1 Background

My research is about studying multiplicative structures in homotopy theory.

One of the fundamental goals of algebraic topology is to have a better understanding of the stable homotopy groups of the sphere spectrum. One our most effective computational tools for computing them is the Adams spectral sequence, which uses a homology theory to approximate the homotopy classes of maps between two spectra X and Y. The Adams spectral sequence for a ring spectrum E starts with the E_2 page, given by $E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*Y)$, where E_*X and E_*Y are comodules over E_*E. Under appropriate conditions this converges to $[X, \hat{Y_E}]_{t-s}$. In the specific case where $E = H\mathbb{F}_p$ and $X = Y = \mathbb{S}$, the spectral sequence takes the form $\mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}(\mathbb{S}) \otimes \mathbb{Z}_p$, where $\mathcal{A}_p^* = H\mathbb{F}_{p*}H\mathbb{F}_p$ is the dual Steenrod algebra.

In order to use the Adams spectral sequence to compute the homotopy groups of the sphere spectrum there are three basic steps. First, the E_2 page has to be computed. Since the E_2 page is algebraic, this is a mechanical computation, which can be done by hand or by a computer. Second, the differentials in the spectral sequence have to be computed, which allows us to go from the E_2 page to the E_∞ page. Finally, the E_∞ page is the associated graded of a filtration on the homotopy groups, so in order to recover the actual homotopy group we need to resolve any potential extension problems. My work is most related to the second step of computing differentials.

In general the differentials can be quite challenging to compute, but the Adams spectral sequence has several pieces of additional structure on it that we can use to help in determining differentials. Probably the most important is that the Adams spectral sequence has a multiplicative structure, which makes each page of the spectral sequence into a differential graded algebra. The Leibniz rule can then be used to propagate one known differential to give us information on many other differentials.

Another important piece of additional structure on the Adams spectral sequence are Steenrod operations. Steenrod operations in this context are shadows of homotopy coherently commutative multiplication maps $\mu_n : E\Sigma_n \otimes_{\Sigma_n} Y^{\otimes n} \to Y$, where $E\Sigma_n \otimes_{\Sigma_n} Y^{\otimes n} =: D_n(Y)$ is the nth extended power of Y. However, we actually use a variant of the extended power which uses a subgroup of Σ_n instead to build Steenrod operations. When π is a subgroup of Σ_n, let $D_\pi(Y) := E\Sigma_n \otimes_\pi Y^{\otimes n}$.

Then given a map $f \in [X, Y]$, we can build a map

$$P_\pi f : D_\pi(X) \to D_n(X) \xrightarrow{D_n(f)} D_n(Y) \xrightarrow{\mu_n} Y.$$

Then $P_\pi f$ induces maps from the homotopy groups of $D_\pi X$ to the homotopy groups of Y, or if we think about it another way, an element of of $\pi_k D_\pi(X)$ induces a map $[X, Y] \to \pi_k Y$.

This can all be interpreted in various categories, like chain complexes, spectra and filtered spectra. When p is a prime, if we work with chain complexes over $\mathbb{F}_p$, and consider the case $n = p$ and $\pi = C_p \subset \Sigma_p$, then when $X = \mathbb{F}_p[m]$, we have that $X^{\otimes p} = \mathbb{F}_p[mp]$ is actually invariant under the C_p action, so we get that $D_p(\mathbb{F}_p[m]) = EC_p \otimes_{C_p} \mathbb{F}_p[mp] \simeq BC_p[mp]$, and in the category of chain complexes over $\mathbb{F}_p$, BC_p is a chain complex whose homology computes the C_p-group homology of $\mathbb{F}_p$. Then these group homology classes give operations on the homology of any chain complex K, with a homotopy coherently commutative multiplication map $D_p(K) \to K$. When $K = C^*(A; \mathbb{F}_p)$ is the singular cochain complex of a space A, this constructs the usual Steenrod operations on the cohomology $H^*(A; \mathbb{F}_p)$. This is also how the Steenrod operations on the Adams spectral sequence were originally constructed.

Novikov [11] and Liulevicius [7] introduced Steenrod operations on the cohomology of Hopf agebras and therefore on the E_2 page of the Adams spectra sequence. The fact that the Adams spectral sequence has Steenrod operations is therefore a purely algebraic one.

A series of authors, Kahn [6], Milgram [9], Mäkinen [8] and Bruner [3] (ordered by increasing generality), realized that by lifting the algebraic maps that produce the Steenrod operations in Ext to maps of filtered spectra, one can get information about the differentials of the Steenrod operations.

Bruner [3] showed that if we assume that the ring spectrum carries an H_∞-structure (a structure stronger than merely homotopy commutative, but weaker than E_∞), then the maps of chain complexes that give the power operations on Ext lift to maps of spectra, and that we can then use these maps of spectra to compute the differentials on power operations. Specifically, for a class of topological degree n, the differentials on a Steenrod operation are determined by the attaching map of a corresponding cell in a particular cell structure on $D_{C_p}(S^n)$.

The fact that we can produce and compute Steenrod operations in the Adams spectral sequence is perhaps somewhat surprising though, since the Adams resolution functor is not an E_∞-functor, so the H_∞ structure of our ring spectrum is not being preserved when we take the Adams resolution, but it does give us enough to build the appropriate maps that induce the Steenrod operations on the level of spectra. There has been some work generalizing Bruner's ideas to other contexts like equivariant and motivic homotopy theory. For example, Sean Tilson [13] used Bruner's ideas to produce a formula for d_2 in the C_2 equivariant Adams spectral sequence.

However, Burklund, Hahn and Senger [4] give a construction that suggests an alternative route for generalizing Bruner's work. Their construction gives a recipe for turning an existing E_∞-functor, $T : \mathcal{C} \to \mathcal{C}^{\mathrm{Fil}}$, intertwining that functor with the cobar resolution of a ring object, E, and producing a new E_∞-functor from objects to filtered objects, $\mathrm{Sh}(T;E)$. Concretely, $\mathrm{Sh}(T;E)$ is the composite

$$\mathcal{C} \xrightarrow{\otimes\,\mathrm{cb}(E)} \mathcal{C}^{\Delta} \xrightarrow{T^{\Delta}} \mathcal{C}^{\mathrm{Fil},\Delta} \xrightarrow{\mathrm{Tot}} \mathcal{C}^{\mathrm{Fil}}.$$

If T is the Whitehead tower functor, then the associated spectral sequence for $\mathrm{Sh}(T;E)X$ has E_1 page isomorphic to the E_2 page of the usual E-Adams spectral sequence for X.

Since this construction is an E_∞ one, it gives us an alternate route for constructing the Steenrod operations in the Adams spectral sequence. The advantage of this method is that it gives us a construction and results that generalize to many kinds of Adams-like spectral sequences in a broader variety of contexts. I'll talk about my work on this idea in the next section.

The context in which I'm most interested in applying these ideas is that of equivariant homotopy theory. Equivariant homotopy theory is a version of homotopy theory where we have a group G and we keep track of G-actions as we build the usual objects of homotopy theory like cohomology theories and spectra. Since many objects of interest in homotopy theory come with natural group actions, keeping track of those actions when studying those objects allows us to maintain greater control over those objects and get stronger results. There are several things that are interesting about power operations in the equivariant case. The homotopy groups of a G-spectrum, X, are indexed by a subgroup H of G and an integer n, and are defined by $\pi_n^H(X) = [(G/H)_+ \wedge S^n, X]^G$. For a fixed n, we can assemble the groups for each subgroup into the structure of a Mackey functor, which we can denote $\underline{\pi}_n X$. Consequently, the Adams spectral sequence can be taken to have values in Mackey functors, so perhaps the first thing we'd like to know is what the Steenrod operations and differentials are on elements that live in π_n^H where H is a proper subgroup of G.

Secondly, one can get spheres with nontrivial group actions by taking the one point compactification of a real G-representation V to get a representation sphere, written S^V. Then for a G-spectrum X we can define $\mathrm{RO}(G)$-graded homotopy groups $\pi_\star^G X$ by $\pi_V^G(X) := [S^V, X]^G$. Similarly, the equivariant Adams spectral sequence can also be viewed as having an $\mathrm{RO}(G)$ grading. Now for a class in degree V, the differentials on the Steenrod operations on that class will be determined by the attaching maps in the appropriate cell structure on $D_{C_p}(S^V)$.

Thirdly, one of the interesting features of equivariant homotopy theory is that there are many distinct lifts of the non-equivariant E_∞-operad which can be partially ordered by which norm maps they have [2]. We say that a G-spectrum is a genuine G-E_∞ ring if it is an algebra for the maximum lift. Much like π_0 of an E_∞ spectrum is an ordinary ring, $\underline{\pi}_0$ of a G-E_∞ ring is a Green functor, which is a commutative monoid in Mackey functors, however $\underline{\pi}_0$ of a genuine G-E_∞ ring has the additional structure of norm maps, which make it into an object called a Tambara functor.

Now if E is a genuine G-E_∞ ring, then when T sends genuine G-E_∞ rings to genuine G-E_∞ filtrations, $\mathrm{Sh}(T;E)$ does as well. As a result, we can get interesting differences in structure on the resulting $\mathrm{Sh}(T;E)X$ depending on the choice of T. For example, the equivariant Whitehead tower functor only preserves the weakest G-E_∞ structure, but the regular slice filtration preserves the maximal G-E_∞ structure. Accordingly, the Adams-like spectral sequences associated to the regular slice filtration will be spectral sequences of Tambara functors. I'm very interested in studying the interaction of the Tambara functor structure with the Steenrod operations.

As a result, I'm also interested in the study of the algebra of Tambara functors. However many of the fundamental questions that have been answered for commutative rings become more complicated for Tambara functors and have not yet been answered. For example, in a joint paper (discussed below) my collaborators and I answered the question of which Tambara functors are algebraically closed (Nullstellensatzian). To get a sense of how much is still unknown about Tambara functors, some important questions that don't yet have complete answers that I am interested in are: What does it mean to localize a Tambara functor at a prime ideal? What properties does the category of modules over a Tambara field have? How can we build schemes out of Tambara functors?

2 Thesis Research

2.1 Steenrod operations in Adams-like spectral sequences

The goal of this project was to first reproduce the Steenrod operations in the Adams spectral sequence using the fact that $\mathrm{Sh}(T;E)Y$ is an E_∞ filtered spectrum when T is an E_∞ tower functor and E and Y are E_∞-rings and second work on proving formulas for the differentials of the Steenrod operations similar to those shown by Bruner, and ideally do so in a way that generalizes to other contexts.

If $F_\bullet$ is a filtered spectrum, let $E_r^{s,n}(F_\bullet)$ denote the spectral sequence that we get from $F_\bullet$ by applying $\pi_\bullet$, so $E_1^{s,n}(F_\bullet) = \pi_n F_{s,s+1}$, where $F_{s,s+1}$ is the cofiber of the map $F_{s+1} \to F_s$. There is a filtered spectrum $U_\bullet(r,s,n)$ that represents a class in $E_1^{s,n}(F_\bullet)$ along with choices of lifts that prove that the first $r-1$ differentials are 0, so that it represents a well defined class in $E_r^{s,n}$.

Now when $F_\bullet$ is E_∞, we can apply our general recipe for producing Steenrod operations to a map $\alpha : U_\bullet(r,s,n) \to F_\bullet$ to get a map

$$D_{C_p}(U_\bullet(r,s,n)) = E_{C_p}^+ \otimes_{C_p} U_\bullet(r,s,n)^{\otimes p} \xrightarrow{E_{C_p}^+ \otimes_{C_p} \alpha^{\otimes p}} E_{C_p}^+ \otimes_{C_p} F_\bullet^{\otimes p} \to F_\bullet.$$

Then classes $\gamma : U_\bullet(r',s',n') \to D_{C_p}(U_\bullet(r,s,n))$ give us classes $P\alpha \circ \gamma : U_\bullet(r',s',n') \otimes X \to F_\bullet$. However, there aren't always enough such maps γ to get all of the Steenrod operations that we expect. To see why, consider the case for $p=2$ and $U(1,s,n)$. The associated graded for $D_2(U(1,s,n))$ has a $D_2(S^n)$ in filtration $2s$, an S^{2n-1} in filtration $2s+1$ and a $D_2(S^{n-1})$ in filtration $2s+2$. So we end up getting operations for the elements of the homotopy groups of $D_2(S^n)$, which look more like the homotopy power operations than the Steenrod operations, which should correspond to elements of the homology of $D_2(S^n)$.

However, if $F_\bullet$ is an algebra over another filtered spectrum $F'_\bullet$ (which we can always take to be $F_\bullet$ if there isn't something more useful), then we get something a little more useful for our purposes if we tensor $P\alpha$ with $F'_\bullet$ and multiply back down again. So let $P_{F'}\alpha$ denote the composite $D_{C_p}(U_\bullet(r,s,n)) \otimes F'_\bullet \to F_\bullet \otimes F_\bullet \to F_\bullet$. This map carries quite a bit of useful information, but in particular, it induces a map on the associated gradeds that in filtration ps looks like $F'_{0,1} \otimes D_{C_p}(S_n) \to F_{ps,ps+1}$. Then if we take π_m of both sides, we get $\pi_m(F'_{0,1} \otimes D_{C_p}(S^n)) \to \pi_m F_{ps,ps+1} = E_1^{ps,m}$.

Now in the case that $F_\bullet = \mathrm{Sh}(T;E)Y$ where T is the Whitehead tower and Y and E are connective E_∞ ring spectra, we can take $F'_\bullet$ to be $\mathrm{Sh}(T;E)\mathbb{S}$, in which case we have that $F'_{0,1}$ is an Eilenberg-Maclane spectrum on the group $H^0\left(\pi_0\left(E^{\otimes \bullet+1}\right)\right)$, where the cohomology here is that of the cosimplicial abelian group. In other words given a class α we get a map

$$(P_F\alpha)_* : H_m\left(D_{C_p}(S^n); H^0\left(\pi_0\left(E^{\otimes \bullet+1}\right)\right)\right) \to E_1^{ps,m}.$$

In the specific case that $E = H\mathbb{F}_2$, $Y = \mathbb{S}$, and $p = 2$, then we get that $D_{C_2}(S^n) = \Sigma^n \mathbb{R}P_n^\infty$, and the map becomes

$$(P_F\alpha)_* : H_m\left(\Sigma^n \mathbb{R}P_n^\infty; \mathbb{F}_2\right) \to E_1^{2s,m}.$$

Since $H_m(\Sigma^n \mathbb{R}P_n^\infty; \mathbb{F}_2)$ is 0 for $m < 2n$ and $\mathbb{F}_2$ for $m \geq 2n$, this gives us classes in $E_1^{2s,m}$ for $m \geq 2n$, and using the usual indexing for the Steenrod squares, we can define $\mathrm{Sq}^i(\alpha)$ to be the class in $E_1^{2s,n+i}$, so that $\mathrm{Sq}^n(\alpha)$ is the class α^2 in $E_1^{2s,2n}$.

Returning to the general case, so far I've just discussed what's happening in filtration ps, but the objects $U_\bullet(r,s,n)$ encode d_r, so $P_{F'}\alpha$ also encodes information about the differentials on the Steenrod operations. In particular, a nonzero differential on a class in $D_2(U_\bullet(r,s,n)) \otimes F'_\bullet$ gives us the first possibly nonzero differential on the image of that class in $F_\bullet$.

We can get this information about the differentials by looking at a second filtration on our filtered object induced by the skeletal filtration on EC_p, since this is the filtration used to compute cellular homology, which is the easiest way to get at our Steenrod operations. Essentially a Steenrod operation is given by a cell e_m of $D_{C_p}(S^n)$ whose attaching map vanishes in $F'_{0,1}$ homology. Then understanding the differential comes down to understanding this attaching map, and how it lifts up the filtration. This is where we need to actually use details of the filtration in question.

2.2 Localizations and spectra of Tambara functors

Historically, an extremely useful strategy for proving results about commutative rings has been to simplify problems by using the techniques of algebraic geometry to reduce global questions to local ones.

Nakaoka [10] defined prime ideals for Tambara functors and defined a spectrum of a Tambara functor as a topological space. We'd like to have a sheaf or sheaf-like structure on this to so that we can do something similar to algebraic geometry for Tambara functors.

However, while we can define localizations by adjoining inverses to elements, several things go wrong. Firstly, if a is a single element of a Tambara functor, T, the map $\operatorname{Spec} T[a^{-1}] \to \operatorname{Spec} T$ is not always the inclusion of an open subset. Secondly, the complements of prime ideals are not multiplicatively closed. Thirdly, even when $\operatorname{Spec} T[a^{-1}] \to \operatorname{Spec} T$ is the inclusion of an open subset, these subsets don't form a basis for the topology on $\operatorname{Spec} T$.

In joint work, Ben Spitz and I showed that for $\mathcal{A}_G$ the Burnside Tamabara functor, there is an open subset of $\operatorname{Spec}(\mathcal{A}_G)$, $U \neq \varnothing$, such that for any Tambara functor $T \neq 0$ and map $f : \mathcal{A}_G \to T$ there is a prime ideal $p \in \operatorname{Spec}(\mathcal{A}_G)$ such that $f^*p \notin U$. In other words U despite being nonempty cannot contain any nonempty affine open subsets, so we cannot define a structure sheaf on $\operatorname{Spec} T$ in a manner analogous to that for ordinary commutative rings.

However, the reason for this failure also suggests a possible solution to the problem. The problem is that if T is a G-Tambara functor then $\operatorname{Spec}(T)$ has additional structure beyond that of a topological space, specifically we showed that $\operatorname{Spec}(T)$ contains the spectra of T's restrictions to H-Tambara functors with $N_G(H)/H$ action, $\operatorname{Spec}(R^G_H T)$, as retract subspaces, and the reason our open subset U failed to contain the image of any Tambara spectrum was that an arbitrary open subset doesn't have to respect this additional structure.

2.3 Algebraically closed (Nullstellensatzian) Tamabara functors

Burklund, Schlank and Yuan [5] introduced the concept of Nullstellensatzian objects. A nonterminal object a of a category $\mathcal{C}$ is *Nullstellensatzian* if all nonterminal compact objects $a \to b$ in the undercategory $\mathcal{C}/a$ admit a section $b \to a$.

The name comes from the fact that the Nullstellensatz says that algebraically closed fields are Nullstellensatzian objects in the category of commutative rings, and in fact these are all of the Nullstellensatzian objects.

In the case of Tambara functors we can take this to be the definition of algebraically closed. In joint work [12], Ben Spitz, Noah Wisdom and I showed that the Nullstellensatzian G-Tambara functors are all of the form $C^G_e k$, where C^G_e is the coinduction functor from commutative rings to G-Tambara functors, which is right adjoint to the restriction functor from G-Tambara functors to commutative rings and where k is an ordinary algebrically closed field.

3 Future Research

3.1 Steenrod operations in Adams-like spectral sequences

As mentioned in the introduction, I think a particularly interesting direction to take my work on Steenrod operations in Adams-like spectral sequences is to look at Steenrod operations in the equivariant context, and in particular to look at the spectral sequence that we get from taking our tower functor to be the regular slice filtration functor. Then we'll get a G-E_∞ filtration and the corresponding spectral sequence will be a spectral sequence of Tambara functors.

3.2 Spectra of Tambara functors

I plan to continue joint work with Ben Spitz and Hiroyuki Nakaoka on understanding localizations of Tambara functors at prime ideals and connecting this to the geometry of the spectrum.

Another project I've been working on in the same vein is that of understanding the geometry of the Tambara functor in terms of the Balmer spectrum of its module category. For rings, Balmer showed [1] that the Balmer spectrum of the tensor triangular category of perfect complexes of modules over that ring is naturally isomorphic to the usual prime spectrum of the ring.

If T is a Tambara functor, then the category of modules over T has not just tensor products, but in fact norms that turn it into a sort of categorified Tambara functor. Then if we restrict our tensor triangular ideals to ones that respect this additional structure we would hope that we again get a natural isomorphism of the Tambara spectrum of T with the Balmer spectrum of the category of perfect complexes of modules over T.

This would then shed light on the problem of localizing a Tambara functor at prime ideals, since the quotient of the category of perfect complexes of modules over a ring by a prime tensor-triangular ideal is the category of perfect complexes of modules over the ring localized at the corresponding prime of the ring.

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