Ordinary Λ-adic Eichler–Shimura for Shimura curves

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1. Introduction: Conjectures of Sharifi
2. Ohta's Λ-adic Eichler–Shimura
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Relating geometry and arithmetic

In [Sha11], Sharifi stated conjectures relating the cohomology of modular curves and the arithmetic of cyclotomic fields.
In particular, Sharifi constructs a map

$$\Upsilon : Y \to P$$

where:

- Y is related to the arithmetic of cyclotomic fields, and
- P is related to the cohomology of modular curves.

Sharifi conjectures that this map is an isomorphism. This conjecture is known in many cases. *(Some details given on later slides.)*

Relating geometry and arithmetic

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Goal: to construct an analogous map relating arithmetic and geometry over totally real number fields.

Construction of Υ: Notation

We sketch the construction of the aforementioned map Υ in a restricted setting, following Fukaya–Kato [FK24].

Notation

- $p \geq 5$ is a prime number.
- $K := \mathbb{Q}(\mu_{p^\infty})$.
- $\Gamma := \mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_p^\times$.
- $\Lambda := \mathbb{Z}_p[[\Gamma]]$.
- $X_\infty := \varprojlim_r \mathrm{Cl}(\mathbb{Q}(\mu_{p^r}))[p^\infty] \cong \mathrm{Gal}(L/K)$, where L is the maximal unramified pro-p abelian extension of K.
- $\mathfrak{h} := \varprojlim_r \mathfrak{h}_r^{\mathrm{ord}}$, where $\mathfrak{h}_r$ is the (adjoint cuspidal) Hecke algebra over $\mathbb{Z}_p$ associated with the modular curve $X_1(p^r)$.
- $I = (T(p)^* - 1, T(\ell)^* - (1 + \ell\langle\ell\rangle^*) : \ell \neq p) \subseteq \mathfrak{h}$ is the Eisenstein ideal.
- $H := \varprojlim_r H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}}$, which is a module over $\mathfrak{h}$.

Construction of Υ: Preliminaries

We have a short exact sequence

$$0 \to P \to H/IH \to Q \to 0$$

of $(\mathfrak{h}/I)$-modules, where $P := H^-/IH^-$ and $Q := H^+/IH^+$. We need the following fact:

Fact

$Q \cong \mathfrak{h}/I$ as $\mathfrak{h}$-modules, with a canonical generator.

Construction of Υ

It turns out that $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts trivially on P and Q. The above exact sequence then gives us a homomorphism

$$G_K \to \mathrm{Hom}_{\mathfrak{h}}(Q, P).$$

Furthermore, the action of G_K on H/IH is unramified at all places. Thus, the above homomorphism factors through X_∞. Since Q has a canonical generator as an $(\mathfrak{h}/I)$-module, we obtain a map of Λ-modules:

$$\Upsilon : X_\infty^- \to P.$$

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A result of Ohta

Recall the following key ingredient of the above construction:

Fact

$Q \cong \mathfrak{h}/I$ as $\mathfrak{h}$-modules, with a canonical generator.

The proof of this fact is somewhat technical, but relies crucially on a result of Ohta [Oht95][Oht00]. Let $I_p \subseteq G_{\mathbb{Q}_p}$ be the inertia subgroup. Let $\mathcal{O}$ be the ring of integers of a complete subfield of $\mathbb{C}_p$ containing all roots of unity.

Ohta's Λ-adic Eichler–Shimura

Theorem (Ohta)

There is a short exact sequence of $\mathfrak{h}[G_{\mathbb{Q}_p}]$-modules:

$$0 \to H_{sub} \to H \to H_{quo} \to 0,$$

where $H_{sub} := H^{I_p}$, and there is a canonical isomorphism

$$H_{quo} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O} \cong S(\Lambda_{\mathcal{O}})^{\mathrm{ord}}(-1),$$

where the right hand side denotes ordinary $\Lambda_{\mathcal{O}}$-adic cusp forms.

We briefly discuss some of the key ingredients of Ohta's first proof (away from certain eigenspaces) [Oht95].

Step 1: Good quotients of Jacobians

Let J_r be the Jacobian of $X_1(p^r)$. Let e' be the composition of Hida's idempotent for $T(p)^*$ and projection away from $1, \omega^{-1}$ eigenspaces for the action of $(\mathbb{Z}/p\mathbb{Z})^\times$.

Construct "good quotients" B_r of J_r (following Mazur–Wiles and Tilouine) such that:

1. $e' J_r[p^\infty] \cong e' B_r[p^\infty]$.
2. $B_{r/\mathbb{Q}_p(\mu_{p^r})}$ extends to an abelian scheme over $\mathcal{O}_r := \mathbb{Z}_p[\mu_{p^r}]$, which we denote by $B_{r/\mathcal{O}_r}$.
3. Let $G_r = e' B_{r/\mathcal{O}_r}[p^\infty]$ (a p-divisible group over $\mathcal{O}_r$). Then G_r is ordinary (connected component is of multiplicative type).
4. A "twisted Weil pairing" gives a duality between $T_p(G_r^\circ)$ and $T_p(G_r^{\text{ét}})$.
5. The action of inertia at p is given explicitly on $T_p(G_r^\circ)$ and $T_p(G_r^{\text{ét}})$.

Step 2: Hodge–Tate decomposition

From the connected-étale sequence for G_r, taking $\mathbb{Z}_p$-duals, and taking inverse limits, we obtain our exact sequence

$$0 \to H_{sub} \to H \to H_{quo} \to 0.$$

At finite level,

$$0 \to H_{r,sub} \to H_r \to H_{r,quo} \to 0.$$

The Hodge–Tate decomposition for p-divisible groups over complete DVRs gives

$$H_{r,quo} \hookrightarrow e'(H^0(X_1(p^r), \Omega^1) \otimes K)(-1) \cong e'S_2(p^r, K)(-1).$$

Step 3: Integrality, Control

Ohta uses *q-expansions* to show that the image of the above map is integral (i.e. Fourier coefficients lie in $\mathcal{O}_r$), and uses control theorems (Hida theory) to exactly characterize the image.

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Shimura curves

For the remainder of the talk:

- $F =$ a totally real number field of degree $d > 1$.
- $B =$ a quaternion algebra over F split at exactly one real place of F.
- $\mathfrak{p} =$ a prime of F that splits B, of residue characteristic p.
- $\mathfrak{N}$ a nonzero ideal of $\mathcal{O}_F$ of sufficiently large norm (in particular, coprime to the discriminant of B).

Associated to this data, we may form **Shimura curves**

$$X_r := X_1^B(\mathfrak{N}\mathfrak{p}^r).$$

for each $r \geq 1$.
These Shimura curves do not have smooth integral models, but the work of Carayol [Car86] describes the bad reduction of certain integral models $\mathfrak{X}_r$ over $\mathcal{O}_{F,\mathfrak{p}}$.

Hecke correspondences

- There are Hecke correspondences on these curves analogous to classical Hecke correspondences. In particular, we have $T(\mathfrak{p})$ acting on J_r, the Jacobian of X_r.
- Let $\mathfrak{h}_r^B$ be the (adjoint) Hecke algebra acting on J_r, and let $\mathfrak{h}^B := \varprojlim_r (\mathfrak{h}_r^B)^{\mathrm{ord}}$.

Proposed Generalization

Conjecture

Let

$$H^B(1) := \varprojlim_r H^1_{\text{ét}}((X_r)_{\overline{F}}, \mathbb{Z}_p(1))^{\text{ord}}.$$

Then there is a short exact sequence of $\mathfrak{h}^B[G_{F_\mathfrak{p}}]$-modules

$$0 \to H^B(1)_{sub} \to H^B(1) \to H^B(1)_{quo} \to 0$$

with explicit Galois action on the "sub" and "quo," together with an isomorphism

$$H^B(1)_{quo} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} \cong \varprojlim_r H^0(\mathfrak{X}_r, \Omega^1) \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}.$$

Good quotients of Jacobians of Shimura curves

We construct "good" quotients A_r of J_r analogous to those of Ohta, Tilouine, and Mazur–Wiles.

Proposition (S.)

$$T(\mathfrak{p})^r J_r[p^\infty] \cong T(\mathfrak{p})^r A_r[p^\infty].$$

Construction of good quotients

We fix a maximal $\mathcal{O}_F$-order $O \subset B$. Let

$$\widehat{O} := O \otimes \widehat{\mathbb{Z}}, \qquad \widehat{B} := B \otimes \widehat{Z}.$$

Then $\widehat{O}^\times$ is an open compact subgroup of $\widehat{B}^\times$.
For each ideal $\mathfrak{M}$ of $\mathcal{O}_F$ coprime to $\mathfrak{D}$ (so that $O/\mathfrak{M}O \cong M_2(\mathcal{O}_F/\mathfrak{M})$), we define

$$\widehat{\Gamma}_0(\mathfrak{M}) := \left\{ x \in \widehat{O}^\times : x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\},$$

$$\widehat{\Gamma}_1(\mathfrak{M}) := \left\{ x \in \widehat{O}^\times : x \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}.$$

We define Shimura curves $X_*^B(\mathfrak{M}) := X^B(\widehat{\Gamma}_*(\mathfrak{M}))$ for $* \in \{0, 1\}$.

Construction of good quotients

Let

$$X_r' := X^B(\widehat{\Gamma}_1(\mathfrak{N}\mathfrak{p}^r) \cap \widehat{\Gamma}_0(\mathfrak{p}^{r+1})), \qquad J_r' := \mathrm{Jac}(X_r').$$

Then we have natural maps $X_{r+1} \xrightarrow{\pi_r} X_r' \xrightarrow{\rho_r} X_r$. This gives maps on Jacobians

$$\pi_r^* : J_r' \to J_{r+1} \qquad \text{and} \qquad (\rho_r)_* : J_r' \to J_r.$$

We define $\alpha_1 : J_1 \to A_1$ to be the identity. Inductively, we define

$$K_r := \ker(\alpha_r \circ (\rho_r)_*), \qquad \alpha_{r+1} : J_{r+1} \to A_{r+1} := J_{r+1}/\pi_r^*(K_r)^\circ.$$

Lemma

For each $r \geq 1$, the subgroup-scheme $\ker \alpha_r$ is stable under Hecke operators. Thus, α_r is Hecke equivariant.

Proof idea of the proposition

Proposition (S.)

$$T(\mathfrak{p})^r J_r[p^\infty] \cong T(\mathfrak{p})^r A_r[p^\infty].$$

Next steps

1. Show that these "good quotients" have potentially good reduction.
2. Apply some p-adic Hodge theory to prove the Conjecture above.
3. Use Jacquet–Langlands to relate $\varprojlim_r H^0(\mathfrak{X}_r, \Omega^1)$ to Hilbert modular forms.
4. Use a Λ-adic Eichler–Shimura for Hilbert modular forms...

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CM Points

- Shimura curves do not have cusps, and differentials on Shimura curves do not have natural q-expansions.
- However, Shimura curves still have *CM points* (0-dimensional special cycles).
- Shimura showed that CM points can be used to study algebraicity of certain automorphic forms (power series expansions around CM points).
- Can we similarly detect integrality of modular forms on Shimura curves by taking power series expansions around CM points?

Side project

Now:

- $X =$ smooth projective curve over a number field F. Assume for simplicity that X is geometrically connected.
- $D =$ degree 0 divisors on $X_{\overline{F}}$.
- $P =$ principal divisors on $X_{\overline{F}}$.
- $J = D/P$ (the $\overline{F}$-points of the Jacobian of X).

An interesting short exact sequence

We have a map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "p^r"] & D \arrow[r] \arrow[d, "p^r"] & J \arrow[r] \arrow[d, "p^r"] & 0 \\ 0 \arrow[r] & P \arrow[r] & D \arrow[r] & J \arrow[r] & 0. \end{tikzcd}$$

Since J is divisible, the Snake Lemma gives:

$$0 \to J[p^r] \to P/p^r P \to D/p^r D \to 0.$$

Taking a limit in r, one obtains

$$0 \to T_p J \to P \widehat{\otimes} \mathbb{Z}_p \to D \widehat{\otimes} \mathbb{Z}_p \to 0.$$

Here, for an abelian group A, we denote by $A \widehat{\otimes} \mathbb{Z}_p$ the p-completion of A.

Pulling back CM points

Let C be a set of geometric points of X. We have a subgroup $\mathrm{Div}^0(C) \subset D$ consisting of degree 0 divisors supported on C. Pulling back the previous exact sequence, we get

$$0 \to T_p J \to \mathcal{E} \to \mathrm{Div}^0(C) \widehat{\otimes} \mathbb{Z}_p \to 0.$$

Is this split?

"Theorem" (S.)

If this sequence is split as G_F-modules, then for every $x, y \in C$, we have $[x] - [y] \in J_{tors}$.

(Think Manin–Drinfeld)

Why?

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