K a global field of positive characteristic.

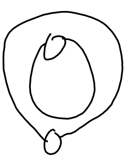
$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1-(N\mathfrak{p})^{-s}}$$

where does $\mathfrak{p}$ range?

primes in some integer ring
not canonical

Also, want "complete" ζ fns, so must also include "∞".

Let $k = \bar{\mathbb{F}}_p \cap K$
Fix $x \in K$ trans $/k$



$\mathfrak{p}$ ranges in non zero primes of $k[x]$ and of their

∞ [in $k[x^{-1}]$ not already accounted for

X a curve / k a finite field

interested in $|X(k_n)|$

(solve equations?)

gen fn $\sum N_n T^{n-1}$

$\exists\ Z(X,T)$ s.t.

$$\frac{d}{dt} \log Z(X,T) = \sum N_n T^{n-1}$$

$$Z(X,T) = \prod_{\substack{p \in X \\ \text{closed pt}}} \frac{1}{1-T^{\deg(p)}}$$

fn fields $\longleftrightarrow$ curves

symmetric. Let $K \longleftrightarrow X$ and take
$$\zeta_K(s) = \prod_{\substack{p \in X \\ \text{closed}}} \frac{1}{1-q^{-s\deg(p)}}$$

$q = |k|$

$$\zeta_K(s) = Z(X,T), \quad T = q^{-s}$$

(Rmk: log-derivative of $\zeta_{\mathbb{Q}}$ is $-\sum \frac{\Lambda(n)}{n^s}$)

Now, as let us understand Z

Riemann-Roch yields

$$Z(X,T) = \frac{P(X,T)}{(1-T)(1-qT)} \quad \in \mathbb{Q}[T],\ \deg 2g$$

(write Z as $\sum_{n\geq 0}$ by geom series)

$$Z\left(X, \frac{1}{qT}\right) = q^{1-g} T^{2-2g} Z(X,T)$$

$$P(X,T) = \prod_{i=1}^{2g} (1 - q_i T) \quad , \quad \frac{1}{q_i} \text{ roots of } Z$$

so poles ζ_K @ $s=0,1$

RH for K: roots on $\mathrm{Re}(s) = \frac{1}{2}$ $\Leftrightarrow$ via $q^{-s} = T$ change of variable

RH for Z: $|q_i| = q^{1/2}$ $\forall i$

Above known to German # theorists < 20's.

RH pfs: Hasse $g=1$ via $End(E)$

Deuring ≥ 2

Deuring arbitrary case via "correspondences"

$$Corr(X, Y) = \{\text{curves } C \subseteq X \times Y\}$$

e.g. graphs
dual fn Severi
pullbacks divisors

General systems!

X a nonsing proj var/$\underbrace{k}_{\text{finite}}$ of dim $= n$

$$Z(X,T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} T^m\right)$$

$$N_m = |X(k_m)|$$

$$\zeta(X,s) = Z(X, q^{-s})$$

1. $Z(X,T) = \dfrac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$, $P_i \in \mathbb{Z}[T]$

$P_0 = 1 - T$, $P_{2n} = 1 - q^n T$

$P_i(T) = \prod_j (1 - \alpha_{ij} T)$ $\quad 1 \leq j \leq 2n-1$

in $\overline{\mathbb{Q}}$

"rationality"

2. "fn'l eq!" $Z\left(X, \frac{1}{q^n T}\right) = \pm q^{\frac{n\chi}{2}} T^{\chi} Z(X,T)$

$\chi =$ euler char $= I(\Delta,\Delta)$ in $X \times X$

$\Leftrightarrow \zeta(X, n-s) = \pm q^{\frac{n\chi}{2} - \chi s} \zeta(X,s)$

3. "RH" $|\alpha_{ij}| = q^{i/2}$ $\forall\, 1 \le i \le 2n-1$

$\Updownarrow$

roots of $P_k(q^{-s})$ on $\mathrm{Re} = \frac{k}{2}$

4. If $X = Y \bmod p$ for Y defined over a subfield of $\mathbb{C}$, then $\deg P_i = b_i(Y(\mathbb{C}))$

e.g. $\mathbb{P}^n$, $N_m = 1 + q^m + \dots + q^{nm}$

$$\zeta = \frac{1}{\prod_{i=0}^{n} 1 - q^{i-s}}$$

Method of Weil.

How to count $N_m = |X(k_m)|$?

$X(k_m) = X(\bar{k})^{\mathrm{Frob}^m}$

Recall Lefschetz fixed point formula. $f: X \to X$

$$I(\Gamma_f, \Delta) = \sum (-1)^i \operatorname{Tr}(\underbrace{f^{(i)}}_{\text{on cohomology}})$$

Rmk. This is what got Grothendieck into the Weil conjectures.

Serre approached him w/ a sheaf theoretic notion

Serre's thesis was SSS, freely model same year as Kodaira

FAC

he and Leray were developing sheaves in at

sheaves were hot

Grothendieck + Serre deeply believed a cohomological answer exists.

Weil cohomology theory

char $K = 0$

char k = whatever

A weil cohom theory is:

$H^* : \text{SmProjVar}/k^{op} \longrightarrow \text{gr } K\text{-Alg}$

s.t.

Tautology
i) $H^*(X)$ fd and supported in $0 \le i \le 2n$, $n = \dim X$

ii) orientation $H^{2n}(X) \xrightarrow{\sim} K$

iii) PD $H^i \otimes H^{2n-i} \longrightarrow H^{2n} \xrightarrow{\sim} K$ a perfect pairing

iv) Künneth $H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$

v) Lefschetz $I(\Delta, \Gamma_f) = \sum (-1)^i \operatorname{Tr}(f^{(i)})$

vi) Let $Z^r(X)$ = codim r alg cycles

$\exists\ \gamma_X : Z^r(X) \longrightarrow H^{2r}(X)$ compatible w/ all

(eg, $Y \subseteq X$ complex mfld, real codim $2r$, $[Y] : H^{2n-2r} \longrightarrow \mathbb{C}$

$w \longmapsto \int_Y w$

$\therefore \in H^{2r}$ by PD)

vii) weak Lefschetz

$W \subset X$ a sm hyperplane section

$j^* : H^i(X) \longrightarrow H^i(W)$

iso $\quad i \leq n-2$

inj $\quad i = n-1$

viii) Hard Lefschetz

W sm hyperplane section

$w = \gamma_X(W) \in H^2 \quad$ (c.f. Kähler form)

$H^{n-i}(X) \longrightarrow H^{n+i}(X) \quad i \geq 0$

$x \longmapsto x w^i$

ok, say this exists.

$$N_m = \sum_i (-1)^i \operatorname{tr}\left((\mathrm{Frob}^m)^{(i)} \right)$$

Rationality. Lemma $\varphi : V \to V / K$ fdvs

$P_\varphi(T) = \det(1 - \varphi T) = \prod (1 - a_i T)$

$\operatorname{Tr}(\varphi^m) = \sum a_i^m$ and $\log \frac{1}{P_\varphi(T)} = \sum_{m=1}^{\infty} \operatorname{Tr}(\varphi^m) \frac{t^m}{m}$

Ps. Under A Gust

$$Z(X,T) = \exp\left(\sum_m N_m \frac{t^m}{m}\right)$$

$$= \exp\left(\sum_m \left(\sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left((\mathrm{Frob}^m)^{(i)}\right)\right)\frac{t^m}{m}\right)$$

$$= \prod_{i=0}^{2d} \underbrace{\left(\exp\left(\sum_m \operatorname{Tr}\left((\mathrm{Frob}^m)^{(i)}\right)\frac{t^m}{m}\right)\right)}_{\det\left(1-\mathrm{Frob}^{(i)}T\right)}^{(-1)^i}$$

So in fact, $P_i(T) = \det\left(1-\mathrm{Frob}^{(i)}T\right)$

and $Z(X,T) = \prod P_i(T)^{(-1)^{i+1}}$ as desired

P_0, P_{2n} are char poly on a 1d space

So far these are poles in K though.

Will in practice have $K = \mathbb{Q}_\ell$, but we still need to prove integrality.

functional eq

$$\mathrm{Frob}^{(i)} : H^i(X, K) \longrightarrow H^i(X, K)$$

$$\mathrm{Frob}_{(i)} : H^i(X, K) \longrightarrow H^i(X, K) \quad \text{on PD}$$

$$\mathrm{Frob}_{(i)} \mathrm{Frob}^{(i)} = q^n$$

$$\sigma(\mathrm{Frob}^{(i)}) = \sigma(\mathrm{Frob}_{(2n-i)})$$

an $\mathrm{Frob}^{(i)} = \dfrac{q^n}{\mathrm{Frob}_{(i)}}$

so if $\sigma = \{\alpha_{i1}, \alpha_{i2}, \dots\}$

then $= \left\{ \dfrac{q^n}{\alpha_{i1}}, \dfrac{q^n}{\alpha_{i2}}, \dots \right\}$

Then use naturality

General idea is to find a Weil cohom theory, esp one w/a comparison to singular cohom

≈ '64 Grothendieck proved all but RH in $H^*(X, \mathbb{Q}_\ell)$

≈ '74 Deligne Weil 1

Lefschetz pencils

"It was as if, in order to get from one point to another, Deligne shot an arrow across the valley and made a high wire and then crossed it" — Lahiri New Yorker

"But he solved it the wrong way" — Artin

Grothendieck dreams are of being "cut deeply in many places"

Betrayal by Deligne

believed deeply in cohomological methods for RH, motivic

Étale

étale $\approx$ covering

Zariski topology is bad

Thm (Grothendieck) $H^i_{sing}(X, A) = 0 \quad \forall i > 0$

Def. $f : X \longrightarrow Y$ étale if df iso @ all pts

Fact. f étale for $X, Y / \mathbb{C} \iff X(\mathbb{C}) \longrightarrow Y(\mathbb{C})$ locally biholo

inverse fn thm

Not a fact. $X \xrightarrow{f} Y$ locally iso in Zariski sense if étale

$$\mathbb{A}^1 - 0 \longrightarrow \mathbb{A}^1 - 0 \qquad \text{char } k \neq 2$$
$$x \longmapsto x^2$$

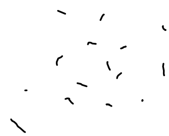
Fact. connected covers of Spec k are Spec K for K/k f.s.

So we need a finer notion than Zariski

but not a topology

e.g. in Weil, what topology

goes on



Recall presheaves are $\underbrace{\tau(X)^{op}}_{\text{why just this category?}} \longrightarrow \mathcal{A}$

$$\mathcal{C}^{op} \longrightarrow \mathcal{A}$$

need also a notion of coverings for sheaves

Def. Grothendieck top / site on $\mathcal{C}$

data: $\forall U$ a set of "coverings" $\{U_i \to U\}$

s.t.

i) $\{U \xrightarrow{id} U\}$ a cover $(U = U)$

ii) $\{U_i \to U\}$, $V \to U \rightsquigarrow \{U_i \times_U V \to V\}$

$\left(\begin{array}{l} V \subseteq U, \ U_i \times_U V \\ \cup U_i \cap V = V \end{array} \right)$

iii) $\{U_i \rightarrow U\}$ $\{U_{ij} \rightarrow U_i\}$ $\rightsquigarrow$ $\{U_{ij} \rightarrow U\}$

$F : \mathcal{C}^{op} \longrightarrow \mathcal{A}$ is a sheaf if

$$F(U) \longrightarrow \prod F(U_i) \rightrightarrows \prod \left(F(U_i \times_U U_j) \right)$$

is an equalizer $\forall \{U_i \rightarrow U\}$,

example

topological site

(little) étale site Et/X

obj = $U \rightarrow X$ étale

mor = mor $/X$

cover = surjective families

(viii) étale site

obj, mor = Sch/X

cov = surj. étale covers

Fact. – $Sh(X_{et})$ has enough injectives

– $F \xmapsto{\Gamma} F(X)$ left exact

Def. $H^i_{et} = R^i\Gamma$

Let $\mathbb{Z}/\ell^k$ = constant sheaf

(!)

$H^i(X_{et}, \mathbb{Z}_\ell) := \varprojlim (X, \mathbb{Z}/\ell^n)$

$\underbrace{H^i(X_{et}, \mathbb{Q}_\ell)}_{\text{a Weil cohom theory}} = H^i(X_{et}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

e.g. $X = \operatorname{Spec} k$

$G = G(k^{sep}/k)$

$\bar{x} = \operatorname{Spec} k^{sep}$

$$\begin{aligned} Sh(X_{\acute{e}t}) &\longrightarrow \operatorname{Disc} G\text{-Mod} \\ F &\longmapsto F_{\bar{x}} \quad \text{stalk} \\ &\qquad \| \\ &\qquad \operatorname*{colim}_{k^{sep}/K/k} F(K) \end{aligned}$$

an equivalence

étale cohom / Spec k $\cong$ Galois cohom of k