Recall, for a smooth proj X over an infinite field k

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\text{Lefschetz fibration} \rightsquigarrow f"'] & X \\
D \cong \mathbb{P}^1 &
\end{tikzcd}$$

The fibers of f are the hyperplane sections of X parameterized by D.

- Rf_* parameterizes cohomology of these fibers by D
- classical algeo tells us that a lot of topology of X is visible in its hyperplane sections

Hence, we love Rf_*.

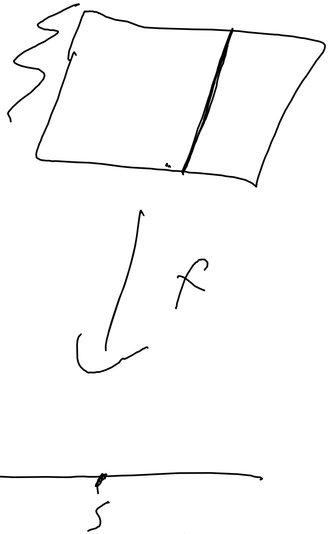
Last time.

Let $A \subseteq D$ be the (finite) set of closed pts s where $\tilde{X}_s$ is singular (w/ a unique odp).

smooth points, $D - A$, are easy, so the remaining $s \in A$ are studied locally

$$D(s) = \operatorname{Spec}(\mathcal{O}_{D,s}^{\text{et}})$$

$$\begin{tikzcd}
\tilde{X}_{D(s)} \arrow[r, hook] \arrow[d, "f_{D(s)}"'] & \tilde{X} \arrow[d, "f"] \\
D(s) \arrow[r, hook] & D
\end{tikzcd}$$



resp. geom/special fibers of $\tilde{X}_{D(s)}$ are the same as $\tilde{X}$

This is cohomologically understand ala the previous talk. We globalize.

Setup

X smooth proj / $k = \bar{k}$ char p

$\dim X = n+1$, $n = 2m+1$,

Let $\Omega = k(D)_{sep}$, $\omega : \operatorname{spec} \Omega \longrightarrow D$.

$K = qf(\mathcal{O}_{D,s}^{sh})$, Then $K_{sep} = \Omega$ where

$$\begin{tikzcd} & D_{(s)} \arrow[d] \\ \operatorname{spec} \Omega \arrow[r, "\omega"'] \arrow[ur, dashed, "\exists \bar{\omega}"] & D \end{tikzcd}$$

$$\rightsquigarrow \pi_1(D_{(s)} - \{s\}, \bar{\omega}) \cong G(\Omega/K) \overset{\rho}{\longrightarrow} \underset{\underset{p \nmid r}{\longleftarrow}}{\lim} \overset{\overset{\hat{\mathbb{Z}}^{(p')}(1)}{\|}}{\mu_r(k)}$$

$$\underset{\cup}{\mathcal{O}_{D,s}^{sh}} \hookrightarrow K \hookrightarrow \Omega$$

where $\sigma(\pi^{1/r}) = \rho(\sigma)\pi^{1/r}$

Let $N_s \subseteq \pi_1(D_{(s)} - \{s\}, \bar{\omega})$ be the kernel.

Def. $\pi_1^{tame}(D-A, \omega)$ is the
$(= \pi_1^t)$

profinite quotient of $\overset{(\hat{\pi}_1)}{\pi_1}(D-A,\omega)$

by all images of N_s under

$\pi_1(D(s)-\{s\}, \bar{\omega}) \longrightarrow \pi_1(D-A, \omega)$

$$\begin{tikzcd}
\pi_1(D(s)-\{s\}, \bar{\omega}) \arrow[r] \arrow[d] \arrow[dr, "O"] & \pi_1(D-A, \omega) \arrow[d] \\
\hat{\mathbb{Z}}^{(n)}(1) \arrow[r, dashed, "\exists \rho_s"'] & \pi_1^{tame}(D-A, \omega)
\end{tikzcd}$$

Fact. π_1^t is topologically generated by the conjugates of $\mathrm{im}(\rho_s)$.

Interlude on π_1

Recall $\pi_1(S,\bar{s}) = \mathrm{Aut}(\mathrm{Fib}_{\bar{s}})^{op}$

$$\mathrm{Fib}_{\bar{s}}(X \to S) = X_{\bar{s}}$$

Let F be lcc on S. Then F is representable by $X \longrightarrow S$, whence $\pi_1(S,\bar{s})$ acts on $F_{\bar{s}}$ continuously.

Fact.

$$\left\{ \begin{array}{c} \text{l.c. sheaves} \\ \text{of } \mathbb{Q}_\ell\text{-modules} \end{array} \right\} \xleftrightarrow{\sim} \{ \text{cont. } \pi_1 \ \mathbb{Q}_\ell\text{-reps} \}$$

$$\begin{tikzcd}
F \arrow[r, mapsto] & F_{\bar{s}} \\
\mathrm{sh}(V) & V \arrow[l, mapsto] \\
\mathrm{const}\ L \arrow[r, mapsto] & \text{trivial}
\end{tikzcd}$$

idea, clear(ish) for finite $\mathbb{Z}/\ell^k$. Then limit to $\mathbb{Z}_\ell$ and tensor to $\mathbb{Q}_\ell$.

Thm 7.1 (Picard-Lefschetz).

Let $V = R^n f_*(\mathbb{Q}_\ell)_w = H^n(\tilde{X}_w, \mathbb{Q}_\ell(-))$, a π_1 representation

i. $R^\nu f_*(\mathbb{Q}_\ell)$ l.c. on $D - A$

ii. $R^\nu f_*(\mathbb{Q}_\ell)$ l.c. on D for $\nu \neq n, n+1$

(and are hence <u>constant</u> as $\pi_1(D, w) = 0$)

iii. Let $s \in A$. $\exists \delta_s \in V(m)$ "vanishing cycle"

$\exists \delta_s^* \in R^{n+1} f_*(\mathbb{Q}_\ell)_s(n-m)$ "covanishing cycle"

well defined up to conjugation/sign

s.t.

$$\lambda \longmapsto \lambda \delta_s^*$$

$$0 \to R^n f_*(\mathbb{Q}_\ell)_s \xrightarrow{sp} R^n f_*(\mathbb{Q}_\ell)_w \to \mathbb{Q}_\ell(n-m) \to R^{n+1} f_*(\mathbb{Q}_\ell)_s \xrightarrow{sp} R^{n+1} f_*(\mathbb{Q}_\ell)_w \to 0$$

$$a \longmapsto \langle a, \delta_s \rangle$$

Recall: δ_s generates ker of sp in deg n

δ_s^* generates cok of sp in deg $n+1$

iv. π_1 trivial on $n+1$, ω

$$\underbrace{\pi_1 \text{ tame on } n, \omega}$$

explicitly, let $u \in \hat{\mathbb{Z}}^{(p)}(1)$

$x \in V$

$$\partial_s(u)x = x - (-1)^m u \langle x, \delta_s \rangle \delta_s$$

"Pf". i. f is smooth and proper over $D - A$,

so $R^v f_* (\mathbb{Q}_\ell)$ is constant on $D - A$

ii. In this range, sp is an iso.

Hence, these are l.c.c. on D

iii. lol

iv. c.f. local results from before

Def. $E := \sum_{\substack{s \in A \\ \sigma \in \pi_1^t}} \mathbb{Q}_\ell(-m/\sigma(\delta_s) \subseteq V$

Fact. All δ_s are conjugate, so

$$E = 0 \iff \exists s \ \delta_s = 0$$

Have $\langle , \rangle : V \otimes V \longrightarrow \mathbb{Q}_\ell(-m)$ by

Poincaré duality.

- alternating as n odd
- π_1^t respects this pairing

Hence, $\pi_1^t \longrightarrow \mathrm{Symp}(V)$

Thm (7.5, Kazhdan–Margulis). Image is open.

Main Thm (7.6). Let $j : D - A \longrightarrow D$

Case 1. $E = 0$.

i. $R^n f_*(\mathbb{Q}_\ell)$ constant

ii. By previous exact sequence,

$$0 \to \underset{SGA}{\bigoplus} \underset{\text{skyscraper}}{\mathbb{Q}_\ell(m-n)_{\{s\}}} \to R^{n+1}f_*(\mathbb{Q}_\ell) \to \underbrace{j_*\Big(\underbrace{sh\big(\underbrace{\underbrace{R^{n+1}f_*(\mathbb{Q}_\ell)_w}_{sh(D)}}_{\pi_1\text{-rep}}\big)}_{l.c.\ sh(D-A)}\Big)}_{sh(D)} \to 0$$

$$\begin{tikzcd} \pi_1\text{-rep} \arrow[d, bend right, "sh((-)_w)"'] \\ l.c.\ sh(D-A) \end{tikzcd}$$

where $R^{n+1}f_*\mathbb{Q}_\ell \longrightarrow j_*(sh(R^{n+1}f_*(\mathbb{Q}_\ell)_w))$ is given as follows.

- both are l.c. on $D-A$, hence controlled by the stalk at ω where this is identity
- for stalks SGA, specialize

Case 2. $E \neq 0$

i. $R^{n+1} f_* (\mathbb{Q}_\ell)$ constant

ii. $R^n f_* (\mathbb{Q}_\ell) = j_* j^* (R^n f_* (\mathbb{Q}_\ell)) = j_* (\mathrm{sh}(U))$

Case 2a. $E \subseteq E^\perp$

$$0 \longrightarrow \underbrace{j_* \mathrm{sh}(E^\perp)}_{\text{constant}} \longrightarrow R^n f_* (\mathbb{Q}_\ell) \longrightarrow \underbrace{j_* (\mathrm{sh}(U/E^\perp))}_{\text{constant}} \xrightarrow[\text{SGA}]{} \bigoplus \mathbb{Q}_\ell (m-n)_{ss} \to 0$$

Case 2b. $E \not\subseteq E^\perp$

$$0 \longrightarrow j_* (\mathrm{sh}(E)) \to R^n f_* (\mathbb{Q}_\ell) \to \underbrace{j_* (\mathrm{sh}(U/E))}_{\text{constant}} \longrightarrow 0$$

$$0 \longrightarrow \underbrace{j_* (\mathrm{sh}(E \cap E^\perp))}_{\text{constant}} \to j_* (\mathrm{sh}(E)) \to \underbrace{j_* (\mathrm{sh}(E / E \cap E^\perp))}_{\text{constant}} \to 0$$

Rmk. Weil $\Rightarrow$ hard Lefschetz $\Rightarrow V = E \oplus E^\perp$, greatly simplifying case 2.

Pf. Case 1. $E = 0 \Rightarrow E^\perp = V$.

π_1^t is trivial on $E^\perp$ by Picard-Lefschetz, and that the $\mathrm{im}(\delta_s)$ control π_1^t

Hence, $V = R^n f_*(\mathbb{Q}_\ell)$ is constant on $D - A$, thus l.c.c. on D and $\pi_1(D, w) = 0$ so it's constant.

As for the exact sequence, do stalkwise.

At w, this is clearly iso.

At s, see the local case from last time

Case 2. $E \neq 0$.

i. π_1 is trivial on $R^{n+1} f_* (\mathbb{Q}_\ell)$ so proceed as before

ii. Picard Lefschetz shows

$$j_* j^* \left(R^n f_* (\mathbb{Q}_\ell)_s\right) = V^{\pi_1(D(s) - \{s\}, \bar{\omega})} = \delta_s^\perp = R^n f_* (\mathbb{Q}_\ell)_s$$

As for constancy, by Picard-Lefschetz we have π_1^t trivial on $E^\perp$, w/E so proceed as before.

Case 2a. Clear at ω

at s ?

FK says $j_*(sh(E))_s = E^{\pi_1(D(s) - \{s\}, \bar{\omega})} = E \cap \delta_s^\perp$

?? $0 \to E^\perp \cap \delta_s^\perp \to \delta_s^\perp \to \dfrac{\delta_s^\perp}{E^\perp \cap \delta_s^\perp} \to \mathbb{Q}_\ell(m-n) \to 0$??

Case 2b. Clear @ ω.

ⓐs

$$E \not\subseteq E^\perp \Rightarrow \mathbb{Q}_\ell(-m)\delta_s \not\subseteq E^\perp \quad \forall s$$

$$\Rightarrow E \not\subseteq \delta_s^\perp \quad \forall s$$

$R^n f_* (\mathbb{Q}_\ell)_s$ is codim 1 in V

so $R^n f_* (\mathbb{Q}_\ell)_s \longrightarrow j_* \big(\underbrace{sh(V/E)_s}_{\text{constant}}\big) = V/E$

is onto

also, $j_*(sh(E))_s \to j_*(sh(E \cap E^\perp))_s$ is given by

$$E \cap \delta_s^\perp \longrightarrow \frac{E \cap \delta_s^\perp}{E \cap E^\perp}$$

Now, say $k = k_0^{sep}$

(e.g. k_0 finite)

Say everything began $/k_0$

$$\begin{tikzcd}
\tilde{X}_0 \arrow[d, "f_0"'] & \tilde{X} \arrow[l] \arrow[d, "f"] & \\
D_0 \arrow[d] & D \arrow[l] \arrow[d] & \operatorname{Spec} \Omega \arrow[l, "\omega"'] \\
\operatorname{Spec} k_0 & \operatorname{Spec} k \arrow[l] &
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[dd] & D(s) \arrow[l] \arrow[dd] & \\
 & & \operatorname{Spec} \Omega \arrow[ul, "\bar{\omega}"'] \arrow[dl, "\bar{\omega}"] \\
D_0 & \underbrace{D_0(s)}_{\substack{\text{non-strict} \\ \text{henselization}}} \arrow[l] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(D(s)-\{s\}, \bar{\omega}) \arrow[r] \arrow[d, hook] & \pi_1(D_0(s)-\{s\}, \bar{\omega}) \arrow[r] \arrow[d, hook] & \pi_1(\operatorname{Spec} k_0, \omega) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1(D-A, \omega) \arrow[r] & \pi_1(D_0-A_0, \omega) \arrow[r] & \pi_1(\operatorname{Spec} k_0, \omega) \arrow[r] & 0
\end{tikzcd}$$

Let $G_s = \operatorname{im}\left(\pi_1(D_0(s)-\{s\}, \bar{\omega}) \to \pi_1(D_0-A_0, \omega)\right)$

$$\pi_1(D-A, \omega)\, G_s = G_s\, \pi_1(D-A, \omega) = \pi_1(D_0-A_0, \omega)$$

Also, for $\sigma \in G_s$ let $\bar{\sigma} \in G(\bar{k}/k_0)$.

Then $\sigma \gamma_s(u) \sigma^{-1} = \gamma_s(\bar{\sigma}(u))$

$V = R^n f_*(\mathbb{Q}_\ell)_{\bar{\omega}} \cong R^n f_{0*}(\mathbb{Q}_\ell)_{\omega}$ is a $\pi_1(D_0-A_0, \omega)$-module

$\Rightarrow \gamma_s(u)x = x - (-1)^m u \langle x, \sigma^{-1}\delta_s\rangle \sigma^{-1}\delta_s$

$\sigma^{-1}\delta_s = \pm \delta_s$

So $\pi_1(D_0-A_0, \omega)\,\delta_s = \pi_1(D-A, \omega)\, G_s\, \delta_s \subseteq E$

Everything above ii has a $\pi_1(D_0 - t_0, w)$ action

A finite extension of k_0 affords a trivial G_S action on δ_S

Another such extension makes

$$\mathbb{Q}_\ell(m-n) \longrightarrow R^{n+1} f_* (\mathbb{Q}_\ell)_S = R^{n+1} f_{0*} (\mathbb{Q}_\ell)_S$$

$\pi_1(\operatorname{Spec} k_0, w)$ equivariant

c.f. Construction of $\delta_S^{\#}$

Prop (7.7). A finite extension of k_0 ensures

i. $\tilde{X} \longrightarrow D$ arises from

$\tilde{X}_0 \longrightarrow D_0 \quad / k_0$

ii. singular pts of f_0 are k_0-rational

iii. Main thm is defined over D_0, $D_0 - t$.