OLD SOLUTIONS

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Feel free to add to, revise, or comment on this document as you see fit. Solutions will be (hopefully) correct, but for sake of time, will probably not be typed up the detail required for the actual exam. Once more problems are added here, I will put some thoughts about common problem types that one should be prepared to deal with (i.e. "show this is an inner product", "use Arzela Ascoli but not really", "know how to compute $exp(A)$" , "iterate a function and show it converges", "do infinite dimensional vector spaces behave well? (no)")

1. S17

Problem 1 (1)**.** Let M be an $n \times n$ real matrix with transpose M^T. Prove that M and MM^T have the same image.

Via the Friedholm alternative, we have $im(M) = ker(M^T)^\perp$ and $im(MM^T) = ker((MM^T)^T)^\perp = ker(MM^T)^\perp$. Since $V = ker(M^T) \oplus ker(M^T)^\perp$, and likewise for $ker(MM^T)$, it is sufficient to prove that $ker(M^T) = ker(MM^T)$. Fix $v \in ker(M^T)$. Then $MM^T * v - M * 0 = 0$, so $v \in ker(MM^T)$. Now, if $v \in ker(MM^T)$, then $MM^Tv = 0$, so $\langle M^Tv, M^Tv\rangle = \langle v, MM^Tv\rangle = \langle v, 0\rangle = 0$, and by property of inner products, this implies that $M^Tv = 0$, and so $v \in ker(M^T)$. Thus, $ker(M^T) = ker(MM^T)$, and so $im(M) = im(MM^T)$.

Problem 2 (2)**.** Let $a, b, c, d \in \mathbb{R}$ and $M \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}$.

(1) Determine conditions on a, b, c, d s.t. there is only one Jordan block for each eigenvalue of M in the Jordan form of M.
(2) Find the Jordan form of M when $a = c = d = 2$ and $b = -2$.

We first compute the characteristic polynomial of M. We see this is $(x-1)^4$. Therefore, the Jordan canonical form of M has only ones on the diagonal, and so has a single Jordan block for eigenvalue 1 iff $nullity(M - I) = 1$. We compute $M - \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 2 & -2 \\ 0 & d & 2 & -2 \end{pmatrix}$. This needs to have exactly three linearly independent rows. We require $c \neq d$ and $a \neq -b$.

If $a = c = d = 2$ and $b = -2$, $nullity(M - I) = 2$, so there are exactly 2 blocks in the Jordan form. The size of the largest block corresponding to eigenvalue 1 is equal to the multiplicity of $x - 1$ in the minimal polynomial of M. We compute that $(X - 1)^2 = 0$, and since $(X - 1) \neq 0$, we have that there are 2 blocks of size 2 in the J.C.F.

Problem 3 (3)**.** Let M be an $n \times n$ real matrix. Suppose M is orthogonal and symmetric

(1) Prove that if M is positive definite, then M is the identity.
(2) Does the answer change if M is only positive semidefinite?

M being real and symmetric implies by spectral theorem that it is diagonalizable with real eigenvalues. M being positive definite implies that each eigenvalue is strictly greater than zero. M being orthogonal implies that the eigenvalues of M must have magnitude 1. This means that the only eigenvalue of M is 1, and by diagonalizability, $M = 1$.

The answer does not change if M is only positive semidefinite. The condition that M is orthogonal prohibits that M has a zero eigenvalue, so it is still true that M can only have eigenvalue 1.

Problem 4 (4)**.** Let $F(x)$ be the determinant of the given matrix (not retyping). Compute $dF/dx(0)$.

Note that $F(x)$ is a polynomial in x, and therefore, if $F(x) = \sum_{i=0}^{n} a_i x^i$, $dF/dx(0) = a_1$ i.e. the coefficient of x. Note also $Det(M) = \sum_{\sigma \in S_5} \prod_{i=1}^{5} sign(\sigma) * M_{i,\sigma(i)}$. We will find all σ such that $\prod_{i=1}^{5} M_{i,\sigma(i)} = x$. For any given σ, exactly one of the $M_{i,\sigma(i)}$ can be x, and the rest must be 1. One can check that there is only one such σ, (not sure how to write this out concisely), where we take $a_{2,1} * a_{3,2} * a_{4,3} * a_{1,4} * a_{5,5} = x$. The sign of this permutation is -1, so our derivative is -1.

Problem 5 (5)**.** Suppose V is a finite dimensional vector space over $\mathbb{C}$ and $T : V \to V$ is a linear transformation. Let $F(X) \in \mathbb{C}[X]$ be a polynomial. Show that $F(T)$ is an invertible transformation iff $F(X)$ and the minimal polynomial of T have no common factors.

If the minimal polynomial of T and $F(X)$ indeed have a common factor, say $(x - \lambda)$, then let v be an eigenvector of T corresponding to λ. $F(X) = q(X) * (X - \lambda)$ for some polynomial $q(X)$, so $F(T)v = q(T)((T - \lambda)v) = 0$, so $F(T)$ is not invertible.

Now, suppose that $F(T)$ has no common factor with the minimal polynomial of T, and write $F(X) = \prod_{i=1}^{n} (x - \lambda_i)^{n_i}$ where the λ_i are distinct. Suppose $F(T)v = 0$ for some vector v (not necessarily nonzero). In particular, $(T - \lambda_1 I) o (T - \lambda_1 I)^{n_1 - 1} \prod_{i=2}^{n} (T - \lambda_i)^{n_i} v) = 0$. Since λ_1 is not an eigenvalue of T (by assumption on no common factors), that means that the inner operator on v must return zero. We repeat this argument on the polynomial of T of smaller degree, until we eventually obtain that $v = 0$.

Problem 6 (6)**.** (1) Let V denote the vector space of real $n \times n$ matrices. Prove that $\langle A, B \rangle - trace(A^T B)$ is an inner product on V.
(2) For $n = 2$, find an orthonormal basis of V.

Note that for any matrix A, $tr(A) = tr(A^T)$. Then $\langle A, B \rangle = tr(A^T B) = tr((A^T B)^T) = tr(B^T A) = \langle B, A \rangle$, and so we have symmetry.

For any matrix A, $\langle A, A \rangle = tr(A^T A)$. We see that the element in the i row and i column of $A^T A$ is by definition $\sum_{j=1}^{n} (A^T)_{i,j} Aj, i = \sum_{j=1}^{n} A_{j,i}^2 \geq 0$. Then $tr(A^T A) = \sum_{i,j} A_{i,j}^2 \geq 0$. We also see that this equals zero iff each $A_{i,j} = 0$ i.e. $A = 0$.

Fix matrices A, B, C and $\lambda \in \mathbb{C}$. Then $\langle A + \lambda C, B\rangle = tr((A + \lambda C)^T B) = tr((A^T + \lambda C^T)B) = tr(A^T B) + \lambda(C^T B) = \lambda\langle A, B\rangle + \langle C, B\rangle$, and so we have linearity.

I will show that the standard basis of V is in fact orthonormal, i.e., $b_{i,j}$ is 1 in the i, j position and zero elsewhere. One checks quickly that $tr((b_{p,q}^T * b_{i,j}) = tr(b_{q,p} b_{i,j})$. From here, one sees that $b_{i,j} b_{j,i}$ is always diagonal with one in one place and zero in the other, so each of these basis elements is unit length. Furthermore, one also can confirm that these basis elements are pairwise orthonormal.

Problem 7 (7)**.** Prove existence and uniqueness of a non-negative continuous function $f : [0, 1] \to [0, 1]$ satisfying $f(x) = 1 - [\int_0^x tf(t)dt]^2$.

Note that the set of continuous functions from $[0, 1]$ to $[0, 1]$ is complete with the supnorm metric. Hence, if I can show that $g(f(x)) = 1 - [\int_0^x tf(t)dt]^2$ is a contraction mapping, then by contraction mapping principle, g has a unique fixed point, and note that f is a fixed point of g iff the given condition is true.

Note first that $g(f(x))$ is always continuous since it is a composition of continuous functions. Since $[\int_0^x tf(t)dt]^2 \geq 0$, $g(f(x))$ is always ≤ 1. Furthermore, since $f(t) \leq 1$ always, $\int_0^x tf(t) \leq \int_0^x t = x^2/2$. For $x \in [0, 1]$, $1 - x^2/2 \in [0, 1]$, and so g is indeed a mapping from this space to itself.

Fix $p(x), q(x)$ in this space. Then $g(p) - g(q) = [\int_0^x tq(t)dt]^2 - [\int_0^x tp(t)dt]^2 = (\int_0^x t(q - p)dt) * (\int_0^x t(q + p)dt$. If $d = sup_{x\in[0,1]}|p(x) - q(x)|$, the absolute value of the first term of this product is less than $\int_0^x t * d = x^2 d/2$. Since both f and g take values in $[0, 1]$, we see $|f + g| \leq 2$, and so the absolute value of the second term in this product is less than $\int_0^x t * 2dt = x^2$. We now have, for any x, $|p(x) - q(x)| \leq x^4 d/2$, and since $x \in [0, 1]$, this is always less than or equal to $d/2$. Then $sup_{x\in[0,1]}|p(x) - q(x)| \leq d/2$ as well, and so this is indeed a contraction mapping.

Problem 8 (8)**.** Show that there is a constant C so that $|\frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx| \leq C \int_0^1 |f''(x)|dx$ for every C^2 function $f : \mathbb{R} \to \mathbb{R}$.

Consider the integral $\int_0^1 f(t)$. We may integrate by parts taking $u = f$ and $v = x - 1/2$ since f is differentiable. Then $\int_0^1 f(x)dx = (f(x) * (x - 1/2)|_0^1 - \int_0^1 (x - 1/2)f'(x)dx$. The first term evaluates to $\frac{f(0)+f(1)}{2}$, and so $|\frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx| \leq |\int_0^1 f'(x) * (x - 1/2)|$. We may integrate by parts on this right integral, with $u = f'(x)$ and $v = (x - 1/2)^2/2$ since f' is differentiable. We compute $\int_0^1 f'(x)(x - 1/2)dx = f'(x)(x - 1/2)^2/2|_0^1 - \int_0^1 f''(x)(x - 1/2)^2/2dx$. The first term is $\frac{f'(1)-f'(0)}{8} = 1/8 \int_0^1 f''(x)dx$ by fundamental theorem of calculus. The second term, since $(x - 1/2)^2 \leq 1/4$ for $x \in 0, 1$, we see that we may take $C = 1/8 + 1/8 = 1/4$.

Problem 9 (9)**.** Let (X, d) be a bounded metric space and let $C(X)$ be the space of bounded continuous real functions on X endowed with the supremum norm. Suppose $C(X)$ is separable.

(1) Show that for every $\varepsilon > 0$, there is a countable set $Z_\varepsilon \subset X$ so that for all $x \in X$, there exists a $z \in Z_\varepsilon$ where for any y in X, $|d(x, y) - d(z, y)| < \varepsilon$.
(2) Show that X is separable.

For part a, observe that for any $x \in X$, the function $f_x = d(x, y)$ is a bounded (since X is bounded) continuous function of y. Since X is separable, let S be a countable set of functions in $C(X)$ where for any $f \in C(X)$, there exists a $g \in S$ such that $sup|f(x) - g(x)| < \varepsilon/2$. Note that since this is true if we take f to be f_x, each f_x is within $\varepsilon/2$ of some $g \in S$. Let S_2 be the subset of S of functions containing at least one f_x within the $\varepsilon/2$ ball around it, and note that this set is also $\varepsilon/2$ dense with the f_x. For each $g \in S_2$, pick some f_y, and let the set of these be S_3. I will show that we may take $Z_\varepsilon = \{y | f_y \in S_3\}$ works. Note first that this set is countable since S_3 is countable. For any f_x, f_x has supnorm distance $\varepsilon/2$ or less from some S_2 member g. Let f_y be the S_3 member corresponding to g. Then by triangle inequality, $|f_x(z) - f_y(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all z, as desired.

Given that part a is true, note that if $x, z \in X$, then $|d(x, y) - d(z, y)| \leq d(x, z)$ by reverse triangle inequality, and since $|d(x, z) - d(z, z)| = d(x, z)$, we see that $sup_{y \in X}|d(x, y) - d(z, y)| = d(x, z)$. For any positive natural number n, part a gives that there exists a countable set $Z_{1/n}$ where for any $x \in X$, there is a $z \in Z_{1/n}$ where $d(x, z) \leq \varepsilon$. Take the union of the $Z_{1/n}$, say Z, and note that since this is a countable union of countable sets, it is countable. Now fix $\varepsilon > 0$ and $x \in X$. Then pick $1/n < \varepsilon$. There exists $z \in Z_{1/n}$, and hence $z \in Z$, such that $d(x, z) \leq \varepsilon$. Thus, Z is a countable dense subset of X, and so X is separable.

Problem 10 (10)**.** Let $K \subset \mathbb{R}^n$ be compact. Suppose that for every $\varepsilon > 0$ and every pair $a, b \in K$ there is an integer $n \geq 1$ and a sequence of points $x_0 ... x_n \in K$ so that $x_0 = 0$, $x_n = b$, and $||x_k - x_{k-1}|| < \varepsilon$ for every $1 \leq k \leq n$

(1) Show K is connected
(2) Show by example that K may not be path connected.

Suppose towards contradiction that K is the disjoint union of two nonempty sets that are both closed relative to K, call these A and B. Fix a point $a \in A$ and $b \in B$. For every $1/m$ with m a positive natural number, there exists a sequence with $x_0 = a$, $x_n = b$, and $||x_k - x_{k-1}|| < 1/m$. But this implies that there exists an $a_m \in A$ and $b_m \in B$ with $||a_m - b_m|| < \varepsilon$ (there must be at least one set change in the sequence since $x_0 \in A$ and $x_n \in B$). Since K is compact, some subsequence of the a_m and b_m, say the a_{m_k} and b_{m_k} must converge. Since A and B are relatively closed in K, the a_{m_k} converge to some $a^* \in A$ and the b_{m_k} converge to some b^* in B. But the condition on the $a_m - b_m$ implies that $a^* - b^* = 0$, and so $a^* = b^*$. This contradicts that A and B are disjoint. Therefore, K must be connected.

To show K is not necessarily path connected, consider the set of points in $\mathbb{R}^2$ given by $\{(x, sin(1/x)|x \in (0, 1]\} \cup \{(0, 0)\}$ i.e. the topologist's sine curve. It is known that this is not path connected. For any $a, b \in K$, if the x component of both a and b is nonzero, then both are in the path connected component of K, and so the given condition is immediately satisfied. If WLOG $a = (0, 0)$ and $b = (x, sin(1/x))$ for some $x \in (0, 1]$, there will always be some zero of $sin(1/x)$ with $x < \varepsilon$ since $sin(1/x)$ has infinitely many periods approaching zero. Let such a zero be x_1. Then x_1 is in the path connected component of K and so we may find a chain connecting it to b via the argument above.

Problem 11 (11)**.** Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions satisfying $|f_n(x)| \leq 1 + \frac{n}{1+n^2x^2}$, and define $F_n(x) = \int_0^x f_n(t)dt$. Show that there is a subsequence $n_k \to \infty$ so that F_{n_k} converges for every $x \in [0, 1]$.

We first show that the F_n are equibounded. Note that $|F_n(x)| \leq \int_0^x |f_n(t)| \leq \int_0^x 1 + \frac{n}{1+n^2t^2} = x + arctan(nx) \leq 1 + \pi/2$ for any x and n.

I will now show that the F_n are equicontinuous on $[1/2, 1]$. By Arzela Ascoli, this implies that some subsequence, the F_{n_k} converges for all $x \in [1/2, 1]$. If $1/2 \leq x < y \leq 1$, then $|F_n(y) - F_n(x)| \leq \int_x^y 1 + \frac{n}{1+n^2t^2} = (y - x) + arctan(ny) - arctan(nx)$. $arctan$ is a continuous function on $[1/2, 1]$, and hence, is uniformly continuous. The sum $(y - x) + arctan(ny) - arctan(nx)$ will also be uniformly continuous on $[1/2, 1]$, and so there exists $\delta > 0$ s.t. $|x - y| < \delta$ implies that $(y - x) + arctan(ny) - arctan(nx) < \varepsilon$. This shows that $|F_n(y) - F_n(x)| < \varepsilon$ for such x, y, so the F_n are equicontinuous on $[1/2, 1]$. By Arzela Ascoli, some subsequence, say the $F_{2,i}$ converges on $[1/2, 1]$.

Now suppose that some subsequence of our original F_n, say the $F_{j,i}$ converges on $[1/j, 1]$. Then $[1/(j+1)/1]$ is also a compact interval, and so this subsequence is also equicontinuous here. Therefore, some subsequence of the $F_{j,i}$, say the $F_{j+1,i}$, converges on $[1/(j+1), 1]$.

We have shown that we can iteratively define subsequences that converge on $[1/j, 1]$ for any $j > 1$. To complete the proof, define our sequence to be $F_{2,1}, F_{3,2}, F_{4,3}$.... Note that this subsequence converges at zero since for any n, $F_n(0) = 0$. Now fix any $x \in (0, 1]$. There exists some $n > 1$ natural number where $1/n < x$. Then for all $i > n$, the $F_{i,i-1}$ were elements of the sequence $F_{n,j}$, and so converge on $[1/n, 1]$, and in particular, at x.

Problem 12 (12)**.** Show that for each $t \in \mathbb{R}$ fixed, that the function $F(y, t) = y^4 + ty^2 + t^2y$ defined for all $y \in \mathbb{R}$ acheives its global minimum at a single point $y_0(t)$

For fixed t, define $f(y) = F(y, t)$, and note that this is continuous. Observe that since this function f, as a polynomial of y, is quartic, $lim_{y\to\infty}F(y, t) = lim_{y\to-\infty}F(y, t) = \infty$. Therefore, there exists a positive M such that $|y| > M$ implies that $F(y, t) > 10$. Consider f where y is restricted to the bounded set $[-M, M]$ and t is fixed. f is a continuous function of y on a compact set, and so by extreme value theorem, it acheives a local minimum for some $y \in [-M, M]$. Furthermore, since $F(0, t) = 0$, we see that the minimum acheived on this set will be less than any value F takes for $|y| > M$. Equivalently, some local minimum of f for $y \in [-M, M]$ is a global minimum of $F(y, t)$. It remains to show that there is exactly one local minimum of f. We compute $f'(y) = 4y^3 + 2yt + t^2$, and $f''(y) = 12y^2 + 2t$. If $t = 0$, then $f(y) = y^4$, and so there is exactly one global minimum at $y = 0$. If $t > 0$, then $f''(y) > 0$, so f cannot obtain a minimum at more than one point. We note that $f'(y)$ must have at least one real root since it is cubic, so there is indeed exactly one local minimum, which is therefore a global minimum. Now suppose $t < 0$, and suppose towards contradiction that y_1 and y_2 are both global minima. Then the polynomial $g(y) = F(y, t) - F(y_1, t)$ has roots at both y_2 and y_1. Neither of these can be a simple root since $g(y) \geq 0$ always, and so $(y_1 - y)^2$ and $(y_2 - y)^2$ both divide g. But since g is monic and degree 4, we see $g = (y_1 - y)^2(y_2 - y)^2$. Then $F(y, t) = (y_1 - y)^2(y_2 - y)^2 + F(y_1, t)$. Equating the coefficients of y^3, we must have $2y_1 + 2y_2 = 0$, and so $y_1 = -y_2$. Equating the coefficients of y, we see $2y_1^2y_2 + 2y_2^2y_1 = t^2$, but the left hand side is also zero. This is a contradiction since $t < 0$. Thus, f obtains a unique local minimum in $[-M, M]$, which is a global minimum.

2. F17

Problem 13 (1)**.** Let $V = \{f(X) = a_0 + a_1X + a_2X^2 + a_3X^3 | a_1, ...a_3 \in \mathbb{C}\}$ be the complex vector space of polynomials in the variable X of degree at most 3.

(1) Show that V is an inner product space with $\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt$
(2) Find an orthonormal basis of V

First, fix $f \in V$. Then $\langle f, f\rangle = \int_{-1}^{1} f(t)\bar{|}f(t)|dt$. The integrand is nonnegative and is continuous. Therefore, this integral is always nonnegative, and is zero iff $f\overline{f} = 0$ everywhere on $[-1, 1]$ iff $f = 0$. For symmetry, fix $g \in V$ as well. Then $\langle f, g\rangle = \int_{-1}^{1} f\overline{g} = \overline{\int_{-1}^{1} \overline{f\overline{g}}} = \overline{\int_{-1}^{1} g\overline{f}} = \langle g, f\rangle$. Finally, fix $h \in V$ and $a \in \mathbb{C}$. Then $\langle af + h, g\rangle = \int_{-1}^{1}(af + h)\overline{g} = a\int_{-1}^{1} f\overline{g} + \int_{-1}^{1} h\overline{g} = a\langle f, g\rangle + \langle h, g\rangle$, all by linearity of the integral. Thus, this defines an inner product.

I won't type out the full Gram Schmidt calculation. Start with a basis for this space, i.e. $1, t, t^2, t^3$. We first normalize basis vector 1 with respect to this product. $\langle 1, 1\rangle = \int_{-1}^{1} 1 = 2$. So if we take $e_1 = 1/\sqrt{2}$, this vector will be unit length. For our next vector, take $b_2 = t - \langle t, 1/\sqrt{2}\rangle * 1/\sqrt{2}$. We note that the inner product in this expression is a symmetric integral of an odd function, and so is zero. So $b_2 = t$. To normalize, we see $\langle t, t\rangle = \int_{-1}^{1} t^2 = 2/3$, so take $e_2 = \sqrt{3/2}t$, and this will be unit length and orthogonal to $1/\sqrt{2}$. The remaining vectors can be computed in the same way, and calculation is made simpler by using the odd and even functions. These polynomials are the Legendre polynomials, and when normalized, the next one is $\sqrt{45/8}(t^2 - 1/3)$.

Problem 14 (2)**.** Let $n \geq 1$ be an integer, and A, B be $n \times n$ matrices.
(1) Show that if A is invertible, AB and BA have the same characteristic polynomial.
(2) Prove or disprove: the same is true even if A is not invertible.

By definition, the characteristic polynomial of AB is $\det(AB - xI)$. Since $det(A) = det(A^{-1})^{-1}$, this is $det(A^{-1})det(AB - xI)det(A)$. By multiplicativity of determinant, this is $det(A^{-1}(AB_xI)A) = det(BA - xI)$, which is exactly the characteristic polynomial of BA. (Note, we have shown that similar matrices have the same characteristic polynomial.)

This is true even if A is not invertible. First, we will show that the invertible matrices are dense in $M_n(\mathbb{C})$. Consider any $\varepsilon > 0$, and the matrix $A - \varepsilon I$. This matrix is non-invertible iff. ε is an eigenvalue of A. A has finitely many eigenvalues, so if $\varepsilon > 0$ and is less than the smallest positive eigenvalue of A, then $A - \varepsilon I$ will always be invertible. Therefore, $|A - (A - \varepsilon I)| = |\varepsilon I| = \varepsilon$, and so indeed, A is a limit of invertible matrices. Let A_n be a sequence of invertible matrices converging to A, and let B be fixed. Consider the function $f : M_n(\mathbb{C}) \to \mathbb{C}[x], f(M) = det(MB - xI) - det(BM - xI)$. This is continuous since it is a composition of continuous functions (in particular, determinant is continuous). For each A_n, A_n is invertible, and we have shown $f(A_n) = 0$. Then since A_n have limit A and since f is continuous, $f(A) = 0$ as well, so $det(AB - xI) = det(BA - xI)$, and this is exactly what we wanted.

Problem 15 (3)**.** Solve the following linear system of differential equations (not going to retype)

If $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$, then the solution is exactly $x(t) = exp(A) * x(0)$, where $x(0)$ is the initial conditions at $t = 0$ (I believe this can be stated without proof, since this is relatively common use of exponentials of matrices. If this seems strange to you, compare to the case where you only have one variable. $x' = ax \to x = e^{at} * x(0)$). So, it remains to compute $exp(A)$. It is easiest to compute the Jordan canonical form. The characteristic polynomial of this matrix is $(x-4)(x-5)$. Since this has two distinct roots, the Jordan canonical form is exactly $D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$. We also compute the matrix P such that $P^{-1}AP = D$. To do so, we need only find the eigenvectors. $A - 4I = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$. Then $(A-4I)v = 0$ iff v is a scalar multiple of $(1,2)$. Likewise, $(A-5I)v = 0$ iff $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} v = 0$ iff v is a scalar multiple of $(1,1)$. If $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, we see $P^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$. We also see that indeed $P^{-1}AP = D$, and so $A = PDP^{-1}$. Then $exp(A) = exp(PDP^{-1}) = Pexp(D)P^{-1}$. But $exp(D)$ is exactly $\begin{pmatrix} e^4 & 0 \\ 0 & e^5 \end{pmatrix}$. Then $Pexp(D) * P^{-1} = \begin{pmatrix} 2e^5 - e^4 & -e^5 + e^4 \\ 2e^5 - 2e^4 & 2e^4 - e^5 \end{pmatrix}$

Problem 16 (4)**.** Let V be a vector space over $\mathbb{R}$ and let V^* be the dual space of V. Let $B = \{e_i\}$ be a basis of V. For each i, define $e_i^\# \in V^*$ by $e_i^\#(e_j) = \delta_{i,j}$.

(1) Show that these $e_i^\#$ are linearly independent in V^*.
(2) Give necessary and sufficient conditions (with proof) on V for these vectors to form a basis of V^*.

Linear independence is defined with finite support, so, after reordering, suppose $a_1e_1^\# + a_2e_2^\# + ...a_ne_n^\# = 0$ i.e. is the zero operator. In particular, for each e_i with $i \in \{1, 2...n\}$, we have $0(e_i) = (a_1e_1^\# + a_2e_2^\# + ...a_ne_n^\#)(e_i) = a_i$. Thus, all coefficients are zero.

We will show that this is a basis iff V is finite dimensional. First, suppose V is indeed finite dimensional. It remains to show that this basis spans V^*. Fix an operator $L \in V^*$. Write $a_i = L(e_i)$ for each of the (finitely many) e_i. Then L and $a_1e_1^\# + a_2e_2^\# + ...a_ne_n^\#$ agree on all vectors of V by linearity, and are thus the same operator. Therefore, L is in the span of B.

If V is infinite dimensional, define $L(e_i) = 1$ for all basis vectors e_i (this is well-defined, since sums of V elements are only defined on finite support). Let $e_1^\#, e_2^\#, ...e_n^\#$ be a finite subset of B that is reindexed. Then $L(e_{n+1}) = 1$, but $(a_1e_1^\# + ...a_ne_n^\#)(e_{n+1}) = 0$. Thus, L is not a (finite) linear combination of B, and so is not in the span of B.

Problem 17 (5)**.** Let V and W be two infinite dimensional vecto spaces over $\mathbb{C}$. Let $Lin_F(V, W)$ be the $\mathbb{C}$ vector space of $\mathbb{C}$ linear maps from V to W

(1) Is $X = \{f \in Lin_F(V, W) | f \text{ has finite rank}\}$ a subspace?
(2) What about $Y = \{f \in Lin_F(V, W) | Ker(f) \text{ has finite dimension}\}$?
(3) What is $X \cap Y$?

(1) Yes. Fix $f, g \in X$ and $a \in \mathbb{C}$. Then $(af + g)v = a(fv) + gv \in im(f) + im(g) \subset span(im(f), im(g))$. Since $im(f)$ and $im(g)$ are finite dimensional, their combined span is as well. So $im(af + g) \subset span(im(f), im(g))$, and hence, is finite dimensional. So $af + g \in X$.
(2) No. Fix $f \in Y$ (if Y is a subspace, it must be nonempty). Then $-f \in Y$ as well (where $f(v) = w \iff -f(v) = -w$). This is because $f(v) = 0 \iff -f(v) = -0 = 0$. Then $f + (-f) = 0$, which has kernel infinite dimensional.

One could also simply say that $0 \notin Y$ implies Y is not a subspace immediately.

Interesting thought: I don't think it is even guaranteed that Y is nonempty if V has a continuum basis (like the set e^{rt} are linearly independent for smooth functions) and W has countably infinite basis like $\mathbb{R}[x]$, it seems that you could not produce any operator with finite dimensional kernel, since the image will always be countable dimension.

(3) This is the empty set. If $Ker(f)$ is finite dimensional, extend a basis of $Ker(f)$ to a basis for V (technically requires axiom of choice, but then again, so does the fact that V has a basis at all). The images of each element that was added in the extension must be linearly independent, or else we have contradicted that these elements were not in the kernel. Therefore, the image will be infinite rank.

Basically an infinite version of rank-nullity theorem, although we technically haven't proven this for the infinite dimensional case.

Problem 18 (6)**.** For each of $F = \mathbb{R}, \mathbb{C}, \mathbb{F}_3$, is it true that every symmetric matrix $A \in M_{2\times 2}(F)$ is diagonalizable?

We use that if A is diagonalizable, then the diagonal entries of D will be the eigenvalues of A with algebraic multiplicity of λ corresponding to the number of times λ appears on the diagonal of D.

(1) Yes. This matrix A is in fact self adjoint, and so is diagonalizable over $\mathbb{C}$. Note that the eigenvalues of A are strictly real since it is self adjoint. If A has only one distinct eigenvalue, this implies that $A = \lambda * I$ for some real λ, since this is its Jordan canonical form. A is similar to itself. If A has two distinct (real) eigenvalues, then note the characteristic polynomial of $A = (x - \lambda_1)(x - \lambda_2)$. This will also be the characteristic polynomial of A when viewed as an operator on $\mathbb{R}^2$, and so A has a real eigenvector for each of the eigenvalues λ_1 and λ_2. These are linearly independent since they correspond to different eigenvalues. Thus, they form a basis of $\mathbb{R}^2$, so A is diagonalizable.

One might note that I couldn't find a quick argument for why diagonalizability over $\mathbb{C}$ implies the same over $\mathbb{R}$. If anyone wants to write that up, be my guest.

(2) No. Consider $A = \begin{pmatrix} 1 & i \\ i & 3 \end{pmatrix}$. The characteristic polynomial of this matrix is $(\lambda - 2)^2$, but $nullity(A - 2I) = 1$, so A cannot have an orthonormal basis of eigenvectors. (One should guess that this is false since symmetry is not self-adjointness in $\mathbb{C}$. To construct such a matrix, one should start by guessing that the off-diagonal elements are i, and hope that the first entry $= 1$ works. Solve for what the last entry should be so that the matrix has one eigenvalue).

(3) No. Consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. The characteristic polynomial of this matrix is $\lambda^2 + 1$ (since $3 = 0$ in this field). But this equation has no roots in $\mathbb{F}_3$ (check each element separately). Therefore, A has no eigenvalues at all, and hence, cannot be diagonalizable. (One might also guess that this is false since this field is not algebraically closed. There are only 18 matrices worth checking, and this seemed like the first one to try.)

Problem 19 (7)**.** Let a_n be non increasing and positive, where $\sum_{n=1}^{\infty} a_n < \infty$. Show that $lim_{n\to\infty} na_n = 0$

Fix $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n < \infty$, there exists a natural number N such that $n, m > N$ implies that $\sum_n^m a_i < \varepsilon$ (since each a_i is positive). Since the a_i are nonincreasing, we have that $(m-n+1) * a_m = \sum_n^m a_M \leq \sum_N^M a_i < \varepsilon$. Therefore, $\varepsilon \geq limsup_m(m-n+1)a_m = limsup_m ma_m - lim(1-n)a_m$. The rightmost limit is zero since $1-n$ is fixed and a_m decreases to zero (since the infintie sum converges). Since limsup $ma_m < \varepsilon$ for any epsilon, we conclude that $limsup ma_m = 0$. But since ma_m is strictly positive, we must have $ma_m \to 0$.

Problem 20 (8)**.** Let $a < b$ be real numbers with $f : [a.b]$ a function s.t. $L(x) = lim_{y\to x} f(y)$ exists for all $x \in [a, b]$.

(1) Show L is continuous on $[a, b]$.
(2) Show that $\{x \in [a, b] : f(x) \neq L(x)\}$ is countable
(3) Show f is Riemann integrable.

(1) Fix $x \in [a, b]$. Then $lim_{z\to x} L(z) = lim_{z\to x} lim_{y\to z} f(y) = lim_{y\to x} f(y) = L(x)$ by definition of L.
(2) I will show that $S_n = \{x \in [a, b] : |f(x) - L(x)| > 1/n\}$ is finite, since the given set will then be a countable union of finite sets, and hence countable. Suppose instead that there are infinitely many $x \in [a, b]$ such that $|f(x) - L(x)| > 1/n$, and let x_i be an infinite sequence of distinct such elements. Since $[a, b]$ is compact, there exists a convergent subsequence, which we call the y_i, that converges to y. Since L is continuous, $L(y) = lim_i L(y_i)$. Since $|f(y_i) - L(y_i)| > 1/n$ for each y_i, we have $lim(L(y_i)) > lim(f(y_i) + 1/n) = L(y) + 1/n$ by definition of L. This is a contradiction, since $1/n$ is not less than zero. Thus, S_n must be finite.
(3) The set from part b is precisely the set of discontinuities of f, and so this is countable. If f is unbounded, let a_n be a sequence of points in $[a, b]$ such that $f(a_n) \geq n$. Since $[a, b]$ is compact, there exists a convergent subsequence, converging to some x. By definition, $L(x) = lim_{y\to x} f(y)$. But since the a_{n_k} become arbitrarily close to x, and $f(a_{n_k})$ does not converge, $L(x)$ cannot be defined. Thus f must be bounded. Since f is bounded and has countably many discontinuities, by Riemann-Lebesgue criterion, f is integrable on $[a, b]$.

Problem 21 (9)**.** Let (X, ρ) be a complete metric space and $f : X \to X$ a function. If f^n is the $n-th$ iterate of f, denote $c_n = sup_{x,y\in X, x\neq y} \frac{\rho(f^n(x), f^n(y))}{\rho(x,y)}$. If $\sum_{n=1}^{\infty} c_n < \infty$, show f has a unique fixed point in X.

For any $x, y \in X$ with $x \neq y$ and any natural number n, the given expression says that $\rho(f^n(x), f^n(y)) \leq \rho(x, y) * c_n$. Fix $x \in X$ and consider the sequence $x, f(x), f^2(x)....$ Then we see, if $i < j$, that $\rho(f^i(x), f^j(x)) = \rho(f^i(x), f^i(f^{j-i}(x))) \leq c_i * \rho(x, f^{j-i}(x))$. Likewise, we use triangle inequality to show that $\rho(x, f^{j-i}(x)) \leq \sum_{n=1}^{j-i} \rho(f^n(x), f^{n-1}(x)) \leq \sum_{n=1}^{j-i} c_{n-1}\rho(f(x), x)$ (we say $c_0 = 1$). Now fix $\varepsilon > 0$ and write $c = \sum_{n=1}^{\infty} c_n$. There exists $N \in \mathbb{N}$ s.t. $i > N$ implies that $c_i < \varepsilon/(c * \rho(f(x), x))$. For any $N < i < j$, we have that $\rho(f^i(x), f^j(x)) \leq c_i\rho(f(x), x) \sum_{n=1}^{j-i} c_n \leq c_i\rho(f(x), x) * c \leq \varepsilon$. This sequence is thus Cauchy, so converges to some y.

I will now show that this y is a fixed point of f, and that it is unique. Fix $\varepsilon > 0$, and choose N so that $i > N$ implies that $\rho(f^i(x), y) < \min \varepsilon/2, \varepsilon/(2c_1)$ (if $c_1 = 0$ then the proof was trivial). Then $\rho(f(y), y) \leq \rho(f(y), f^{i+1}(x)) + \rho(f^{i+1}(x), y) \leq c_1 * \rho(y, f^i(x)) + \rho(f^{i+1}(x), y) \leq \varepsilon/2, \varepsilon/2$. Since this is true for any ε, we have $\rho(y, f(y)) = 0$, and so $y = f(y)$.

For uniqueness, suppose that z is another fixed point of f. Then for any positive integer n, $\rho(y, z) = \rho(f^n(y), f^n(z)) \leq c_n \rho(y, z)$. Since the c_n must converge to zero, this will be false unless $\rho(y, z) = 0$ i.e. $y = z$.

Problem 22 (10)**.** Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ a continuous function s.t. $\int_a^b f(x)x^n dx = 0$ for each integer $n \geq 0$. Prove $f = 0$

By Stone-Weierstrauss theorem, the set of polynomials with real coefficients is dense in the set of continuous functions on $[a, b]$ For any $p(x)$ a real polynomial, write $p(x) = \sum_{i=1}^n a_i x^i$. Then $\int_a^b f(x)p(x) = \sum_{i=0}^n a_i \int_a^b f(x)x^i = 0$. f is a continuous function on a compact set, so it is bounded by some $M > 0$. Fix an $\varepsilon > 0$, and choose a $p(x)$ such that $sup|p - f| < \varepsilon/(M(b - a))$. Then $\int_a^b f^2 = \int_a^b fp + \int_a^b f(f - p) = \int_a^b f(f - p) \leq \int_a^b M * \varepsilon/(M(b - a)) = \varepsilon$. Therefore, $\int f^2 = 0$. Since f^2 is continuous and positive, this implies that $f^2 = 0$, and so $f = 0$.

Problem 23 (11)**.** Prove Young's inequality: If $p, q \in (1, \infty)$ and $1/p + 1/q = 1$, then for any $a, b \geq 0$, $ab \leq a^p/p + b^q/q$

Rewriting, we must show $a^p/p + b^q/q - ab \geq 0$. Fix $b \geq 0$, and consider the continuous function $f(a) = a^p/p + b^q/q - ab$. Since $p > 1$, $lim_{a\to\infty} f(a) = \infty$, so for some $M > 0$, $a \geq M$ implies that $f(a) > 1$. Consider the compact interval $[0, M]$, and since f is continuous on this compact set, it obtains a minimum on this set. I will show that for a local minimum inside this set, $f(x) = 0$. Since $f(0)$, $f(M) > 0$, 0 is the global minimum for $f(a)$. Since this is true for any b, we have $a^p/p + b^q/q - ab \geq 0$ always, which is what we wanted. It remains to compute the local minima of f inside $[0, M]$. The derivative of f is $a^{p-1} - b$. This is zero whenever $a^{p-1} = b$. Furthermore, $f''(a) \geq 0$, so any critical point is a local minimum. We compute $f(b^{1/(p-1)}) = b^{p/(p-1)}/p + b^q/q - b^{1/(p-1)}b = b^q/p + b^q/q - b^q = 0$, and hence, the global minimum of f is zero.

Note: Most online sources give a slicker proof using that log is concave and increasing. I don't see how anyone would think of that on the exam, although the condition on p and q may suggest concavity to some people. I am sure there are other solutions.

Problem 24 (12)**.** Let X be a compact metric space and $C(X)$ be the space of continuous real-valued functions on X endowed with the supremum norm. Let $F \subset C(X)$ be non-empty. Prove that F is compact iff F is closed, bounded, and equicontinuous.

If F is compact, then F is immediately closed and bounded. It remains to show it is equicontinuous. Fix $\varepsilon > 0$. Let S_δ be the set of $f \in F$ such that $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \varepsilon$. Since each f is continuous on a compact set, it is uniformly continuous, and so the set of S_δ forms a cover of F. Note that if $\delta_1 < \delta_2$ that $S_{\delta_1} \supset S_{\delta_2}$. So for our finite subcover, some S_δ is in fact a cover of F, implying that every $f \in F$ satisfies $|x - y| < \delta$ implies $|f(x) - f(y)| \leq \varepsilon$. F is thus equicontinuous.

For the reverse direction, we will show F is closed and totally bounded, since this implies compactness in a metric space. We have already assumed F is closed. Fix $\varepsilon > 0$. We will construct a finite epsilon net of F. Since F is equicontinuous, there exists $\delta > 0$ where $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/3$ for all $x, y \in X$ and $f \in F$. Since X is compact, there exists a finite δ net of X, say the a_i. For all $f \in F$, consider the vector in $\mathbb{R}^n$ of $(f(a_1), f(a_2), f(a_3)...)$. F is equibounded, so each component of these vectors are bounded, and hence each vector is bounded in magnitude. These vectors form a bounded subset of $\mathbb{R}^n$, and so by compactness of this set, there exists a finite $\varepsilon/3$ net of these vectors. Equivalently, there exists a finite set of f_i where for any $g \in F$, $|g(a_j) - f_i(a_j)| \leq \varepsilon/3$ for each a_j and f_i. Fix $x \in X$. Then there exists an a_j where $|x - a_j| < \delta$. There also exists an f_i where $|f_i(a_j) - g(a_j)| < \varepsilon/3$ for all j. So $|g(x) - f_i(x)| \leq ||g(x) - g(a_j)| + |g(a_j) - f_i(a_j)| + |f_i(a_j) - f_i(x)|$. Since $|a_j - x| < \delta$, the first and last term are less than $\varepsilon/3$, and the middle is less than $\varepsilon/3$ by choice of f_i and a_j. This implies that the f_i are a finite ε net of F, and so F is totally bounded, and hence also compact.

3. S18

Problem 25 (1)**.**] Prove that $e^t, e^{2t}...e^{nt}$ are linearly independent in the space of continuous functions on the interval $[1, 2]$.

We argue by induction on n. The base case $n = 1$ is vacuous. Suppose that the first n of these functions are linearly independent. Suppose that $a_1e^t + a_2e^{2t} + ...a_{n+1}e^{(n+1)t} = 0$ for some a_i and for all t in this interval. Taking derivatives of both sides, we obtain $a_1e^t + 2a_2e^{2t} + ...na_ne^{nt} + (n+1)a_{n+1}e^{(n+1)t} = 0$. Subtracting $n+1$ of the first equation from the second, we have $(1-(n+1))a_1e^t + (2-(n+1))a_2e^{2t} + ...(n-(n+1))e^{nt} = 0$. Simplifying, $-na_1e^t + (-n+1)a_2e^{2t} + ... - 1a_ne^{nt} = 0$. By induction, the coefficients $b_i = (i - n - 1)a_i$ are all zero. Since $i > n + 1$ for these i, this implies all a_i are zero. Our original equation now reads $a_{n+1}e^{(n+1)t} = 0$ for all t, so $a_{n+1} = 0$. This set is thus linearly independent.

Problem 26 (2)**.** Let A and B be two real 5×5 real matrices such that $A^2 = A$, $B^2 = B$, and $1 - (A + B)$ is invertible. Prove $rank(A) = rank(B)$.

Observe that $A(1-A-B) = A - A^2 - AB = -AB$. Likewise, $(1-A-B)B = B - B^2 - AB = -AB = A(1-A-B)$. Therefore, $rank(A(1-A-B)) = rank((1-A-B)B)$. Since $(1-A-B)$ is invertible, $rank(A) = rank(A(1-A-B)) = rank((1-A-B)B) = rank(B)$. (To see this last step, one can use Sylvester's rank inequality, or just note that the rank of $(1-A-B)|_W = dim(W)$).

Problem 27 (3)**.** Let $A = (a_{i,j})$ be a complex $n \times n$ matrix. Suppose $e^A = 1 + A + A^2$. Prove or disprove: $A = 0$

This is false. Every $M_n(\mathbb{C})$ contains a nonzero nilpotent matrix A where $A^2 = 0$ (i.e. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and its extensions). For this matrix, $e^A = 1 + A + A^2/2! + ... = 1 + A$. Likewise, $1 + A + A^2 = 1 + A$.

For an alternative solution, we show that the real equation $f(x) = e^x - x^2 - x - 1 = 0$ has a positive solution. Note that this function on the left side is continuous since it is the difference of continuous functions. Then $f(1) < 0$, $f(10) > 0$, and so by I.V.T., there exists a $c \in (1, 10)$ s.t. $f(c) = 0$. Now, consider the diagonal matrix D where each diagonal entry is c. One sees that $e^D = 1 + D + D^2$.

Problem 28 (4)**.** Consider the following matrices (I'm not retyping these, go look at the problem sheet). Which pairs are similar over $\mathbb{R}$?

Two real matrices are similar over $\mathbb{R}$ iff they have the same real Jordan canonical form. Each of these matrices is triangular, so we read off that the characteristic polynomial of each matrix is $(x-1)^3$. There are thus three possibilities for the real canonical form: 3 blocks of size 1, one block of size 2 and one block of size 1, or 1 block of size 3. Therefore, the Jordan form for each matrix is determined by the total number of blocks.
Note that the number of blocks in the J.C.F. for eigenvalue λ is equal to $nullity(X - \lambda I)$. For matrices A through E, we see that this nullity is 2, so these are all similar. For matrix F, this nullity is 1, so this is not similar to any of the other matrices.

Problem 29 (5)**.** Let A, B be two positive definite 2×2 matrices. Prove or disprove:

(1) A+B is positive definite
(2) AB+BA is positive definite

The first claim is true. Fix any nonzero $v \in \mathbb{C}^2$. Then $\langle x, (A+B)x \rangle = \langle x, Ax \rangle + \langle x, Bx \rangle$ by linearity, and each term is strictly greater than zero by positive definiteness of A, B.
The second claim is false. Consider $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. This is positive definite since it is symmetric and both eigenvalues (2 and 1) are positive. Let $B = \begin{pmatrix} 1/8 & 1/\sqrt{2} \\ 1/\sqrt{2} & 9/2 \end{pmatrix}$. The characteristic polynomial of this matrix is $\lambda^2 - 37\lambda/8 + 1/16$. By Descartes' rule of signs, this polynomial has only positive roots, and so B is symmetric with positive eigenvalues, and so is positive definite. We compute $AB + BA = \begin{pmatrix} 1/2 & 3/\sqrt{2} \\ 3/\sqrt{2} & 9 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 - 19\lambda/2$, and so *zero* is an eigenvalue of this matrix, and so this matrix is not positive definite.

The example seems rather strange, but the idea was that if we can force the constant coefficient of the characteristic polynomial to be zero or less, then we are done. To do so, if we let B be a matrix that is "almost" not positive definite, and scale its rows and columns, we ought to be able to make it no longer positive definite.

Problem 30 (6)**.** Compute the determinant of the following matrix: (not retyped for sake of time)

Note that subtracting a multiple of one row from another does not change the determinant (this is equivalent to left multiplying by a particular elementary matrix with determinant one). We subtract multiples of the first row from each remaining row to obtain $\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & 0 & 0 & 0 & -31 \\ 0 & 0 & 0 & -31 & -62 \\ 0 & 0 & -31 & -62 & -124 \\ 0 & -31 & -62 & -124 & -248 \end{pmatrix}$. We compute the determinant by taking a column sum along the first column. The determinant is $1 * det(A_{1,1}) + 0 * det(A_{2,1}) + 0 * .... = 1 * det(A_{1,1})$. The determinant of $A_{1,1}$ can be computed by taking repeated column sums along the first column, yielding that $det(A) = 1 * (-1 * -31) * (-31) * (-1 * -31) * -31 = 31^4$.

Problem 31 (7)**.** Prove that for each $p \in \mathbb{N}$, the infinite series $\sum_{n=1}^{\infty} \frac{\sin(\pi n/p)}{n}$ converges.

Let $a_n = \sin(\pi n/p)$ and $b_n = 1/n$. By theorem, if a_n has bounded partial sums, b_n is nonincreasing, and $b_n \to 0$, then $\sum_{n=1}^{\infty} a_n b_n$ converges. $1/n$ is indeed nonincreasing and converges to zero, so it remains to show that $a_n = \sin(\pi n/p)$ has bounded partial sums. Observe that $\sin(\pi n/p) = -\sin(\pi(-n)/p) = -\sin(\pi(2p-n)/p)$. Consider the sum $\sum_{n=1}^{2p} \sin(\pi n/p)$. For each $n \in [1, 2p] \backslash \{p, 2p\}$, we see that $2p - n \in [1, 2p] \backslash \{p, 2p\}$ and is distinct from n. Therefore, $\sum_{n=1}^{2p} \sin(\pi n/p) = \sin(\pi) + \sin(2\pi) + \sum_{n=1, n \neq p, n \neq 2p} \sin(\pi * n/p) = 0$. Likewise, we compute that for any positive integer k, $\sum_{n=1}^{2pk} \sin(\pi n/p) = 0$ since the summand is periodic in n with period $2p$. Now, for any positive integer m, by Euclidean algorithm, write $m = 2p * q + r$ for integers q and r and $r < 2p$. Then $\sum_{n=1}^{m} \sin(2\pi * n/p) = \sum_{n=1}^{2p} \sin(2\pi * n/p) + \sum_{n=2p+1}^{m} \sin(2\pi * n/p) = \sum_{n=2p+1}^{m} \sin(2\pi * n/p)$. Since $|\sin(2\pi * n/p)| \leq 1$, we see $|\sum_{n=1}^{m} \sin(2\pi * n/p)| \leq \sum_{n=2p+1}^{m} |\sin(2\pi * n/p)| = r < 2p$. Therefore, every partial sum of the a_n is bounded in $[-2p, 2p]$. The conditions of our theorem are satisfied, so $\sum_{n=1}^{\infty} a_n b_n$ converges.

Problem 32 (8)**.** This problem is notoriously hard and I don't have the bravery to try to understand and type it up. Sylvester posted his solution on CCLE, although it is unlikely anyone could write it up in full detail for the exam. It just seems like an unreasonable and uninstructive problem.

Problem 33 (9)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be non-decreasing (but not necessarily continuous). Prove f is Riemann integrable on any finite interval $(a, b) \subset \mathbb{R}$. DO NOT PROVE USING LEBESGUE THEORY UNLESS YOU ALSO PROVE LEBESGUE THEORY.

To prove using Lebesgue theory, one would first have to demonstrate that f has countably many discontinuities (assign a distinct rational number to each jump, or use the trick Sylvester showed on CCLE), and then prove that this implies that we can find partitions that minimize the contribution of each jump (create an interval around each jump of size less than 2^{-j} if this is jump number j).

A direct proof is more straightforward. Fix such (a, b) and let P_n be the partition of (a, b) into n uniform segments, i.e., $x_0 = a, x_1 = a + (b-a)/n, x_2 = a + (b-a) * 2/n, x_n = b$. Then the upper sum $U(f, P_n) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) * sup_{x \in [x_i, x_{i+1}]} f(x)$. Since f is nondecreasing, this supremum is always $f(x_{i+1})$, and by choice of partition, $x_{i+1} - x_i$ is always $(b-a)/n$. Then $U(f, P_n) = \sum_{i=1}^{n-1} f(x_{i+1}) * (b-a)/n$. Likewise, we compute $L(f, P_n) = \sum_{i=1}^{n-1} f(x_i) * (b-a)/n$. Therefore, $U(f, P_n) - L(f, P_n) = (f(b) - f(a)) * (b-a)/n$. The limit of the sequence $U(f, P_n) - L(f, P_n)$ as $n \to \infty$ is therefore zero, and so f is Riemann integrable.

Problem 34 (10)**.** Let $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be nonempty, open, and connected. Suppose $f : U \to \mathbb{R}$ is such that all first partial derivatives of f exist and vanish at each point of U. Prove f is constant.

I will first show f is constant on each open ball B contained in U. Fix any point $x \in U$, and since U is open, let $B(x, r)$ be the ball of radius r around x, and suppose this ball is contained in U. Fix $y \in B(x, r)$. Write $x = (x_1, x_2, ... x_n)$ and $y = (x_1 + h_1, x_2 + h_2, ... x_n + h_n)$. Then $f(y) - f(x) = (f(y) - f(x_1 + h_1, x_2 + h_2, ... x_n)) + (f(x_1 + h_1, ... x_n) - f(x_1 + h_1, ... x_{n-1}, x_n)) + (f(x_1 + h_1, x_2 ... x_n) - f(x))$, where for any term in this sum, the arguments of f differ in only one coordinate. Note that each $(x_1+h_1, x_2+ ....., x_i+h_i, x_{i+1}, x_{i+2}..., x_n)$ is contained in $B(x,r)$ since their distance from x must be less than $d(x,y)$. By fundamental theorem of calculus, one computes that $f(x_1+h_1, x_2+ ....., x_i+h_i, x_{i+1}, x_{i+2}..., x_n) - f(x_1+h_1, x_2+ ....., x_i, x_{i+1}, x_{i+2}..., x_n) = \int_0^{h_i} \partial_i f(x_1+h_1, x_2+ ....., x_i+s, x_{i+1}, x_{i+2}..., x_n)ds = \int_0^{h_i} 0 = 0$ since the partial derivatives vanish everywhere in U. This means that each term from our sum is zero, and so $f(y) = f(x)$. Therefore, f is constant on any open ball contained in U.

We now use an interesting characterization of connected sets (an exercise in the week two analysis notes). Let C be the open cover of U consisting of every open ball contained in U. Since U is connected, for any B_a and B_b, there exists a sequence of balls $B_1 = B_a$, $B_2 ... B_n = B_b$ where $B_i \cap B_{i+1} \neq \emptyset$. Now, fix any $x, y \in U$, and pick a B_x in C containing x and a B_y containing y in C. There exists a chain between them as described above. Then since $B_x \cap B_2 \neq \emptyset$, and since f is constant on each of B_x and B_2, we must have that $f(c) = f(x)$ for each $c \in B_2$. Continuing in this way, $f(c_i) = f(x)$ for each $c_i \in B_i$. In particular, since $y \in B_n$, $f(y) = f(x)$, as desired.

There is probably a way to do this with path connectedness that I am not seeing. Since U is nonempty, open, and connected, it is path connected. Every directional derivative vanishes since each partial one does. Is there a quick way we can integrate along a path connecting x and y?

Problem 35 (11)**.** Let (X, ρ) be a compact metric space and let $f : X \to X$ be an isometry. Prove $f(X) = X$.

Suppose towards contradiction that $f(X)$ is a proper subset of X, and pick $x_0 \in X \backslash f(X)$. f is an isometry, and so maps open sets to open sets (and hence, closed sets to closed sets), and so we see that $f(X)$ is clopen. This implies that there exists an $r > 0$ such that $B(x,r) \subset X \backslash f(X)$ since $X \backslash f(X)$ is open. One sees that $f(B(x,r)) = B(f(x), r)$. Note that we may repeat these arguments to show that, if f^i is f applied i times, that $f^i(B(x,r)) = B(f^i(x), r)$. Since X is compact, some subsequence of the $f^i(x)$ must converge. But since each $f^i(x) \in f^i(X) \backslash f^{i+1}(X)$, the balls $B(f^i(x), r)$ are actually disjoint. This is a contradiction, since the $f^i(x)$ are always separated by distance at least r.

Problem 36 (12)**.** Let F be a family of real-valued functions on a compact metric space taking values in $[-1, 1]$. Prove that if F is equicontinuous then the function $g(x) = \sup\{f(x) : f \in F\}$ is continuous.

Pick $\varepsilon > 0$ and x in our metric space. By equicontinuity of F and compactness of our metric space, there exists $\delta > 0$ such that $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$ for any x, y in our space and $f \in F$. Now suppose $|x - y| < \delta$. By definition of supremum, there exists a function $f_a \in F$ such that $g(x) \leq f_a(x) + \varepsilon/2$. By equicontinuity, $f_a(x) + \varepsilon/2 \leq f_a(y) + \varepsilon/2 + \varepsilon/2 \leq g(y) + \varepsilon$. Therefore, $g(x) - g(y) \leq \varepsilon$.

Likewise, there exists $f_b \in F$ such that $g(y) \leq f_b(y) + \varepsilon/2$. We likewise compute that $g(y) \leq g(x) + \varepsilon$. Thus, we have $|g(x) - g(y)| \leq \varepsilon$, so g is continuous.

Problem 37 (1)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of nonnegative numbers such that
$$\sum_{n\geq 1} a_n \quad \text{diverges.}$$
Show that
$$\sum_{n\geq 1} \frac{a_n}{2a_n+1} \quad \text{diverges.}$$

Suppose that there exists some positive integer N such that $a_n < 1$ for all $n \geq N$. Then for $n \geq N$, we have
$$\frac{a_n}{2a_n+1} > \frac{1}{3}a_n.$$
Since the series $\sum_{n=N}^{\infty} a_n$ diverges and $a_n \geq 0$ for all n, we have that the set of partial sums $\sum_{n=N}^{M} a_n$ is unbounded. Since
$$\sum_{n=N}^{M} \frac{a_n}{2a_n+1} > \frac{1}{3}\sum_{n=N}^{M} a_n,$$
Then the partial sums $\sum_{n=N}^{M} \frac{a_n}{2a_n+1}$ are unbounded, so the series $\sum_{n=N}^{\infty} \frac{a_n}{2a_n+1}$ diverges, so the series $\sum_{n=1}^{\infty} \frac{a_n}{2a_n+1}$ diverges.
Suppose instead that there are infinitely many n such that $a_n \geq 1$. For such n,
$$\frac{a_n}{2a_n+1} = \frac{1}{2} - \frac{1/2}{2a_n+1} \geq \frac{1}{2} - \frac{1/2}{3} = \frac{1}{3}.$$
Since $a_n/(2a_n+1) \geq 1/3$ for infinitely many n, the limit of $a_n/(2a_n+1)$ cannot be zero, so the series diverges again.

Problem 38 (2)**.** Let A be a connected subset of $\mathbb{R}^n$ such that the complement of A is the union of two separated sets B and C, that is
$$\mathbb{R}^n \setminus A = B \cup C \quad \text{with} \quad \overline{B} \cap C = B \cap \overline{C} = \varnothing.$$
Show that $A \cup B$ is a connected subset of $\mathbb{R}^n$.

Let D be a nonempty subset of $A \cup B$ that is relatively open and closed in $A \cup B$. Then $D \cap A$ is relatively open and closed in A, and since A is connected, we either have $A \subseteq D$ or $D \cap A = \varnothing$.
Suppose first that $D \cap A = \varnothing$. Then $D \subseteq B$, and $(A \cup B) \setminus D$ is also relatively open and closed in $A \cup B$, and $A \subseteq (A \cup B) \setminus D$. So we can replace D with $(A \cup B) \setminus D$ and get $A \subseteq D$.
We claim that $D \cup C$ is open and closed in $\mathbb{R}^n$. First, $D \cup C$ is closed: let $x \in \overline{C \cup D}$. If $x \notin \overline{C}$ and $x \notin \overline{D}$, then there exists $\varepsilon > 0$ such that the ε-ball around x doesn't meet either D or C, contradicting x being in the closure of $D \cup C$. So $x \in \overline{D} \cup \overline{C}$. If $x \in \overline{C}$, then $x \notin B$. So $x \in A \subseteq D$ or $x \in C$, so $x \in D \cup C$. If $x \notin \overline{C}$, then $x \in \overline{D}$. If $x \in A \cup B$, then $x \in D$ because D is relatively closed in $A \cup B$. Otherwise, $x \in C$. In any case, $x \in D \cup C$. This shows that $D \cup C$ is closed in $\mathbb{R}^n$.
Now we show that $D \cup C$ is open in $\mathbb{R}^n$. Let $x \in D \cup C$. Case 1: $x \in D$. Since D is relatively open in $A \cup B$, there exists $\varepsilon_1 > 0$ such that $|y - x| < \varepsilon_1$ and $y \in A \cup B$ implies $y \in D$. Then for any $y \in \mathbb{R}^n$ with $|y - x| < \varepsilon_1$, either $y \in D$ or $y \notin A \cup B$, in which case $y \in C$. So $D \cup C$ contains the ε_1-ball centered at x. Case 2: $x \in C$. Then $x \notin \overline{B}$, so there exists $\varepsilon_2 > 0$ such that $|y - x| < \varepsilon_2$ implies $y \notin B$. Then $y \in A \cup C \subseteq D \cup C$. So $D \cup C$ contains an ε_2-ball centered at x. Since $D \cup C$ contains an open ball around each point, we have $D \cup C$ is open in $\mathbb{R}^n$.

Since $\mathbb{R}^n$ is connected, we must have $D \cup C = \mathbb{R}^n$. Then since C is disjoint from $A \cup B$,

$$D \cap (A \cup B) = (D \cup C) \cap (A \cup B) = A \cup B.$$

So $A \cup B = D$. We have shown that the only nonempty subset of $A \cup B$ that is both relatively open and closed in $A \cup B$ is all of $A \cup B$, so $A \cup B$ is connected.

Problem 39 (3)**.** Let $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to [0,1]$ be two Riemann integrable functions. Assume that

$$|g(x) - g(y)| \geq \alpha |x - y| \quad \text{for any} \quad x, y \in [0,1]$$

and some fixed $\alpha \in (0,1)$. Show that $f \circ g$ is Riemann integrable.

We rely on the Riemann-Lebesgue theorem, which says that a bounded function on $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. By measure zero, we mean for all $\varepsilon > 0$, the set of discontinuities can be covered by countably many open intervals whose lengths sum to less than ε. Let $\ell(I)$ denote the length of an interval I, so that $\ell((a,b)) = b - a$.
Note that if g is continuous at x and f is continuous at $g(x)$, then $f \circ g$ is continuous at x. Let $D_{f\circ g}$, D_f, and D_g be the sets of discontinuities of $f \circ g$, f, and g respectively. So $x \in D_{f\circ g}$ if and only if $x \in D_g$ or $g(x) \in D_f$, so

$$D_{f\circ g} = D_g \cup g^{-1}(D_f).$$

Let $\varepsilon > 0$. Since D_g has measure zero, there exist open intervals $I_1, I_2, \cdots \subseteq \mathbb{R}$ such that $\sum_{k=1}^{\infty} \ell(I_k) < \varepsilon/2$ and $D_g \subseteq \bigcup_k I_k$. Since D_f has measure zero, there exist open intervals $I_1', I_2', \cdots \subseteq \mathbb{R}$ such that $\sum_{k=1}^{\infty} \ell(I_k') < \alpha\varepsilon/2$ and $D_g \subseteq \bigcup_k I_k'$.
Consider an open interval (a,b). If $x, y \in g^{-1}((a,b))$, then

$$|x - y| \leq \frac{1}{\alpha} |g(x) - g(y)| < \frac{b-a}{\alpha}.$$

It follows that $g^{-1}((a,b))$ is contained in an open interval of length $(b-a)/\alpha$. For each I_k', let J_k be an open interval containing $g^{-1}(I_k')$ such that $\ell(J_k) = \ell(I_k')/\alpha$. For all $x \in g^{-1}(D_f)$, we have $g(x) \in D_f$, so $g(x) \in I_k'$ for some k, so $x \in g^{-1}(I_k')$, so $x \in J_k$. This shows that

$$g^{-1}(D_f) \subseteq \bigcup_k J_k,$$

and

$$\sum_{k=1}^{\infty} \ell(J_k) = \frac{1}{\alpha} \sum_{k=1}^{\infty} \ell(I_k') < \varepsilon/2.$$

Then

$$D_{f\circ g} \subseteq \bigcup_k I_k \cup \bigcup_k J_k,$$

and

$$\sum_k \ell(I_k) + \sum_k \ell(J_k) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We have covered D_f with countably many open intervals whose lengths sum to less than ε. Since $\varepsilon > 0$ was arbitrary, $D_{f\circ g}$ has measure zero. Also, f is bounded, so $f \circ g$ is bounded. It follows from Lebesgue's criterion for Riemann integrability that $f \circ g$ is Riemann integrable.

Problem 40 (4)**.** Let $f : [0,1] \to \mathbb{R}$ be a continuous function on the closed interval $[0,1]$ and differentiable on the open interval $(0,1)$. Assume that $f(0) = 0$ and f' is a decreasing function on $(0,1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is a decreasing function on $(0,1)$.

Suppose for the sake of contradiction that g is not decreasing on $(0,1)$. Then for some $0 < x < y < 1$, we have $g(y) \geq g(x)$. Since g is differentiable on $(0,1)$, by the Mean Value Theorem, there exists $c \in (x,y)$ such that

$$g'(c) = \frac{f(y) - f(x)}{x - y} \geq 0.$$

Note that

$$g'(c) = \frac{cf'(c) - f(c)}{c^2}.$$

So $cf'(c) \geq f(c)$. Since f is continuous on $[0,c]$ and differentiable on $(0,c)$, the mean value theorem again says that there exists $d \in (0,c)$ such that

$$f'(d) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c} \leq f'(c).$$

So $d < c$ and $f'(d) \leq f'(c)$. But this contradicts f' decreasing on $(0,1)$. So the original assumption was false, and g is decreasing on $(0,1)$.

Problem 41 (5)**.** Let $B := \{x \in \mathbb{R}^n : |x| \leq 1\}$ and let $g : \partial B \to \mathbb{R}$ be a 1-Lipschitz function.

(a) Show that the function $f : B \to \mathbb{R}$ given by

$$f(x) := \inf_{y \in \partial B} [g(y) + |x - y|]$$

is 1-Lipschitz.

(b) Show that the set $M(g) := \{h : B \to \mathbb{R} \mid h \text{ is 1-Lipschitz and } h|_{\partial B} = g\}$ is compact in the space of continuous functions on B endowed with the supremum norm.

(a) Let $x_1, x_2 \in B$. The function $y \mapsto g(y) + |x_k - y|$ is continuous on the set ∂B. Also, ∂B is closed and bounded, so is compact. So for $k = 1, 2$, there exists some $y_k \in \partial B$ such that

$$f(x_k) = \inf_{y \in \partial B} [g(y) + |x_k - y|] = g(y_k) + |x_k - y_k|.$$

By definition of infimum, we have

$$f(x_2) = g(y_2) + |x_2 - y_2| \leq g(y_1) + |x_2 - y_1|.$$

Then by the reverse triangle inequality,

$$f(x_2) - f(x_1) \leq g(y_1) + |x_2 - y_1| - g(y_1) - |x_1 - y_1| \leq |x_2 - x_1|.$$

By interchanging x_2 and x_1,

$$f(x_1) - f(x_2) \leq |x_2 - x_1|.$$

So

$$|f(x_1) - f(x_2)| \leq |x_2 - x_1|$$

and f is 1-Lipschitz.

(b) The Arzelà-Ascoli Theorem states that for a compact metric space K and a subset $F \subseteq C(K, \mathbb{R})$ of the continuous functions from $K \to \mathbb{R}$ endowed with the supremum norm, we have F compact if and only if F is closed, uniformly bounded, and equicontinuous. In this case, $K = B$ is compact because B is closed and bounded in $\mathbb{R}^n$. Furthermore, every function in $M(g)$ is 1-Lipschitz, so continuous, so $M(g) \subseteq C(B, \mathbb{R})$. So it suffices to show that $M(g)$ is equicontinuous, and closed and bounded with respect to the supremum norm.

Let $\varepsilon > 0$. Suppose $|x - y| < \varepsilon$ for some $x, y \in B$. Let $h \in M(g)$. Since h is 1-Lipschitz, we have

$$|h(x) - h(y)| \leq |x - y| < \varepsilon.$$

This shows that $M(g)$ is equicontinuous.

Let $h_1, h_2, \cdots \in M(g)$ be a sequence with h_n converging uniformly to some continuous function $h : B \to \mathbb{R}$ (equivalently, converging in the supremum norm). In particular, $h_n \to h$ pointwise. Since all h_n are 1-Lipschitz, for any $x, y \in B$, we have

$$|h(x) - h(h)| = \lim_n |h_n(x) - h_n(y)| \leq \lim_n |x - y| = |x - y|.$$

So h is 1-Lipschitz. Let $x \in \partial B$. Then $h_n(x) = g(x)$ for all n. So $h(x) = \lim_n h_n(x) = \lim_n g(x) = g(x)$. So $h|_{\partial B} = g$. So $h \in M(g)$. Since $M(g)$ contains all of its limit points, $M(g)$ is closed.

For all $x, y \in B$, we have $|x - y| \leq |x| + |y| \leq 2$. Fix some $x_0 \in \partial B$. Then for all $x \in B$ and all $h \in M(g)$, we have

$$|h(x)| \leq |h(x) - h(x_0)| + |h(x_0)| \leq |x - x_0| + |g(x_0)| \leq 2 + |g(x_0)|.$$

So $M(g)$ is uniformly bounded by $2 + |g(x_0)|$. So $M(g)$ is closed and bounded with respect to the supremum norm, and equicontinuous, so is compact with respect to the supremum norm.

Problem 42 (6)**.** For $x \in (0, \infty)$, let

$$F(x) = \int_0^\infty \frac{1 - e^{-tx}}{t^{3/2}}\, dt.$$

Show that $F : (0, \infty) \to (0, \infty)$ is well-defined, bijective, of class C^1, and that its inverse is of class C^1.

By using u-substitution, we let $u(t) = tx$ for $x > 0$ fixed and $t > 0$. Then for $\varepsilon > 0$ and $M > \varepsilon$,

$$\int_\varepsilon^M \frac{1 - e^{-tx}}{t^{\frac{3}{2}}}\, dt = \sqrt{x} \int_{\varepsilon x}^{Mx} \frac{1 - e^{-u}}{u^{3/2}}\, du$$

Let

$$C(\varepsilon, M) := \int_\varepsilon^M \frac{1 - e^{-u}}{u^{3/2}}\, du.$$

The integrand is nonnegative for $u > 0$. To show that $C(\varepsilon, M)$ converges as $\varepsilon \to 0$ and $M \to \infty$, it suffices to show that $C(\varepsilon, M)$ is bounded above by some constant, since $C(\varepsilon, M)$ is monotonically increasing as both ε decreases and M increases. For $u > 0$, by the mean value theorem, there exists some $v > 0$ such that $(1 - e^{-u})/u = e^{-v} \leq 1$. So $1 - e^{-u} \leq u$. Then

$$\int_\varepsilon^1 \frac{1 - e^{-u}}{u^{3/2}}\, du \leq \int_\varepsilon^1 u^{-1/2}\, du = [2u^{1/2}]_\varepsilon^1 \leq 2.$$

For $u \geq 1$, we have $1 - e^{-u} \leq 1 - e^{-1}$. So

$$\int_1^M \frac{1-e^{-u}}{u^{3/2}}\, du \leq (1-e^{-1}) \int_1^M u^{-3/2}\, du = (1-e^{-1})(2-2M^{-1/2}) \leq 2 - 2e^{-1} \leq 2.$$

This shows that

$$\int_\varepsilon^M \frac{1-e^{-u}}{u^{3/2}}\, du \leq 4.$$

So $C(\varepsilon, M)$ converges to some constant $C > 0$ as $\varepsilon \to 0$ and $M \to \infty$. Then for all $x > 0$, we have $C(\varepsilon x, Mx) \to C$ as well. So

$$F(x) = \int_0^\infty \frac{1-e^{-tx}}{t^{\frac{3}{2}}} dt = C\sqrt{x}$$

Is well-defined. This function is bijective $(0, \infty) \to (0, \infty)$, with inverse $G(y) = (y/C)^2$. We check that G and F are mutually inverse: for $x > 0$ and $y > 0$,

$$G(F(x)) = (F(x)/C)^2 = \sqrt{x}^2 = x, \qquad F(G(y)) = C\sqrt{G(y)} = C(y/C) = y.$$

We have $F'(x) = \frac{C}{2\sqrt{x}}$ is continuous on $(0, \infty)$, so F is C^1, and $G'(x) = 2y/C^2$ is also continuous, so G is C^1 as well.

Problem 43 (7)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the property that

$$T(T(x)) = T(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Show that there exists $1 \leq m \leq n$ and a basis of $\mathbb{R}^n$ such that in this basis the entries of T satisfy

$$T_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } 1 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Note: this problem is technically false, since the conclusion would imply $T_{11} = 1$ always, but T could be the zero map.
Let $m = \operatorname{rank}(T)$, and let $v_1, \ldots, v_m$ be a basis of $\operatorname{im}(T) \subseteq \mathbb{R}^n$. So there exists $x_1, \ldots, x_m \in \mathbb{R}^n$ such that $T(x_k) = v_k$ for all k. By the Rank-Nullity Theorem, $\dim(\ker(T)) = n - \operatorname{rank}(T) = n - m$. So let $v_{m+1}, \ldots, v_n$ be a basis of $\ker(T)$. We claim that $v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$. It suffices to show that these n vectors span $\mathbb{R}^n$, since $\mathbb{R}^n$ has dimension n. Let $x \in \mathbb{R}^n$. Let $x_0 = x - T(x)$. Then

$$T(x_0) = T(x) - T(T(x)) = 0.$$

So $x_0 \in \ker(T) = \operatorname{span}\{v_{m+1}, \ldots, v_n\}$, and $T(x) \in \operatorname{im}(T) = \operatorname{span}\{v_1, \ldots, v_m\}$, so $x = T(x) + x_0 \in \operatorname{span}\{v_1, \ldots, v_n\}$. So $v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$.
In this basis, for $1 \leq i \leq m$, we have

$$T(v_i) = T(T(x_i)) = T(x_i) = v_i.$$

So $T_{ij} = 1$ if $i = j$ and $1 \leq i \leq m$, and $T_{ij} = 0$ if $1 \leq j \leq m$ and $i \neq j$. Also, for $j > m$, we have $v_j \in \ker(T)$, so $T(v_j) = 0$, so $T_{ij} = 0$ if $j > m$.

Problem 44 (8)**.** Let X be an $n \times n$ symmetric (real) matrix and $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$. Define

$$G = (X - z)^{-1}.$$

Show that

$$\sum_{1\le j\le n} |G_{ij}|^2 = \frac{\operatorname{Im}(G_{ii})}{\operatorname{Im}(z)}$$

By the Spectral Theorem, since X is real and symmetric, X is diagonalizable with real eigenvalues. So z is not an eigenvalue of X, so $X - z$ has trivial kernel, so is invertible. So G is well-defined. We have

$$I = G(X - z) = (X - z)G.$$

So

$$GX = G(X - z) + zG = (X - z)G + zG = XG.$$

So G and X commute. Let G^* denote the conjugate transpose (adjoint) of G. Then taking adjoints of the above equation, since X is real and symmetric, $X^* = X$, so

$$G^*X = G^*X^* = (XG)^* = (GX)^* = X^*G^* = XG^*.$$

So X also commutes with G^*. Also,

$$GX = I + zG,$$

and

$$G^*X = (XG)^* = (GX)^* = I + \overline{z}G^*.$$

So

$$\begin{aligned} z\sum_{1\le j\le n} |G_{ij}|^2 &= z\sum_{1\le j\le n} G_{ij}\overline{G_{ij}} \\ &= z\sum_{1\le j\le n} G_{ij}(G^*)_{ji} \\ &= (zGG^*)_{ii} \\ &= ((GX - I)G^*)_{ii} \\ &= (GXG^* - G^*)_{ii} \\ &= (GG^*X - G)_{ii} + G_{ii} - G^*_{ii} \\ &= (G(G^*X - I))_{ii} + G_{ii} - \overline{G_{ii}} \\ &= \overline{z}(GG^*)_{ii} + 2\operatorname{Im}(G_{ii}) \\ &= 2\operatorname{Im}(G_{ii}) + \overline{z}\sum_{1\le j\le n} |G_{ij}|^2. \end{aligned}$$

Then

$$\begin{aligned} 2\operatorname{Im}(z)\sum_{1\le j\le n} |G_{ij}|^2 &= z\sum_{1\le j\le n} |G_{ij}|^2 - \overline{z}\sum_{1\le j\le n} |G_{ij}|^2 \\ &= 2\operatorname{Im}(G_{ii}), \end{aligned}$$

so

$$\sum_{1\le j\le n} |G_{ij}|^2 = \frac{2\operatorname{Im}(G_{ii})}{2\operatorname{Im}(z)} = \frac{\operatorname{Im}(G_{ii})}{\operatorname{Im}(z)}.$$

Problem 45 (9)**.** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be linearly independent elements of the vector space (over $\mathbb{R}$) of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}$. Show that for any $v \in \mathbb{R}^n$, there exists v_1 and v_2

such that

$$v = v_1 + v_2, \quad f(v) = f(v_1), \quad \text{and} \quad g(v) = g(v_2).$$

Since f and g are linearly independent, neither can be the zero map, so $\text{rank}(f) = \text{rank}(g) = 1$. By the Rank-Nullity theorem, we have $\dim\ker f = \dim\ker g = n - \text{rank}(f) = n - 1$.
Let $V = \ker f \cap \ker g$. Suppose for the sake of contradiction that $\dim V > n-2$. Since $V \subseteq \ker f$, we have $\dim(V) \le n-1$, so $\dim(V) = n-1$. But since $\dim\ker f = n-1$, we have $V = \ker f$. Similarly, $V = \ker g$, so $\ker f = \ker g$. Let $v \in \mathbb{R}^n$ be such that $f(v) \neq 0$. Let $\lambda = g(v)/f(v)$, so $g(v) = \lambda f(v)$. Now let $w \in \mathbb{R}^n$. Then

$$f\left(w - \frac{f(w)}{f(v)}v\right) = f(w) - f(w) = 0.$$

So $w - \frac{f(w)}{f(v)}v \in \ker f = \ker g$, so

$$0 = g\left(w - \frac{f(w)}{f(v)}v\right) = g(w) - \frac{f(w)}{f(v)}g(v) = g(w) - \frac{f(w)}{f(v)}\lambda f(v) = g(w) - \lambda f(w).$$

So $g(w) = \lambda f(w)$ for all $w \in \mathbb{R}^n$. But this contradicts f and g being linearly independent. So our assumption was wrong, and $\dim V \le n-2$.
Furthermore, the inclusion-exclusion formula gives

$$\dim(\ker f + \ker g) = \dim\ker f + \dim\ker g - \dim(\ker f \cap \ker g) \ge n - 1 + n - 1 - (n-2) = n.$$

So $\ker f + \ker g = \mathbb{R}^n$, so for all $v \in \mathbb{R}^n$, there exists $v_1 \in \ker g$ and $v_2 \in \ker f$ such that $v = v_1 + v_2$, and

$$f(v) = f(v_1) + f(v_2) = f(v_1),$$

and

$$g(v) = g(v_1) + g(v_2) = g(v_2).$$

Problem 46 (10)**.** Let $A := \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Calculate $\lim_{n\to\infty} A^n$.

Since A is lower triangular, we can read the eigenvalues off the diagonal as 1, 1/2 and 1/3. Since there are 3 distinct eigenvalues, A is diagonalizable. Let v_1, v_2, v_3 be eigenvectors for 1, 1/2, and 1/3, respectively. Then the matrix P with columns v_k is invertible, and $A = PDP^{-1}$, where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Specifically, we calculate that v_1 can be $(1,1,1)$, v_2 can be $(0,1,2)$, and v_3 can be $(0,0,1)$. Note that

$$D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 0 \\ 0 & 0 & 3^{-n} \end{bmatrix}$$

Since matrix multiplication is continuous,

$$\lim_n A^n = \lim_n PD^nP^{-1} = P(\lim_n D^n)P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}.$$

Then

$$\lim_n A^n = \begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} = \boxed{\begin{bmatrix}1&0&0\\1&0&0\\1&0&0\end{bmatrix}}.$$

Problem 47 (11)**.** Let V be the space of all 3×3 real matrices that are skew-symmetric, i.e. $A^t = -A$, where A^t denotes the transpose of A. Prove that the expression

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(\mathrm{AB^t})$$

defines an inner product on V. Exhibit an orthonormal basis of V with respect to this inner product.

The trace of a matrix's transpose is the same as the original trace, so

$$\langle B, A\rangle = \frac{1}{2}\,\mathrm{Tr}(BA^t) = \frac{1}{2}\,\mathrm{Tr}((AB^t)^t) = \frac{1}{2}\,\mathrm{Tr}(AB^t) = \langle A, B\rangle.$$

So the form is symmetric.
Now for any $A_1, A_2, B \in V$ and $c_1, c_2 \in \mathbb{R}$, by linearity of matrix multiplication and trace,

$$\begin{aligned}\langle c_1A_1 + c_2A_2, B\rangle &= \frac{1}{2}\,\mathrm{Tr}((c_1A_1 + c_2A_2)B^t)\\ &= \frac{1}{2}\,\mathrm{Tr}(c_1A_1B^t + c_2A_2B^t)\\ &= c_1\frac{1}{2}\,\mathrm{Tr}(A_1B^t) + c_2\frac{1}{2}\,\mathrm{Tr}(A_2B^t)\\ &= c_1\langle A_1, B\rangle + c_2\langle A_2, B\rangle.\end{aligned}$$

So the form is linear in the first coordinate (so linear in the second coordinate by symmetry). For any $A \in V$,

$$\begin{aligned}\langle A, A\rangle &= \frac{1}{2}\,\mathrm{Tr}(AA^t)\\ &= \frac{1}{2}\sum_{i=1}^{n}(AA^t)_{ii}\\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(A^t)_{ji}\\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}^2\\ &\geq 0,\end{aligned}$$

with equality if and only if $A_{ij} = 0$ for all i, j, if and only if $A = 0$. This completes the proof that the form is an inner product.
For an orthonormal basis of V, let

$$A_1 := \begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \quad A_2 := \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \quad A_3 := \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}.$$

Each of A_k is visibly skew-symmetric. It suffices to show that they are orthonormal (which implies they are linearly independent) and that they span V. Note that $A_kA_k^t$ is diagonal, with two 1s and a zero on the diagonal, so $\langle A_k, A_k \rangle = \frac{1}{2}\operatorname{Tr}(A_k A_k^t) = \frac{1}{2}(2) = 1$. It is straightforward to calculate that the A_k are mutually orthogonal. To see that they span V, suppose $A \in V$. Then for $i = 1, 2, 3$, we have $A_{ii} = -A_{ii}^t = -A_{ii}$, so $A_{ii} = 0$. For $j \neq i$, $A_{ij} = -A_{ji}$. So

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix} = A_{12}A_1 + A_{13}A_2 = A_{23}A_3.$$

This completes the problem.

Problem 48 (12)**.** Let V be a finite-dimensional vector space. Let $T : V \to V$ be a linear transformation such that $T(W) \subseteq W$ for every subspace W of V with $\dim(W) = \dim(V) - 1$. Prove that T is a scalar multiple of the identity.

Let $n = \dim(V)$. First, we claim that every nonzero vector of V is an eigenvector. Suppose that some $v \neq 0$ is not an eigenvector. Then $v, T(v)$ are linearly independent. Let $v_1 = v$, $v_2 = T(v)$, and extend to a basis $v_1, \ldots, v_n$ of V. Let $W = \operatorname{span}\{v_1, v_3, \ldots, v_n\}$. Then $\dim(W) = n - 1$, so $T(W) \subseteq W$. But $T(v) = v_2 \notin W$, a contradiction. It follows that for every nonzero vector v, there exists a scalar λ such that $T(v) = \lambda v$. A priori, the scalar λ depends on v, but we will prove it does not. Let $v_1, \ldots, v_n$ be a basis of V. Then for each k, there exists a scalar λ_k such that $T(v_k) = \lambda_k$. Consider $\sum_k v_k$. On one hand, there exists some scalar λ such that

$$T\left(\sum_k v_k\right) = \lambda \sum_k v_k = \sum_k \lambda v_k.$$

On the other hand,

$$T\left(\sum_k v_k\right) = \sum_k T(v_k) = \sum_k \lambda_k v_k.$$

Then

$$0 = \sum_k (\lambda - \lambda_k) v_k.$$

Since the v_k are linearly independent, we must have $\lambda - \lambda_k = 0$. So $\lambda_k = \lambda$ for all k. Then $T(v_k) = \lambda v_k$ for all k. For any $v \in V$, there are scalars a_k such that $v = \sum_k a_k v_k$. Then

$$T(v) = T\left(\sum_k a_k v_k\right) = \sum_k a_k \lambda v_k = \lambda v.$$

So $T = \lambda \mathrm{id}$.

5. S19

Problem 49 (1)**.** Consider $C([0,1])$, and let X be the subset of $C([0,1])$ of all 1 lipschitz functions f such that $f(0) = 0$. Show X is connected and complete.

For completeness, it is sufficient to show this space is closed, since the given space is already complete. If $f_n \in X$ converge uniformly to some $f \in C([0,1])$, we see that $f(0) = 0$. Likewise, for any $x, y \in [0,1]$, we have $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$ by triangle inequality. For n large enough, the first and last term are both less than $\varepsilon/2$ by uniform convergence. The middle term is less than or equal to $|x - y|$ for all n. Then $|f(x) - f(y)| \leq |x - y| + \varepsilon$ for all $\varepsilon > 0$, and so $|f(x) - f(y)| \leq |x - y|$. Thus, $f \in X$, so X is closed, and so complete.

For connectedness, we will show convexity. For any $t \in [0,1]$, and $f, g \in X$, one checks quickly that $tf + (1-t)g \in X$.

Problem 50 (2)**.** Let $a, b \in \mathbb{R}$ be two real numbers and consider a sequence a_n defined recursively:

$a_0 = a, a_1 = b, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ when $n \geq 2$. Prove that $lim_{n\to\infty}$ exists and compute its value.

First, observe that both a_0 and a_1 can be written in the form $c * a + (1-c) * b$ for some $c \in [0,1]$. Suppose that this is true for some a_{n-2} and a_{n-1}. Then $a_n = \frac{1}{2}(c_{n-1}a + c_{n-2}a + (1-c_{n-1})b + (1-c_{n-2})b) = da + (1-d)b$ where $d = \frac{c_{n-2}+c_{n-1}}{2}$, and note $d \in [0,1]$ as well. So, write each a_i as $c_i * a + (1-c_i) * b$ for some $c_i \in [0,1]$. I will show that the a_i have limit $\frac{a+2b}{3}$. Note $a_i - \frac{a+2b}{3} = (c_i - \frac{1}{3})a + (\frac{1}{3} - c_i)b$. It is sufficient then to show that $c_i \to 1/3$. I will show that for i even and at least 2, $c_i = \frac{2^{i-1}+1}{3*2^{i-1}}$, and for i odd and at least 3, $c_i = \frac{2^{i-1}-1}{3*2^{i-1}}$. From here, it is easy to see that $limc_i = 1/3$. It is true for $c_2 = 1/2$ and $c_3 = 1/4$. If n is even and at least 4, then $c_n = (c_{n-1} + c_{n-2})/2 = \frac{2^{n-1}+1}{3*2^{n-1}}$, as desired. Similar can be shown for n odd and at least 5.

Problem 51 (3)**.** Let $a, b \in \mathbb{R}$ with $a \leq b$ and let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Assume there is $\delta > 0$ such that $f(x) \geq \delta$ for all $x \in [a.b]$. Show that $1/f$ is Riemann integrable.

Fix $\varepsilon > 0$. Since f is Riemann integrable, let P_n be a partition of $[a,b]$ such that $U(f,P_n) - L(f,P_n) < \varepsilon/\delta^2$. On P_n, we have that $U(1/f,P_n) - L(1/f,P_n) = \sum_{[x_i,x_{i+1}]\in P_n}(x_{i+1}-x_i)sup_{x,y\in[x_i,x_{i+1}]}|1/f(x) - 1/f(y)| = \sum_{[x_i,x_{i+1}]\in P_n}(x_{i+1}-x_i)sup_{x,y\in[x_i,x_{i+1}]}|\frac{f(y)-f(x)}{f(y)*f(x)}| \leq \sum_{[x_i,x_{i+1}]\in P_n}(x_{i+1}-x_i)sup_{x,y\in[x_i,x_{i+1}]}|\frac{f(y)-f(x)}{\delta^2}| < \delta^2 * (U(f,P_n) - L(f,P_n)) = \varepsilon$.

Problem 52 (4)**.** Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a,b] \to \mathbb{R}$ be continuous functions differentiable on (a,b). Show there exists $\zeta \in (a,b)$ such that $(f(b) - f(a))g'(\zeta) = (g(b) - g(a))f'(\zeta)$

Consider the function $h(x) = g(x) * (f(b) - f(a)) - f(x) * (g(b) - g(a))$. As this is a difference of functions that are continuous on $[a,b]$ and differentiable on (a,b), it has both of these properties as well. We see that $h(b) = g(a)f(b) - g(b)f(a)$. Likewise, $h(a) = g(a)f(b) - g(b)f(a) = h(b)$. By Rolle's theorem, there exists a point $\zeta \in (a,b)$ such that $h'(\zeta) = g'(\zeta)(f(b) - f(a)) - f'(\zeta)(g(b) - g(a)) = 0$. This is equivalent to the given condition.

Problem 53 (5)**.** Show that each metric space can be embedded isometrically into a Banach space.

Let $C_b(D)$ be the space of bounded continuous functions on X into $\mathbb{R}$ with the standard supremum norm, and note that this space is complete. If D is empty we are trivially done, so if not, fix $x_0 \in D$, and define $S(x) = f_x(y) = d(x,y) - d(x_0,y)$. This is continuous since the metric is, and is bounded by reverse triangle inequality. For isometry, $(S(x) - S(z))(y) = d(x,y) - d(z,y)$. By reverse triangle inequality, this is less than or equal to $d(x,z)$, and taking $y = z$ gives that $|S(x) - S(y)| \geq d(x,z)$, and so $|S(x) - S(z)| = d(x,z)$, as desired.

Note: This is in the week 2 notes. I have no idea how someone would just come up with this on their own, and I am 90 percent positive it is only in the notes because Sylvester saw this question and got worried that there would be more. It seems like the sort of question that a professor just thought was a neat fact and wasn't actually a suitable question. I wouldn't worry about whether you could solve this from scratch, but rather, keep in mind that it could be similar to future questions.

Problem 54 (6)**.** For $n \geq 1$, let $f_n : \mathbb{R}^+ \to \mathbb{R}$, $f_n(x) = \frac{x}{n^2} e^{-x/n}$. Show that the f_n converge uniformly to zero, but $lim_{n\to\infty} \int_0^\infty f_n(x) = 1$.

Note each f_n is nonnegative. For each f_n, we compute $f_n'(x) = \frac{1}{n^2} e^{-x/n} + \frac{-x}{n^3} e^{-x/n} = \frac{n-x}{n^3} e^{-x/n}$. Therefore, f_n has a critical point iff $x = n$. We also check that $f_n''(x) = \frac{-1}{n^3} e^{-x/n} - \frac{n-x}{n^4} e^{-x/n} = \frac{x-2n}{n^4} e^{-x/n}$, which is negative when $x = n$, and so this is a local maximum. We check $f_n(0) = 0$ and $f_n(kn)$ for integer k is $\frac{k}{ne^k}$. The root test shows that this sequence converges as $k \to \infty$, and since $f_n'(x)$ is negative for $x > n$, we see that $lim_{x\to\infty} f_n(x) = 0$. Therefore, the global maximum of $f_n(x)$ is $f_n(n) = \frac{1}{ne}$.

Now pick $\varepsilon > 0$. For all n satisfying $1/n < e * \varepsilon$, and all x positive, we have $|f_n(x) - 0(x)| \leq \varepsilon$, and so we have uniform convergence.
We calculate the given integrals via integration by parts. First, a u substitution gives that $\int_0^\infty f_n(x) = \int_0^\infty u * e^{-u} du$. Taking $u_2 = u$, $dv = e^{-u}$, this expression is equal to $lim_{x\to\infty} - ue^{-u}|_0^x - \int_0^x -e^{-u} = lim_{x\to\infty} - xe^{-x} + 1 = 1$ by our above calculation. So, $lim_{n\to\infty} \int_0^\infty f_n(x) = lim_{n\to\infty} 1 = 1$.

Problem 55 (7)**.** For $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -5 & 3 \\ 1 & -1 & 2 \end{pmatrix}$, write A^{-1} as a polynomial in A with real coefficients.

We compute the characteristic polynomial of A to be $x^3 + 2x^2 + 13x - 1$ (I'm not typing out this calculation). By Cayley-Hamilton theorem, $A^3 + 2A^2 + 13A - 1 = 0$. Applying A^{-1} to both sides, $A^2 + 2A + 13 = A^{-1}$.

Problem 56 (8)**.** Let V be the vector space of all 2×2 matrices with real entries. Let W be the subspace generaged by all matrices of the form $AB - BA$ for $A, B \in V$. What is the dimension of W? Justify your answer

We note that for any matrix of the form $AB - BA$, $tr(AB - BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0$. Any matrix in the span of such matrices must also be traceless, and so this subspace is a supspace of the kernel of the trace operator. We know that the trace operator has kernel 3, since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are all in this space, are linearly independent, but I is not in this space, so $3 \leq dim(ker(tr)) < 4$ i.e. $dim(ker(tr)) = 3$. In particular, $dim(W) \leq 3$. We will exhibit three linearly independent matrices in W, and so $dim(W) = 3$. First, taking $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives that $A \in W$. Now, taking $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ gives that $D \in W$. Finally, taking $M = A + D$ and $N = B + D$ gives that $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in W$. We see that this matrix, and A, D are linearly independent, so $dim(W) = 3$.

Problem 57 (9)**.** Let V be the vector space over $\mathbb{C}$ of all complex polynomials of degree at most 10. Let $D : V \to V$ be the differentiation operator. find all eigenvalues and eigenvectors of e^D on V.

Observe that D is nilpotent with $D^{11} = 0$ but $D^{10} \neq 0$. We note that $e^D = \sum_{i=0}^{10} D^i/i!$. We see that $e^D(c) = c$ where c is constant (and hence, eigenvalue 1). I will show that in fact the nonzero constant functions (a subspace of degree 1 over $\mathbb{C}$) are the only eigenvectors. Let f be any polynomial in V with degree m more than 1. Then $e^D(f) = f + \sum_{i=1}^{m-1} D^i(f)/i!$, and note that this sum term will be a polynomial of degree exactly $m-1$. The leading coefficient of $e^D(f)$ is the leading coefficient of f, and hence, if f is an eigenvector, it must have eigenvalue 1. This implies that $\sum_{i=1}^{m-1} D^i(f)/i! = 0$, which is impossible, since each $D^i(f)$ has distinct degree.

Problem 58 (10)**.** Let A be an $n \times n$ complex diagonalizable matrix and I the $n \times n$ identity matrix. Show $M = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$ is not diagonalizable.

We use that a matrix is diagonalizable over $\mathbb{C}$ iff its minimal polynomial is squarefree. For any positive integer n, we compute that $M^n = \begin{pmatrix} A^n & nA^{n-1} \\ 0 & A^n \end{pmatrix}$. Then, for any polynomial $\sum_{i=0}^{n} a_i * x^i$ with $a_i \in \mathbb{C}$, we have that $p(M) = \begin{pmatrix} p(A) & \sum_{i=0}^{n} a_i * i * A^{n-1} \\ 0 & p(A) \end{pmatrix}$. We have $p(M)$ is zero iff $p(A)$ is zero and the upper right component is zero. But $p(A)$ is zero iff the characteristic polynomial of A, $m_A(x)$, divides $p(x)$. We also note that the upper right corner is exactly equal to $p'(A)$. This is zero exactly when $m_A(x)|p'(A)$. Since m_A is squarefree, m_A divides p and p' if and only if $m^2|p$. This implies that the minimal polynomial of M must be divisible by m_A^2, and so cannot be squarefree. M is not diagonalizable.

Problem 59 (11)**.** Let A be an $n \times n$ complex matrix. Prove that $rank(A) = rank(A^2)$ iff $lim_{\lambda \to 0}(A + \lambda I)^{-1}A$ exists.

Problem 60 (12)**.** Let $A = (a_{i,j})$ be an $n \times n$ real matrix whose diagonal entries are all at least 1, and such that $\sum_{i \neq j} a_{i,j}^2 < 1$. Prove A^{-1} exists.

Write $A = D + B$ where D is the diagonal component of A and B is the off diagonal component of B. We note that D is positive definite since its eigenvalues are along the diagonal and all positive. We note also that the condition on the remaining terms implies that $||B|| < 1$, since $(B^T Bv)_i = \sum_j (B^T B)_{i,j} v_j \leq \sqrt{\sum_j b_{i,j}^2}|v|$ by Cauchy Schwarz, and therefore, $||B^T B|| < |v| \sum_{i,j} b_{i,j}^2 < |v|$. This implies that $|\langle Bv, Bv \rangle| = | < B^T Bv, v\rangle \leq |v| * |B^T Bv| < |v|^2$, again by Cauchy-Schwarz.

Now, suppose $Av = Dv + Bv = 0$ for some v. Then $Dv = -Bv$. But if $v \neq 0$, we have $|Dv| > |v|$, but $|-Bv| < |v|$, a contradiction. Thus, $v = 0$, and A is invertible.

Note: This was an old Berkeley problem (week 4 Thursday), but I think it is the first time it has appeared on a test for UCLA. No idea what to think about that. Maybe keep in mind that old Berkeley probs could appear?

Problem 61 (1)**.** Let A be an invertible $n \times n$ matrix with real entries and let e_1 be the first standard basis vector. For each $\lambda \in \mathbb{R}$, define $A_\lambda(x) = Ax + \lambda\langle e_1, x\rangle e_1$. Show that A_λ is invertible iff $1 + \lambda\langle e_1, A^{-1}e_1\rangle \neq 0$.

Consider the set of vectors $A^{-1}e_i$. Since A^{-1} has full rank and the e_i are a basis, the $A^{-1}e_i$ are a basis as well. For each $A^{-1}e_i$, we see that $A_\lambda(A^{-1}e_i) = e_i + \lambda\langle e_1, A^{-1}e_i\rangle e_1$. From here, we see that the $A_\lambda(A^{-1}e_i)$ for $i \geq 2$ are linearly independent, and $A_\lambda(A^{-1}e_1)$ is in their span iff $A_\lambda(A^{-1}e_i) = 0$. Therefore, A_λ is invertible iff $A_\lambda(A^{-1}e_1) \neq 0$ iff $(1 + \lambda\langle e_1, A^{-1}e_1\rangle)e_1 \neq 0$ iff the given condition is true.

Problem 62 (2)**.** Find a real symmetric matrix A so that $A^2 + A =$ (not typing it out)

We can see that $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ works.

To find this matrix, the real symmetry condition suggests looking at eigenvalues and eigenvectors of A. Such an eigenvector would also be an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$. Finding eigenvectors of $A^2 + A$ isn't difficult. From here, you can compute what Ae_i for each e_i is, and this gives the above matrix for a certain choice of the eigenvalues.

Problem 63 (4)**.** Let V be a vector space and $1 \leq n < dim(V)$ be an integer. Let $\{V_i\}$ be a collection of n dimensional subspaces of V with the property that $dim(V_i \cap V_j) = n - 1$ for every $i \neq j$. Show that at least one of the following holds:
All V_i share a common $n - 1$ dimensional subspace.
There is an $n + 1$ dimensional subspace of V containing all V_i.

Problem 64 (5)**.** Show that an $x \times n$ matrix A with real entries obeys $lim_{k\to\infty}||A^k|| = 0$ iff all (possibly complex) eigenvalues have modulus strictly less than 1.

Note that the operator norm of any real matrix A, when viewed as an operator of a real vector space, is the same as the operator norm when viewed as an operator over a complex space. This is because the singular values remain the same, and the operator norm is the largest singular value modulus. Therefore, it is sufficient to show this equivalence holds when A is viewed as an operator over a complex vector space. If A has any eigenvalue of modulus ≥ 1, then fix a unit length eigenvector v corresponding to this λ. Then $A^k v = \lambda^k v$, which has norm at least one. The operator norm is the supremum of such vector norms, and so must be at least one for all k. This implies that the limit cannot go to zero.

Now, if all eigenvalues have modulus less than one, consider the Jordan canonical form of A. One checks that its entries are vanishing, and so the operator norm of $J^k \to 0$. Then $||A^k|| \leq ||P|| * ||J^k|| * ||P^{-1}|| \to 0$, where P is the invertible matrix satisfying $PJP^{-1} = A$.

Problem 65 (6)**.** Let M_n be the vector space of $n \times n$ matrices with real entries. Given $B \in M_n$, we define a linear transformation $L_b : M_n \to M_n$ by $L_B(A) = B^T AB$.
Prove L_B is invertible iff B is.

Find $rank(L_B)$ when B is diagonal.

Find $rank(L_B)$ in general.

For the first part, if B is invertible, then so is B^{-T}, and we see that $L_B(A)^{-1} = (B^T)^{-1}AB^{-1}$. If B is not invertible, then B^T is not invertible. This implies that $B^T v = 0$ for some nonzero v. Take A to be the matrix each of whose columns is v. Then $L_B(A) = 0$, so L_B is not invertible.

Let $E_{i,j}$ be the standard basis of M_n i.e. $E_{i,j}$ is only nonzero in row i column j, where it is one. Then $rank(L_B)$ is precisely the number of linearly independent $L_B(E_{i,j})$. Suppose that $B_{i,i} = \lambda_i$. Then we compute that $L_B(E_{i,j}) = \lambda_i * \lambda_j * E_{i,j}$. The set of nonzero $L_B(E_{i,j})$ is linearly independent since the $E_{i,j}$ are linearly independent, and so $L_B(E_{i,j})$ is nonzero iff λ_i and λ_j are both nonzero. Note that $rank(B)$ is exactly the number of nonzero λ_i, and so there are $rank(B)^2$ eigenmatrices not corresponding to the eigenvalue zero. Thus, $rank(L_B) = rank(B)^2$.

Problem 66 (7)**.** Show that the equation $x = cos(x)$ has exactly one solution on $[0, 1]$

$x = cos(x)$ iff $f(x) = x - cos(x) = 0$, so we show that $f(x)$ has exactly one root on $[0, 1]$. Note that f is continuous since it is a difference of continuous functions. $f(0) = 0 - cos(0) = -1$. $f(1) = 1 - cos(1)$. Since $1 \in [0, \pi/2]$, we have that $0 < cos(1) < 1$, and so $f(1) > 0$. Then $0 \in (f(0), f(1))$. By I.V.T., there exists a $c \in (0, 1)$ such that $f(c) = 0$. For uniqueness, observe that $f'(x) = 1 + sin(x)$. For $x \in [0, 1]$, $sin(x) > -1$, and so $f'(x)$ is positive. This means that $f(x)$ is strictly increasing on $[0, 1]$, and hence injective.

Problem 67 (8)**.** Show that $sup_{0<h\leq 1} \sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} < \infty$

Write $\frac{h}{1+n^2h^2} = h * f(nh)$ where $f(x) = \frac{1}{1+x^2}$. Note that f is an even function, and therefore, $h * f(nh) = h * f((-n) * h)$. The given sum is then equal to $h * f(0) + 2 * \sum_{n=1}^{\infty} h * f(nh)$. It is sufficient to show that $sup_{0<h\leq 1} \sum_{n\in\mathbb{Z}^+} h * f(nh) < \infty$ since $h * f(0) = h < 1$. Fix $N > 0$. Then $\sum_{n=1}^{N} h * f(nh) = L(f, P_n)$ where $f = \frac{1}{1+x^2}$ and P_n partitions $[0, Nh]$ into N uniformly sized blocks, where we have used that f is decreasing on the positive reals. But $L(f, P_n) \leq \int_0^{Nh} f(x)dx = arctan(Nh) - arctan(0) < \pi/2$. Since $\sum_{n=1}^{N} h * f(nh)$ is increasing in N and is bounded above by $\pi/2$, by MCT, it converges to some positive real less than or equal to $\pi/2$. Since this was independent of h, the supremum over all such h of this sum will also be no greater than $\pi/2$.

Problem 68 (9)**.**
(1) Show that the relation $(2 + x + y) = z^2 + e^x + e^y$ determines z as a smooth function of (x, y) in some neighborhood of the origin.
(2) Show that $(0, 0)$ is a critical point of $z(x, y)$ and determine its nature (minimum, maximum, etc.)

Problem 69 (11)**.** Let X denote the set of non-decreasing functions $f : [0,1] \to [0,1]$ with the supnorm metric.
Prove (X, d) is complete.
Prove it is not compact.

Let $f_n(x)$ be a Cauchy sequence of functions in X. Note that this implies that $f_n(x)$ is a Cauchy sequence of real numbers for each $x \in [0,1]$, and by completeness of $\mathbb{R}$, $f_n(x)$ converges pointwise to some function f. It remains to show that $f \in X$. We see that for any x, $f(x) = lim_n f_n(x)$ by pointwise convergence, and so $f(x) \in [0,1]$. Now, fix $0 \leq x < y \leq 1$. Then $f(y) - f(x) = lim_n f_n(y) - f_n(x)$. Since $f_n(y) - f_n(x)$ is nonnegative, so to is the limit, so we have f is nondecreasing.

Since compactness implies separability in metric spaces, it is sufficient to show that X is not separable. Consider the functions $f_r(x) = 0$ if $x \leq r$ and 1 if $x > r$, where $r \in [0,1]$. Then $sup|f_{r_1}(x) - f_{r_2}(x)| \leq 1$, and if $r_1 < r_2$, then $f_{r_1}(r_1) - f_{r_2}(r_1) = -1$, and so $sup|f_{r_1}(x) - f_{r_2}(x)| = 1$ for all distinct r_1, r_2. Note that there are uncountably many f_r since $[0,1]$ is uncountable. There can be no countable dense subset of X, since this would imply that the balls $B(f_{r_1}, 1/2)$ all contain a distinct element of our dense subset, which contradicts the countability of such a set. X is not separable, and hence, not compact.

Problem 70 (12)**.** Let $l^\infty(\mathbb{Z})$ denote the space of bounded functions $x : \mathbb{Z} \to \mathbb{R}$ together with the metric $d(x, y) = sup_{n\in\mathbb{Z}}|x(n) - y(n)|$. Show that a function $f : l^\infty(\mathbb{Z}) \to \mathbb{R}$ is continuous iff its restriction to any compact subset of $l^\infty(\mathbb{Z})$ is continuous.

For the forward direction, assume f is continuous, and let K be a compact subset of $L^\infty(\mathbb{Z})$. For any point $x \in K$, f is continuous at x since any sequence of K elements x_n converging to x is a sequence of $L^\infty(\mathbb{Z})$ elements converging to x. By continuity of f in $l^\infty(\mathbb{Z})$, $f(x_n) \to f(x)$, and so f is continuous on K.

For the reverse direction, assume f is continuous on any compact $K \subset l^\infty(\mathbb{Z})$. Let x_n be a sequence of $l^\infty(\mathbb{Z})$ elements converging to $x \in l^\infty(\mathbb{Z})$. I will show that $K = \{x_n\} \cup \{x\}$ is compact. For any infinite sequence y_n in K, either the y_n have only finitely many distinct values, in which case some subsequence is constant, or the y_n have infinitely many distinct values, in which case some subsequence must converge to x. K is sequentially compact, and hence, compact. By assumption, f is continuous on K, and so $lim_{n\to\infty} f(x_n) = f(x)$. f is thus sequentially continuous, and so continuous.

Note that the ambient space did not matter at all. The conclusion is actually generally true for metric spaces.

7. S20

8. F20

Problem 71 (1)**.** Let M be an $n \times n$ matrix with rational entries such that $M^2 = 2I$.
Prove n is even.
Give an example of such matrix M for $n = 2$.

The minimal polynomial of M must divide $x^2 - 2$. But $M \neq \pm\sqrt{2}I$, and so $x^2 - 2$ is the minimal polynomial of M. Note also that the characteristic polynomial must divide some power of the minimal polynomial since λ is a (complex) root of the minimal polynomial iff it is a root of the characteristic polynomial. This implies that the characteristic polynomial is exactly $(x^2-2)^i$ for some positive integer i, and hence, n must be even since this polynomial is degree $2i$.

We see that the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ works. (To find this, just write out the general expression for a 2 by 2 matrix and solve for possible entries)

Problem 72 (2)**.** Let A be an orthogonal $n \times n$ matrix.

Prove A^3 is orthogonal
Prove $A + \frac{1}{2}I$ is invertible.

A^3 is orthogonal iff $(A^3)^T A^3 = I$. But $(A^3)^T = (A^T)^3$. We see that $A^T * A^T * A^T * A * A * A = I$ by orthogonality of A.

$A + \frac{1}{2}I$ is invertible iff $1/2$ is not an eigenvlaue of A. I will show that orthogonal matrices can only have eigenvalues of modulus 1. Fix an eigenvector of A, v, corresponding to eigenvalue λ. Then $\lambda\langle v, v\rangle = \langle Av, v\rangle = \langle v, A^T v\rangle = \langle v, A^{-1}v\rangle$. Since $A^{-1}Av = v$, we have that v is an eigenvector of A^{-1} with eigenvalue $1/\lambda$. (λ must be nonzero or else A is not invertible at all). Then $\langle v, A^{-1}v\rangle = \frac{1}{\lambda}\langle v, v\rangle$. We have that $\lambda^2 = 1$, as desired.

Problem 73 (3)**.** Let M be a complex 4×4 matrix such that $M^6 = M^4 = 2M^3 - M^2$. Describe all possible J.C.F. of M.

The condition $M^4 - 2M^3 + M^2 = 0$ gives that the minimal polynomial of M divides $x^2(x-1)^2$, and so the only possible eigenvalues of M are 0 and 1. Note also that the power of $x - \lambda$ in the minimal polynomial is equal to the size of the largest Jordan Block for that eigenvalue. In particular, there can be no Jordan Blocks of size 3 or more for either possible eigenvalue. I will now show that there can be no block for eigenvalue 1 of size greater than 1. Let B be such a block. Note that $(B^n)_{i,j} = n$ for all n, and so $M^6 \neq M^4$. All blocks for eigenvalue zero are nilpotent with order no greater than 4, and so $B^6 = B^4$ for such blocks. This leaves us with only a few possible matrices, as described below.

(1) 4 zero's on the diagonal, 4 blocks
(2) 4 zeros on the diagonal, 1 block of size 2, 2 blocks of size 1
(3) 4 zero's on the diagonal, 2 blocks of size 2
(4) 3 zeros on the diagonal and one 1, all blocks of size 1.
(5) 3 zeros on the diagonal, one block of size 2 for the zeros
(6) 2 zeros and 2 ones on the diagonal, all blocks size 1.
(7) 2 zeros and 2 ones on the diagonal, block of zeros is of size 2.
(8) 1 zero on the diagonal and 3 ones, all blocks size 1.
(9) 4 ones on the diagonal, all blocks size 1.

Problem 74 (4)**.** Let A be a 2×2 real matrix with eigenvalues 2 and -1. Consider the set X of 2×2 real matrices C such that $C = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ is diagonalizable over $\mathbb{C}$. Prove X is a 2 dimensional subspace of $M_n(\mathbb{R})$

A matrix of this form is diagonalizable iff its minimal polynomial is squarefree. We compute that the characteristic polynomial of C is $\mu_A(x)^2$, where $\mu_A(x)$ is the characteristic polynomial of A. But since A is a 2×2 matrix with eigenvalues 2 and -1, $\mu_A(x)$ is exactly $(x-2)(x+1)$. We see that $C - 2I$ and $C + I$ are nonzero since $A - 2I$ and $A + I$ are nonzero, and so C is diagonalizable iff $(C-2I)(C+I) = 0$. We compute that $(C-2I)(C+2I) = \begin{pmatrix} 0 & AB + BA - B \\ 0 & 0 \end{pmatrix}$, and so C is diagonalizable iff $B = AB + BA$. Write $A = PDP^{-1}$ where D is the diagonal matrix with 2 and -1 on the diagonal. Write $F = P^{-1}BP$. We see that $B = AB + BA$ iff $F = DF + FD$, and each choice of such FC yields a unique such B, and each such B yields a unique such C. Write $F = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then we must have $\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 4w & x \\ y & -2z \end{pmatrix}$. We see that $w = z = 0$ is required, and any choice of x and y are valid. This is then indeed a dimension 2 subspace.

Problem 75 (5)**.** Let $K = \mathbb{F}_p$ be a finite field with p elements where p is prime. Let $V = K^9$ be a vector space, and let $W \subset V$ be a subspace of V such that $dim(W) = 5$. Compute the number of subspaces $U \subset V$ such that $dim(U) = 6$ and $dim(W \cap U) = 3$.

Problem 76 (6)**.** Let $v_1, \ldots. v_k \in \mathbb{R}^n$ satisfy $\langle v_i, v_j \rangle < 0$ for all $1 \leq i < j \leq k$. Prove $k \leq n+1$.

We argue by induction on n. The case for $n = 1$ is trivial. Suppose that the claim is true for all $n \leq N$ for some fixed N. Suppose towards contradiction that in $\mathbb{R}^{N+1}$, there exists $N+2$ vectors v_1 through v_2 such that $\langle v_i, v_j \rangle < 0$ for $i < j$. Consider the $N+1$ vectors $w_i = v_i - \langle v_i, v_{N+2} \rangle v_{N+2}$. We see that these vectors inhabit $v_{N+2}^{\perp}$, which is an N dimensional subspace of $\mathbb{R}^{N+1}$. However, for $i < j$, we see $\langle w_i, w_j \rangle = \langle v_i, v_j \rangle - 2\langle v_i, v_{N+2} \rangle \langle v_{N+2}, v_j \rangle + \langle v_i, v_{N+2} \rangle \langle v_j, v_{N+2} \rangle \langle v_{N+2}, v_{N+2} \rangle$. The first two terms are negative by assumption, and the last term can be made to have arbitrarily small absolute value by rescaling v_{N+2}. If v_{N+2} is rescaled such that the (finitely many) $\langle w_i, w_j \rangle < 0$, then we have a contradiction, because $v_{N+2}^{\perp}$ is isomorphic to $\mathbb{R}^N$, but we have found $N+1$ vectors with mutually negative inner product in this space.

Problem 77 (9)**.** Let $f(x)$ be a real and continuous function on $[0, 1]$. Show that $lim_{n \to \infty}(n+1) \int_0^1 x^n f(x) dx = f(1)$.

Fix $\varepsilon > 0$. Since f is continuous on the compact set $[0, 1]$, it is bounded. In particular, there exists positive M such that $f(x) < M$ for all x and $f(x) > -M$ for all x. Write $\zeta_n = (\varepsilon/M)^{1/(n+1)}$. For any n, observe that the given integral is equal to $(n+1)\int_0^{\zeta_n} x^n f(x) + (n+1)\int_{\zeta_n}^1 x^n f(x)$. The first integral is less than ε and greater than $-\varepsilon$ by using the boundedness of f. Since x^n is positive on $(\zeta_n, 1)$ and x^n and f are both continuous, by Mean Value theorem for integrals, $(n+1)\int_{\zeta_n}^1 x^n f(x) = (n+1) f(c_n) \int_{\zeta_n}^1 x^n = f(c_n) * (1 - \varepsilon/M)$ for some $c_n \in (\zeta_n, 1)$. Taking limits, since $c_n \to 1$ and since f is continuous, we have that the given limit is in the interval $(f(1)*(1-\varepsilon/M) - \varepsilon, f(1)*(1-\varepsilon/M) + \varepsilon)$ for any $\varepsilon > 0$. This implies that the given limit is indeed $f(1)$.

Problem 78 (10)**.** Let f_n be a uniformly bounded equicontinuous sequence of real valued functions on a compact metric space X with distance function d. Define the functions g_n from $X \to \mathbb{R}$ by $g_n = max\{f_1(x), \ldots f_n(x)\}$. Show that g_n converges uniformly.

Note that for any x, $g_{n+1}(x) \geq g_n(x)$. Note also that the g_n converge pointwise at any x since the sequence $g_n(x)$ is a bounded (by uniform boundedness of f_n) monotone sequence of real numbers.

By Dini's theorem, since X is compact, it is sufficient to show that the limit function g is continuous.

First, we will show that $g(x) = sup(f_1(x), f_2(x)...)$. The supremum is defined since f_n are uniformly bounded. Note that $g(x) \geq f_n(x)$ for all n, and so $g(x) \geq sup(f_1(x), f_2(x)...)$. Since each $g_n(x) \leq sup(f_1(x)...)$, we have that the limit $g(x) \leq sup(f_1(x)...)$ as well.

We now show continuity of g. Fix $x, y \in X$. Then $g(x) - g(y) = sup_i(f_i(x) - g(y)) \leq sup_i(f_i(x) - f_i(y))$. Fix $\varepsilon > 0$. By equicontinuity of the f_n, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| \leq \varepsilon$ when $|x - y| \leq \delta$. Then for this delta, we have $|g(x) - g(y)| \leq \varepsilon$ as well, implying continuity of g.

Problem 79 (11)**.** For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$, and assume that these functions are uniformly bounded. Let X be a countable subset of $\mathbb{R}$. Show that the sequence f_n has a subsequence that converges pointwise for all $x \in X$.

Enumerate the X elements as $x_1, x_2....$ Consider the sequence $f_1(x_1), f_2(x_1), f_3(x_1)...$ This is a bounded sequence of real numbers since the f_n are uniformly bounded. Therefore, by compactness, some subsequence of the $f_1(x_1)$, $f_2(x_1)$... converges, i.e., some subsequence of the f_n converge pointwise at x_1. Relabel this subsequence as $f_{1,1}, f_{1,2}, f_{1,3}....$ Now suppose that we have a sequence g_n that converges for x_i for all $1 \leq i \leq k$ for some k. Then we may apply the above argument to find a subsequence of the g_n that converge at x_{k+1} as well. Applying this argument to our sequence $f_{1,1}$, $f_{1,2}$... repeatedly, we obtain nested subsequences, where $f_{k,1}, f_{k,2}$... converges for all x_i with $i \leq k$.
Now, define $f_k = f_{k,k}$. Fix $x_i \in X$. For $k \geq i$, we see that the f_k are a subsequence of $f_{i,1}$, $f_{i,2}$..., and this sequence was defined to converge at x_i. Thus, the f_k are a subsequence of the f_n that converge at all $x \in X4$.

Problem 80 (12)**.** Let X be an open convex subset of $\mathbb{R}^n$. Let $f : X \to \mathbb{R}$ be a differentiable function.
Show that for any $x, y \in X$, there is a point z lying on the line segment from x to y for which $f(y) - f(x) = \nabla f(z) * (y - x)$.

Use part a to show that if the partial derivatives of f are bounded, then f is uniformly continuous on X.

Fix $x, y \in X$. Define $g(t) : [0, 1] \to \mathbb{R}$ where $g(t) = f((1-t)*x+t*y)$. Note that g is continuous since it is a composition of continuous functions, and since X is convex, $(1-t)*x+t*y \in X$ for all t in this range. By Mean Value Theorem, there exists a $t^* \in [0, 1]$ such that $g'(t^*)*(1-0) = f(1)-f(0)$. Now, observe that $f(1) - f(0) = f(y) - f(x)$. By chain rule, $g'(t^*) = \nabla f((1 - t^*)x + t^*y) * (y - x)$. Taking $z = (1 - t^*)x + t^*y$ works.

Suppose that the partial derivatives of f are bounded. Then $|\nabla f| \leq M$ for some positive M. Fix $\varepsilon > 0$. Then if $|y - x| \leq \varepsilon/M$, we have that $|f(y) - f(x)| \leq |\nabla f| * |(y - x)|$ (Cauchy Shchwarz) $< \varepsilon$, as desired.