- Integrality
- Functional Equation

$\Bigg\}$ Milne 27

- Reductions $\}$ Milne 28

$\boxed{\text{Integrality}}$

Goal: $Z(X_0, t) = \dfrac{P(t)}{Q(t)}$ P, Q constant term 1 w/ $\mathbb{Z}$ coefficients and coprime.

Def. $N_m(x) = |\mathbb{F}_q(K(x), \mathbb{F}_{q^m})|$ $x \in X_0$ closed

$= |\{\mathbb{F}_{q^m}$ points of X_0 with image $x\}|$

$$= \begin{cases} \deg(x) & \deg(x) \mid m \\ 0 & \text{o/w} \end{cases}$$

Of course, $N_m = \sum_{\deg(x) \mid m} N_m(x)$

$$\log \frac{1}{1-t^{\deg(x)}} = \sum \frac{t^{n\deg(x)}}{n}$$

$$= \sum N_m(x) \frac{t^m}{m}$$

$$Z(X,t) = \exp\left(\sum_m N_m \frac{t^m}{m} \right)$$

$$= \exp\left(\sum_x \sum_m N_m(x) \frac{t^m}{m} \right)$$

$$= \exp\left(\sum_x \log \frac{1}{1-t^{\deg(x)}} \right)$$

$$= \prod_x \frac{1}{1-t^{\deg(x)}}$$

$$\in 1 + t\,\mathbb{Z}[[t]]$$

Lemma, ℓ prime,

$$1+t\mathbb{Z}_\ell[[t]] \ni f(t) = \left.\begin{array}{c}\frac{g(t)}{h(t)}\end{array}\right\} \in 1+t\,\mathbb{Q}_\ell[t]$$

coprime

Then $g, h \in \mathbb{Z}_\ell[t]$

Pf, work in the splitting field of h.

$$h(t) = \prod(1-c_i t)$$

If $|c_i|_\ell > 1$, $|c_i|_\ell^{-1} < 1$

so $f(c_i^{-1})$ converges

$$\underbrace{f(c_i^{-1})\, h(c_i^{-1})}_{=0} = \underbrace{g(c_i^{-1})}_{\neq 0}$$

So $|c_i|_\ell \leq 1$ for all i, so $h \in \mathbb{Z}_\ell[t]$

work with $\frac{1}{f}$ for g.

This proves integrality, as $\bigcap_\ell \mathbb{Z}_\ell = \mathbb{Z}$

Functional Equation

Recall. $Z(X_0, \frac{1}{q^d t}) = \pm q^{d\chi/2} t^{\chi} Z(X_0, t)$

$\chi = \Delta\cdot\Delta = \sum (-1)^r b_r$.

Lemma. $\sigma(F^*|H^r) = \dfrac{q^d}{\sigma(F^*|H^{2d-r})}$

Pf. By Poincaré duality, $\sigma(F^*|H^r) = \sigma(F_*|H^{2d-r})$

$= \dfrac{q^d}{\sigma(F^*|H^{2d-r})}$

as $F^* = \dfrac{q^d}{F_*}$ by projection and degree of F. $\square$

Pf of full eq. Recall

$$Z(X_0, t) = \prod_{r=0}^{2d} P_r(t)^{(-1)^{r+1}}$$

where $P_r(t) = \det(1 - tF|H^r)$

First, $P_r\left(\frac{1}{q^d t}\right) = \prod_{\alpha | H^r} \left(1 - q^{-d} t^{-1} \alpha\right)$

$$= (-1)^{\beta_r} (q^d t)^{-\beta_r} \prod_{\alpha | H^r} \left(\alpha - q^d t\right)$$

$$= \text{'' \quad ''} \left(\prod_{\alpha | H^r} \alpha\right)\left(\prod_{\alpha | H^r} \left(1 - \frac{q^d}{\alpha} t\right)\right)$$

$$= \text{'' \quad ''} \det(f|H^r) \prod_{\gamma_i | H^{2d-r}} \left(1 - \gamma t\right)$$

$$= (-1)^{\beta_r} (q^d t)^{-\beta_r} \det(f|H^r)\, P_{2d-r}(t)$$

Now, we seek $\det(f|H^r)$. These are hard alone, but we pair them.

Let $r \neq d$. $\det(f|H^r)\det(f|H^{2d-r}) = (q^d)^{\beta_r}$

$\det(f|H^d)$? For an eigenvalue α of $f|H^d$, there are 3 disjoint cases

$\alpha \neq q^d/\alpha$ } come in pairs

$\alpha = q^{d/2}$

$\alpha = -q^{d/2}$

so $\det(f|H^d) = \pm q^{\beta_d d/2}$

Hence,

$$Z\left(\mathcal{X}_0, \frac{1}{\xi^d t}\right) = \prod_r P_r\left(\frac{1}{\xi^d t}\right)^{(-1)^{r+1}}$$

$$= \prod_r \left((-1)^{\beta_r} (\xi^d t)^{-\beta_r} \det(f|H^r)\, P_{2d-r}(t) \right)^{(-1)^{r+1}}$$

$$= Z(\mathcal{X}_0, t) \prod_r \left((-1)^{\beta_r} (q^d t)^{-\beta_r} \det(f|H^r) \right)^{(-1)^{r+1}}$$

$$= \pm Z(\mathcal{X}_0, t) (\xi^d t)^{\chi} \underbrace{\prod_r \det(f|H^r)^{(-1)^{r+1}}}$$

$r = d$ contributes $\left(\pm q^{d\beta_d/2}\right)^{(-1)^{d+1}}$

$r \neq d$ contributes $\prod_{r=0}^{d-1} q^{(-1)^{r+1}\beta_r d}$

$$= \prod_{r \neq d} q^{(-1)^{r+1}\beta_r \frac{d}{2}}$$

$$= \pm Z(\mathcal{X}_0, t) (q^d t)^{\chi} q^{-\frac{d\chi}{2}} Z(\mathcal{X}_0, t) \qquad \square$$

Reductions

Recall, X_0 nonsingular projective / $\mathbb{F}_q$

Then all $\alpha \in \sigma(F|H^r)$ are algebraic with all conjugates having complex norm $q^{r/2}$.

Lil' reduction, RH holds over $\mathbb{F}_q$ $\Leftrightarrow$ it holds over $\mathbb{F}_{q^m}$

Pf, $f : X \to X$ from $\mathbb{F}_q$

$f_m : X \to X$ from $\mathbb{F}_{q^m}$

Then $f_m = f^m$

so $\sigma(f_m) = \sigma(f)^m$

Main reduction, We reduce to the following:

Let X_0 nonsingular projective even dimensional variety

(*) and $\alpha \in \sigma(F|H^d)$

Then α is algebraic and

$$q^{\frac{d-1}{2}} < |\alpha'| < q^{\frac{d+1}{2}}$$

for all conjugates α' of α.

Pf. Let X_0 sm proj dim $= d$ / $\mathbb{F}_q$

$\alpha \in \sigma(F | H^d)$

By Künneth, $H^d(X, \mathbb{Q}_\ell)^{\otimes m}$ is a summand of

$H^{md}(X^m, \mathbb{Q}_\ell)$

so $\alpha^m \in \sigma(F | H^{dm}(X^m, \mathbb{Q}_\ell))$

Let m be even. By $(*)$ we have

$$q^{\frac{md-1}{2}} < |\alpha'|^m < q^{\frac{md+1}{2}}$$

$$\Big\downarrow m \to \infty$$

$$|\alpha'| = q^{\frac{d}{2}}$$

So we have shown the case for the middle cohomology. We show the general case by induction.

$d = 0$. :)

$d = 1$, only nontrivial one is H^1, which is the middle cohomology

So let $d \geq 2$.

How to go forward? Must relate cohomology of subvarieties to cohomology of X.

Gysin sequence

(X, Z) a smooth pair of codimension c, i.e. codim c per connected component

$U = X \setminus Z$.

Then

$$H^r(X, F) \longrightarrow H^r(U, F) \quad 0 \leq r < 2c-1$$

$$0 \to H^{2c-1}(X, F) \to H^{2c-1}(U, F) \to H^{r-2c}(Z, F(-c)) \to H^r(X, F) \to H^r(U, F) \to \dots$$

Arises from $H^{r-2c}(Z, F(-c)) \cong H^r_Z(X, F)$

where $H^r_Z(X, F)$ are the derived functors of

$$F \longmapsto \left(\ker(\Gamma(X, F) \longrightarrow \Gamma(Z, F)) \right)$$ and the LES $U \xrightarrow{j} X \xleftarrow{i} Z$

$$\dots \to H^r_Z(X, F) \to H^r(X, F) \longrightarrow H^r(U, F) \to .$$

c.f. $0 \to j_! j^* \mathbb{Z} \to \mathbb{Z} \to i_* i^* \mathbb{Z} \to 0$

repr $F(U)$ repr $F(X)$ repr $\ker(F(X) \to F(Z))$

We seek to understand $\sigma(F|H^r)$.

By PD, WLOG $r > d$.

Let $Z = X \cap H$ a smooth hyperplane section.

By hyp. in,

$$\to H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \xrightarrow{i_*} H^r(X, \mathbb{Q}_\ell) \to H^r(U, \mathbb{Q}_\ell) \to \dots$$

U is affine, so $H^r(U, \mathbb{Q}_\ell) = 0$ for $r > d$

(cf weak Lefschetz)

Hence, $i_* : H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \longrightarrow H^r(X, \mathbb{Q}_\ell)$

is onto for $r > d$.

Fact. $F^* i_* = q^{\dim X - \dim Z} i_* F^*$

commute with Frob as i defined over $\mathbb{F}_q$

Pf. $F_* F^* i_* = q^{\dim X} i_*$

$F_* q^{\dim X - \dim Y} i_* F^* = q^{\dim X - \dim Y} \overbrace{i_* F_*} F^*$

$= q^{\dim X} i_*$

The eigenvalues of $F|H^{r-2}(Z, \mathbb{Q}_\ell)$ satisfy RH by induction, so they have norm $q^{\frac{r-2}{2}}$.

Hence, the eigenvalues of $F|H^r(X, \mathbb{Q}_\ell)$ have norm $q^{\frac{r}{2}}$. $\square$