from $\operatorname{Cot}(G_{r,\mathcal{O}_\infty})$ is given by pulling back differentials along the composition

$$\operatorname{Spec}(\mathbb{Z}_p[\mu_{Np^R}][[q]] \to \mathfrak{X}_r^{sm} \to \mathfrak{J}_r \to B_{r/\mathbb{Z}_p[\mu_{Np^r}]}.$$

The q-expansion of f is therefore in $\mathcal{O}_\infty[[q]]$.

A similar argument replacing the ∞ cusp with the 0 cusp gives the statement on $w_r f$. $\square$

Corollary 3.5. *We have natural injections, for all $r \geq 1$:*

$$H_{r,quo}\otimes\mathcal{O}_\infty \hookrightarrow eS_2^*(\Gamma_1(Np^r),\mathcal{O}_\infty)(-1).$$

Here, the superscript "$*$" denotes the image under w_r. If we set (under Ohta's trace maps)

$$\mathfrak{S} := \varprojlim_r eS_2^*(\Gamma_1(Np^r),\mathcal{O}_\infty),$$

then we have an injection

$$H_{quo}\widehat{\otimes}\mathcal{O}_\infty \hookrightarrow \mathfrak{S}(-1).$$

4. Surjectivity onto ordinary Λ-adic cusp forms

Theorem 4.1. *The maps in Corollary 3.5 are isomorphisms.*

In this section, we sketch the proof of this Theorem.

Lemma 4.2. *The $\Lambda_{\mathcal{O}_\infty}$-modules $H_{quo}\widehat{\otimes}\mathcal{O}_\infty$ and $\mathfrak{S}$ are free of finite rank.*

Proof sketch. In fact, Ohta shows that $H\widehat{\otimes}\mathcal{O}_\infty$ is free of finite rank using results of Hida [2, Thm 3.1(ii)]. Furthermore, since we have an injection as in Corollary 3.5, it suffices to show that $\mathfrak{S}$ is free of finite rank (why?). This follows from the Theorem below, which we will also use. $\square$

Theorem 4.3. [5, Thm 2.3.6] *There is a canonical isomorphism*

$$e\mathfrak{S} \cong eS(N,\Lambda_{\mathcal{O}_\infty}).$$

Remark 4.4. This is only a special case of [5, Thm 2.3.6]. Ohta proves a similar result in all weights $k \geq 2$.

In the remainder of this section, we only roughly sketch Ohta's arguments. To show surjectivity, Ohta uses control Theorems of Hida to reduce to showing that

$$H_{1,quo} \otimes \mathcal{O}_\infty \cong eS_2^*(\Gamma_1(Np), \mathcal{O}_\infty).$$

By Proposition 3.1, it suffices to show that we have an isomorphism

$$e\,\mathrm{Cot}(B_{1,\mathbb{Z}[\mu_{Np}]}) \cong eS_2^*(\Gamma_1(Np), \mathbb{Z}[\mu_{Np}]).$$

Let k be the residue field of $\mathbb{Z}_p[\mu_{Np}]$. We denote by $\overline{X_1}$ the special fiber of $\mathfrak{X}_1$ over k. It is known that $\overline{X_1}$ has two irreducible components C_0 and C_∞ that intersect the sections of $\mathfrak{X}_1$ corresponding to the cusps 0 and ∞, respectively. Furthermore, C_0 and C_∞ meet transversally precisely at the supersingular points.

Ohta shows that the quotient $J_1 \to B_1$ factors through the "Mazur–Wiles" good quotient A_1, and that we have an isomorphism $eA_1[p^\infty] \cong eB_1[p^\infty]$.

The result then follows from the combination of the following two Propositions:

Proposition 4.5. [5, Prop 3.4.7] *The quotient map $J_1 \to A_1$ induces isomorphisms*

$$\mathrm{Cot}(A_{1,\mathbb{Z}_p[\mu_{Np}]})^{(i)} \cong H^0(\mathfrak{X}_1, \Omega^1)^{(i)}$$

for all $i \not\equiv 0 \pmod{p-1}$ (here, the superscript "(i)" denotes the ω^i-eigenspace). Furthermore, let

$$S_2' := \{f \in S_2(\Gamma_1(Np), \mathbb{Z}_p[\mu_{Np}]) : w_1 f \in \mathbb{Z}_p[\mu_{Np}][[q]]\}.$$

Then q-expansions give an isomorphism

$$H^0(\mathfrak{X}_1, \Omega^1)^{(i)} \cong (S_2')^{(i)}.$$

Proposition 4.6. [5, Prop 3.4.8] *We have an equality*

$$eS_2' = eS_2^*(\Gamma_1(Np), \mathbb{Z}[\mu_{Np}]).$$

5. Final remarks

Above, we only really considered the action of I_p on H_{sub} and H_{quo}. In Section 3.5 of [5], Ohta considers the action of the full decomposition group $G_{\mathbb{Q}_p}$, thus finishing the proof of Theorem 1.1. In this report, we chose to focus on the relationship between Ohta's Eichler–Shimura cohomology groups and Hida's Λ-adic cusp forms, so we omit the description of the Galois action on these groups.

One might like to generalize these results to other settings (e.g. Shimura curves). One obstruction might seem to be Ohta's use of the geometry of the special fiber of $\mathfrak{X}_1$. However, work of Carayol [1] has a description of certain integral models of Shimura curves that might be useful. Another serious obstruction seems to be Ohta's use of q-expansions to detect integrality of cusp forms as in Proposition 3.4. One workaround would be the development of so-called "Serre–Tate expansions" at CM points of modular forms on Shimura curves (see, for example, [3, §4] for Shimura curves over $\mathbb{Q}$). We hope to explore these ideas further.

References

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