NIM AND OTHER GAMES

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1. Overview

In the first section, we discuss Nim and introduce some ideas without using too much math notation, but you can skip to Section 3, where I explain the questions I'm interested in.

2. Nim

Recall the game of Nim. There are n piles of stones with $a_1, \dots, a_n \geq 0$ stones in them, respectively. Two players, player A and player B, take turns. On each player's turn, they pick a pile and remove any nonzero number of stones from the pile they have chosen. The player to take away the last stone wins.

We observe that it is impossible to draw in Nim. The game must terminate, since there are finitely many stones in play, and each turn reduces the number of stones in play by at least 1. At the end of the game, one of the players removes the last stone, so one player must win.

What would "optimal play" look like in this game? Playing optimally should mean that if a player has a strategy that leads to guaranteed victory, then that player will use that strategy. If there is no way for a given player to win, assuming the other player plays optimally, then we are indifferent between any moves we could make. If a given player has a strategy to guarantee victory, we say that player starts their turn in a **winning state**. Otherwise, we say the player is in a **losing state**. Note that in a given state s, if there are no possible moves, then there are no moves that guarantee victory, so s is a losing state. For example, in Nim, the state with no stones is a losing state, which agrees with the fact that the player who takes the last stones wins.

For example, if, at the start of person A's turn, there is one pile left, then A can win by removing all of the stones in this pile. If there are two piles, then A would not want to take away a whole pile, since that would allow person B to remove the last pile.

From this discussion, it is clear that player A is in a winning state if and only if player A can make a move that puts player B in a losing state, and player A is in a losing state if and only if any move player A makes will put player B in a winning state. A priori, we might not expect that a given state has to be winning or losing. But for Nim, it turns out to be a well-defined classification.

Lemma. *For Nim, every possible starting state of the game is either winning or losing in a well-defined way.*

Proof. We represent a state of the game as $(a_1, \dots, a_n)$, where a_k is a nonnegative integer for each k. We then proceed by induction on $a_1 + \cdots + a_n$. If $a_1 + \cdots + a_n = 0$, then as $a_k \geq 0$ for all k, we must have $a_k = 0$ for all k. From above, we know this is a losing state. Now suppose that for some positive integer T, all states with $a_1 + \cdots + a_n < T$ can be classified as winning or losing. Now suppose $a_1 + \cdots + a_n = T$. Let $S(a_1, \dots, a_n)$ be the set of possible states one move away from $(a_1, \dots, a_n)$. Every possible move reduces $a_1 + \cdots + a_n$, so each state in $S(a_1, \dots, a_n)$ can be classified as winning or losing by the induction hypothesis. If all of these states are winning states,

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then $(a_1, \ldots, a_n)$ is a losing state. Otherwise, there exists a state $s \in S(a_1, \ldots, a_n)$ that is losing. Then $(a_1, \ldots, a_n)$ is winning. By induction, every state is either winning or losing. □

For an explanation of the winning strategy of Nim, see Section 4.

3. Games More Generally

For our purposes, a "game" will mean something a little more general.

Definition 1. A **game** is a triple $(S, \to, W)$, where S is a set of **states**, "$\to$" is a relation on S (i.e. a subset of $S \times S$, where the statement "$a \to b$" means (a, b) is in the subset) and W is a subset of S such that for all $a \in S$, we have $a \in W$ if and only if there exists $b \in S$ such that $b \notin S$ and $a \to b$. We call W the set of **winning states**, and we call $L := S \setminus W$ the set of **losing states**. If $a \to b$, we say that $a \to b$ is a **possible move**, and we say that we can move from a to b.

Note that we don't require games to actually end. For example, a game could be $S = \{0, 1\}$, and $0 \to 1$ and $1 \to 0$ and $1 \in W$. This satisfies the definition, but "playing" the game would never end, since the gameplay would look like

$$0 \to 1 \to 0 \to 1 \to \cdots$$

I don't really care about the end of a game, but I do care about winning or losing!

Because I like abstract nonsense, I'll make another definition.

Definition 2. A **cogame** is a triple $(S, \to, M)$, where S is a set of **states**, "$\to$" is a relation on S, and M is a subset of S such that for all $a \in S$, we have $a \in M$ if and only if there exists $b \in S$ such that $b \notin M$ and $b \to a$. We call M the set of **cowinning states**.

The only thing different in this definition is that the order of the relation has been reversed.

Is Nim a cogame with $M = W$? On one hand, if there exists $b \in S$ such that $b \notin W$ and $b \to a$, then $b \in L$, so $a \in W$. If $a \in L$, then $b \to a$ implies $b \in W$, a contradiction. But the converse is not so clear. We must show that for any $a \in W$, there exists $b \in L$ with $b \to a$. In other words, does every winning state come from a losing state? This turns out to be true (see Section 4).

Here's a more fun game related to Nim. Let $e_1, \ldots, e_n$ be the usual basis for $\mathbb{R}^n$, and let

$$S = (\mathbb{R}_{\geq 0})^n := \{\sum_k a_k e_k : a_k \in \mathbb{R}, a_k \geq 0.\}$$

For $x, y \in S$, we say $x \to y$ if $x - y = r e_k$ for some k and some $r > 0$. Intuitively, a move is decreasing one coordinate by some nonzero amount, but now that amount can be arbitrarily small. Is there a unique game structure on this? It turns out the answer is no for $n = 2$. In fact, any order-preserving bijection $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\phi(0) = 0$ defines a game structure by declaring

$$W = \{(a, b) \in S : b \neq \phi(a)\}.$$

Indeed, suppose $(a, b) \in W$. Then $b > \phi(a)$ or $b < \phi(a)$. If $b > \phi(a)$, then decrease b to $\phi(a)$ so that $(a, \phi(a)) \notin W$. If $b < \phi(a)$, then $a > \phi^{-1}(b)$, so decrease a to $\phi^{-1}(b)$ so that $b = \phi(\phi^{-1}(b))$ and $(\phi^{-1}(b), b) \notin W$. Conversely, suppose $b = \phi(a)$. Then changing b to $b' \neq b$ will result in $b' \neq \phi(a)$, and changing a to a' will result in $b \neq \phi(a')$. This shows that this is a game. More concretely, $\phi(x) = x$ and $\phi(x) = x^2$ give two different game structures on S. Can you generalize this to $n > 2$? This also happens to be a cogame. Is every game structure on this set a cogame?

Some relations don't admit any game structure. For example if the relation is $a \to b$ for all $a, b \in S$, then if a state a was losing, we have $a \to a$, so a is winning, a contradiction. But if all states are winning, then no state is winning, since a winning state requires a losing state. This is an apparent paradox, so we conclude that no game with this relation exists. It is then a necessary condition that there exists $a \in S$ with $a \not\to a$.

4. Appendix: Winning Nim

For this section, we denote states of Nim by finite sequences $(a_1, \ldots, a_n)$, where a_k is a nonnegative integer for all k. We define an operation $\oplus$ on nonnegative integers as follows: write a and b in binary: $a = \sum a_j 2^j$ and $b = \sum b_j 2^j$, where $a_j, b_j \in \{0, 1\}$ for all j. Then define

$$a \oplus b := \sum (a_j - b_j)^2 2^j.$$

Lemma. *The operation $\oplus$ makes $\mathbb{Z}_{\geq 0}$ into an abelian group in which every element is its own inverse.*

Proof. For $a_j, b_j, c_j \in \{0, 1\}$, we have $(a_j - b_j)^2$ is just addition modulo 2. So $(a_j - b_j)^2 = (b_j - a_j)^2$, and $((a_j - b_j)^2 - c_j)^2 = ((b_j - c_j)^2 - a_j)^2$, and $(a_j - a_j)^2 = 0$. Combining these facts with associativity of usual addition gives commutativity, associativity, and involutivity. Oh, and 0 is the identity. $\square$

Theorem. *In Nim, a state $(a_1, \ldots, a_n)$ is winning if and only if*

$$a_1 \oplus \cdots \oplus a_n \neq 0.$$

Proof. Similar to above, we induct on $a_1 + \cdots + a_n$. For $a_1 + \cdots + a_n = 0$, we have $a_1 = \cdots = a_n = 0$, so $(a_1, \ldots, a_n)$ is losing and $a_1 \oplus \cdots \oplus a_n = 0$.

Now suppose the result holds for $a_1 + \cdots + a_n < T$ for some positive integer T. Suppose that $a_1 \oplus \cdots \oplus a_n = 0$. We show that any move results in a nonzero $\oplus$-sum. Without loss of generality, any move changes a_1 to $a_1' \neq a_1$. Then we are assuming that

$$a_1 \oplus \cdots \oplus a_n = a_1' \oplus \cdots \oplus a_n.$$

Then

$$a_1 \oplus a_1' = a_1 \oplus a_1' \oplus (a_2 \oplus \cdots \oplus a_n) \oplus (a_2 \oplus \cdots \oplus a_n) = (a_1 \oplus \cdots \oplus a_n) \oplus (a_1' \oplus \cdots \oplus a_n) = 0.$$

So

$$a_1 = a_1 \oplus a_1' \oplus a_1' = a_1'.$$

But this contradicts $a_1 \neq a_1'$. It follows that any move makes the $\oplus$-sum nonzero, and by the inductive hypothesis, all possible moves result in winning states, so the given state $(a_1, \ldots, a_n)$ is losing.

Suppose that $a_1 \oplus \cdots \oplus a_n \neq 0$. We show that there exists a move that makes the $\oplus$-sum zero. Let $a_0 := a_1 \oplus \cdots \oplus a_n$. The leading nonzero binary digit of a_0 must come from some a_k. Then $a_k \oplus a_0$ has leading nonzero binary digit farther to the right than a_k, so $a_k \oplus a_0$ is less than a_k. For our move, we decrease a_k to $a_k \oplus a_0$. Without loss of generality, $k = 1$. Then

$$(a_1 \oplus a_0) \oplus a_2 \oplus \cdots \oplus a_n = a_0 \oplus (a_1 \oplus \cdots \oplus a_n) = a_0 \oplus a_0 = 0.$$

By induction hypothesis, $(a_1 \oplus a_0, a_2, \ldots, a_n)$ is a losing state, so $(a_1, \ldots, a_n)$ is winning. The result then holds by induction. $\square$

Here's the way to apply this in real life, through an example.

Example 1. Suppose the state of the game is $(3, 4, 7)$ and it is player B's turn. We write

$$3 = 1 + 2, \quad 4 = 4, \quad 7 = 1 + 2 + 4$$

and count if there are an even or odd number of each power of 2 in these expansions. In this case, there are an even number of each power, so $3 \oplus 4 \oplus 7$ is a losing state. Say player B changes the state to $(3, 4, 6)$. Now

$$3 = 1 + 2, \quad 4 = 4, \quad 6 = 2 + 4.$$

So $3 \oplus 4 \oplus 6 = 1$, so player A already knows they will win. There is a 1 in the binary expansion of 3, so we decrease 3 to $3 \oplus 1 = 2$. So player A changes the state to $(2, 4, 6)$. Again, any move made by player B will still result in player A winning, so player B impatiently changes the state to $(2, 4, 0)$. Then player A calculates $2 \oplus 4 = 6 = 2 + 4$, so player A decreases 4 to $4 \oplus 6 = 2$. So the new state is $(2, 2)$. Player B at this point knows that they are definitely doomed to loose, so they change the state to $(2, 1)$. Then player A changes it to $(1, 1)$, then player B changes it to $(1, 0)$, and player A takes the last stone and wins.

From all of this, we can show that Nim is a cogame.

Proposition. *Nim is a cogame with $M = W$.*

Proof. Let $(a_1, \ldots, a_n) \in W$. Then $a_1 \oplus \cdots \oplus a_n \neq 0$. Let $a_{n+1} := a_1 \oplus \cdots \oplus a_n$. Then $(a_1, \ldots, a_n, a_{n+1}) \in L$, as

$$a_1 \oplus \cdots \oplus a_n \oplus a_{n+1} = a_{n+1} \oplus a_{n+1} = 0.$$

Furthermore, we have $(a_1, \ldots, a_n, a_{n+1}) \to (a_1, \ldots, a_n)$ by just subtracting a_{n+1} from the $n + 1$ pile. □