Refs Szamuely

Milne

Conrad

SGA (lol)

SAG (lmao)

Stacks

Topology . Recall

$$\text{pted Conn Cov}/S \longleftrightarrow \text{subgroups of } \pi_1(S, \bar{s})$$

$$\begin{matrix} X \\ \downarrow p \\ S \end{matrix} \longmapsto \operatorname{im}(p_*)$$

$$\tilde{S}_{\bar{s}}/H \longleftarrow\!\!\shortmid\ H$$

$$\left(\text{or } \begin{matrix} X \\ \downarrow p \\ S \end{matrix} \longmapsto \operatorname{Aut}\left(\tilde{S}_{\bar{s}}/X\right) \right)$$

u.p. <-> conjugacy

normal <——> normal

yadda yadda. Whatever shit

Topology v2. Dieck

$$\begin{array}{c} X \\ \downarrow p \\ S \end{array} \quad \text{cover} \qquad X_{\bar{s}} = p^{-1}[\bar{s}]$$

$\bar{s}_1, \bar{s}_2 \in S$. $\gamma : \bar{s}_1 \longrightarrow \bar{s}_2$ in S

$$\Big\}$$

$X_{\bar{s}_1} \longrightarrow X_{\bar{s}_2}$ in Set

ergo $X \rightsquigarrow T_X : \Pi_1(S) \longrightarrow Set$

i.e. $Cov/_S \longrightarrow Fun(\Pi_1(S), Set)$

Thm. equivalence

what is the universal cover? $\tilde{S}_{\bar{s}} \longleftrightarrow \underbrace{\Pi_1(S)(\bar{s}, -)}_{S(\bar{s}, -)}$

Rmk. $\pi_1(S, \bar{s})^{op} = \Pi_1(S)(\bar{s}, \bar{s})$

$= Aut_{\pi_1(S)}(\bar{s})$

$= Aut(S(\bar{s}, -))$

$= Aut(\tilde{S}_{\bar{s}})$

$=$ Deck transformation

Rmk. $\mathrm{Hom}_S(\widetilde{S}_{\bar{s}}, X) = \mathrm{Nat}(S(\bar{s}, -), T_X)$

$$= T_X(\bar{s})$$

Def. $\mathrm{Fib}_{\bar{s}} : \mathrm{Cov}/S \longrightarrow \mathrm{Set}$

$$X \longmapsto X_{\bar{s}}$$

Above says $\mathrm{Fib}_{\bar{s}} = \mathrm{Hom}_S(\widetilde{S}_{\bar{s}}, -)$

Now, S connected

$$\begin{tikzcd} \mathrm{Cov}/S \arrow[r, "\sim"] \arrow[drr, "\sim"] & \mathrm{Fun}(\Pi_1(S), \mathrm{Set}) \arrow[r, "\sim"] & \mathrm{Fun}(\pi_1(S, \bar{s}), \mathrm{Set}) \arrow[d, equal] \\ & & \pi_1(S, \bar{s}) - \mathrm{Set} \end{tikzcd}$$

conn $\longleftrightarrow$ transitive

(= coset space up to base pt choice)

Change of b.p.

$$\bar s_1 \xrightarrow{\gamma} \bar s_2 \quad \rightsquigarrow \quad \begin{tikzcd} \pi_1(S,\bar s_1) \arrow[r, "\sim"] \arrow[d, leftrightarrow, "\text{Tanaka}"] & \pi_1(S,\bar s_2) \\ \mathrm{Fib}_{\bar s_1} \arrow[r, "\sim"] & \mathrm{Fib}_{\bar s_2} \end{tikzcd}$$

htpy seq.

$$1 \to \pi_1(X,\bar x) \to \pi_1(S,\bar s) \to \pi_0(F_{\bar s},\bar x) \to \pi_0(X,\bar x) \to \pi_0(S,\bar s)$$

Cor. $X \to S$ a right princ. G-cover, connected

$$1 \to \pi_1(X,\bar x) \to \pi_1(S,\bar s) \to G \to 1$$

Étale

cover $\rightsquigarrow$ Fét

pt $\rightsquigarrow$ geom pt

S a scheme, $\bar{s} : \operatorname{Spec} \Omega \longrightarrow S$ a geom pt

($\operatorname{Spec} \Omega = \Omega^{sep}$)

Def. $\mathrm{Fib}_{\bar{s}} : \mathrm{F\acute{e}t}/S \longrightarrow \mathrm{Set}$

$X \longmapsto X \times_S \operatorname{Spec} \Omega$

the geom fiber over $\bar{s}$

Def. $\pi_1(S, \bar{s})^{op} = \mathrm{Aut}(\mathrm{Fib}_{\bar{s}})$

Rmk.
$$\begin{tikzcd} \mathrm{Fib}_{\bar{s}} : \mathrm{F\acute{e}t}/S \arrow[r] \arrow[dr] & \pi_1(S,\bar{s})\text{-}\mathrm{Set} \arrow[d] \\ & \mathrm{Set} \end{tikzcd}$$

Example

$S = \mathrm{Spec}(k)$

$$\begin{array}{c} X \\ \downarrow \\ S \end{array} \quad \text{f\'et flat } X = \mathrm{Spec}(A), \ A \text{ a f\'et } k\text{-alg}$$

$\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot$

each a f.s. $/k$

e.g. $X = \mathrm{Spec}\, L$, $\bar{s} \leftrightarrow k \hookrightarrow k^{sep}$

$$\mathrm{Fib}_{\bar{s}}(X) = \mathrm{Spec}\, L \otimes_k k^{sep}$$

$$= \mathrm{Hom}_k(L, k^{sep}) \qquad (p.\ \epsilon.\ \tau)$$

so k^{sep} repr $\mathrm{Fib}_{\bar{s}}$ where $\pi_1(S, \underline{\bar{s}}) = G(\overbrace{k^{sep}}^{\text{via } \bar{s}\,!}/k)$

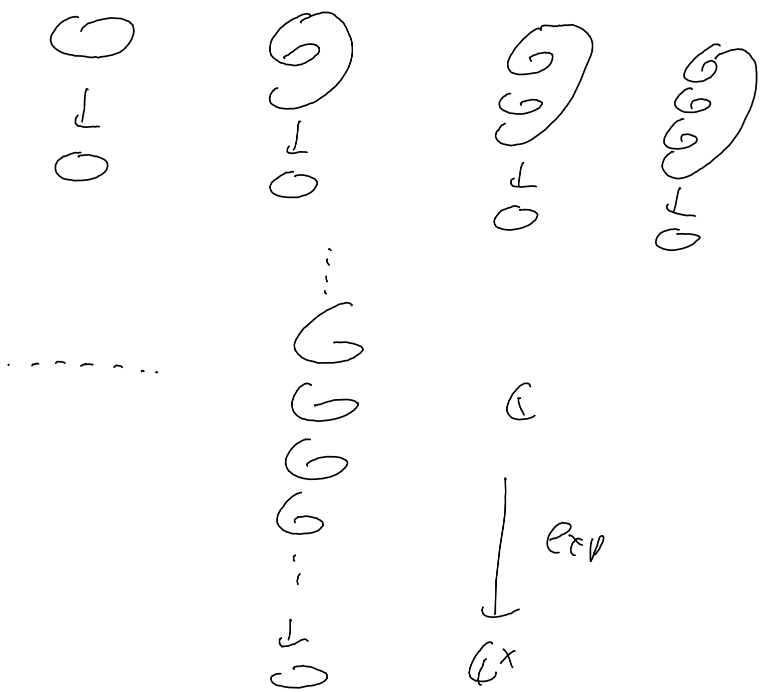
$\hookrightarrow$ choice of sep. closure!

Is π_1 repr as in the topological case?

what's w/ finiteness (étale + f.d. $\Rightarrow$ quasi-finite)

e.g., $\mathbb{A}^1 - 0 \xrightarrow{f \mapsto f^n} \mathbb{A}^1 - 0$

$(\mathbb{C}^\times \longrightarrow \mathbb{C}^\times)$



WOOPS. So we cannot do this. But we're algebraists, so we assemble all our objects into one mega object.

Preserve all these finite parts together

Pro Representability

So a repr object of $Fib_{\bar{s}}$ needn't exist

what do we do?

what did we do when limits of cauchy sequences didn't converge?

Def. $F : \mathcal{C} \to Set$ is pro-represented by a directed system $\tilde{S}$ if $\operatorname{colim}_{X \in \tilde{S}} Hom(X, Y) = F(Y)$.

e.g. Recall for $Fib_{\bar{s}} : Fet/S \longrightarrow Set$, $S = Spec\, k$, that we showed $Fib_{\bar{s}}(Spec\, A) = Hom_k(A, k^{sep})$

so $Fib_{\bar{s}}$ repr by k^{sep}, not in Fet/S.

But A f.et $\therefore$ $Hom_k(A, k^{sep}) = \bigcup_{\substack{k \subset L/k \\ \text{Galois}}} Hom_k(A, L)$

so $Fib_{\bar{s}}$ is pro-repr by $\tilde{S}_{\bar{s}} = (L/k \text{ fin, Galois})$.

Mimic this in general, so need a notion of Galois

Lemma (rigidity).

$$\begin{tikzcd} & X \arrow[d, "\text{fét}"] \\ Y \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] \arrow[r] & S \end{tikzcd}$$

$f(\bar{y}) = g(\bar{y})$, Y connected

$\Downarrow$

$f = g$

Pf. $X \to X \times_S X$ open, closed so cuts out a clopen subset

$Y \xrightarrow{(f,g)} X \times_S X$ touches Δ so $\subseteq \Delta$.

Cor. $\mathrm{Aut}(X/S) \circlearrowright \mathrm{Fib}_{\bar{s}}(X)$ faithfully for $X \xrightarrow{\text{fét}} S$ connected

Cor. $|\mathrm{Aut}(X/S)| \leq \deg(X/S)$

Def. Galois if $=$

Rmk. Hence, for $H \leq \mathrm{Aut}(X/S)$, can form X/H.

$$\begin{tikzcd} X \arrow[d, "\text{fét}"] \\ S \end{tikzcd} \implies \begin{tikzcd} X \arrow[d] \arrow[r, "\text{fét}"] & X/H \arrow[dl, "\text{fét}"] \\ S & \end{tikzcd}$$

Thus, $\mathrm{Fib}_{\bar{s}}$ is pro-repr by the directed system $\tilde{S}$

defined as follows,

first, $\forall\, X \longrightarrow S$ fét Galois,

let $\bar{x} \in \mathrm{Fib}_{\bar{s}}(X)$.

Say $Y \xrightarrow{\varphi} X$ of Galois cov, where $X \le Y$

$\exists!\ \sigma \in \mathrm{Aut}(Y/S)$ s.t. $\varphi \circ \sigma(\bar{y}) = \bar{x}$

Let $\varphi_{X,Y} = \varphi \circ \sigma$

so $\varphi_{X,X} : X \longrightarrow X$ and $\mathrm{Fib}_{\bar{s}}(\varphi_{X,X})(\bar{y}) = \bar{x}$.

why directed?

Lemma, $X \longrightarrow S$ fét connected, $\exists\, \pi : Z \longrightarrow X$ s.t.

$Z \longrightarrow S$ Galois and universal, i.e.

$$\begin{tikzcd}
Z \arrow[r] & X \arrow[r] & S \\
Z' \arrow[u, dotted, "\exists!"] \arrow[ur] \arrow[urr, "\text{Galois}"'] & &
\end{tikzcd}$$

Galois as permute comp orbits

π := some projection

Pf. Let $\bar{x}_1, \dots, \bar{x}_n$ be geom pts of $X_{\bar{s}}$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$. Z = conn comp of $\bar{x}$ in X^n/S.

Hence,

$$\begin{tikzcd} X & & Y \\ & X \times_S Y \arrow[ul] \arrow[ur] & \\ & C \arrow[u, "\text{conn comp.}"'] & \\ & Z \arrow[u, "\text{Galois cover}"'] & \end{tikzcd}$$

$\therefore$ directed

Prof thm. $Y \in \mathrm{FEt}/S$, $X \to S$ Galois

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Fib}_{\bar{\xi}}(Y)$$

$$\varphi \longmapsto \mathrm{Fib}_{\bar{\xi}}(\varphi)(\bar{x})$$

$\rightsquigarrow$ claim $\varinjlim_{X \in \xi} \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Fib}_{\bar{\xi}}(Y)$

Inverse. WLOG Y connected, $Z \xrightarrow{\pi} Y$ a Galois closure.

Let $\bar{y} \in \mathrm{Fib}_{\bar{\xi}}(Y)$. Then $\exists!\, \sigma \in \mathrm{Aut}(Z/S)$ s.t.

$\mathrm{Fib}_{\bar{\xi}}(\pi \circ \sigma)(\bar{z}) = \bar{y}$. Send $\bar{y} \longmapsto \pi \circ \sigma$.

$$\begin{tikzcd} Z \arrow[r, "\pi"] & Y & \bar{y} \\ Z \arrow[u, "\sigma"] & \bar{z} \arrow[ur, bend right, mapsto] & \end{tikzcd}$$

Now we return to π_1.

want this direct system to yield one for π_1's

Computation

Lemma. $X \xrightarrow{\pi} Y$ Galois $/S$.

$\exists$ $\mathrm{Aut}(X/S) \longrightarrow \mathrm{Aut}(Y/S)$, surjective

Pf. $\sigma \in \mathrm{Aut}(X/S)$. Say $\bar{x} \longmapsto \bar{s}$

$$\begin{tikzcd} \bar{x} \arrow[rrr, mapsto] \arrow[ddd] & & & \sigma(\bar{x}) \arrow[ddd] \\ & X \arrow[r, "\sigma"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] & \\ & Y \arrow[r, dashed, "\exists!\,\tau?"] & Y & \\ \pi(\bar{x}) \arrow[rrr, mapsto, "\exists!\,\tau"'] & & & \pi(\sigma(\bar{x})) \end{tikzcd}$$

$\tau \circ \pi = \pi \circ \sigma$ by rigidity $\square$

Thm. $\pi_1(S,\bar{s})^{op} = \varprojlim_{\tilde{x}\in\tilde{S}} \mathrm{Aut}(X/S)$

Pf. $\mathrm{Aut}(\mathrm{fib}_{\bar{s}}) = \mathrm{Aut}(\tilde{S})$

$$= \varprojlim_{\tilde{x}\in\tilde{S}} \mathrm{Aut}(X/S).$$

Thm. (étale Galois correspondence)

- $\pi_1(S,\bar{s})$ profinite.

- $\mathrm{fib}_{\bar{s}}: \mathrm{Fét}/S \xrightarrow{\sim} \mathrm{Fin\,Disc}\ \pi_1(S,\bar{s})\text{-Sets}$

connected $\longleftrightarrow$ transitive

Galois $\longleftrightarrow$ finite quotient of π_1

Pf. - ✓

- Let $E \in \mathrm{Fin\,Disc}$. WLOG transitive. $U = \mathrm{Stab}(*) \trianglelefteq^{o} \pi_1$.
$V \subseteq U$ basic, so $\pi_1/V = \mathrm{Aut}(X/S)$.
$\mathrm{fib}_{\bar{s}}(X/(U/V)) = E.$

Rmk. Galois theory but allow disconnectedness + no pointless assumption.

Computation. S integral normal

$K = K(S)$

$\bar{s}$: Spec $K^{sep} \longrightarrow$ Spec K

K_S = composition of $K^{sep}/L/K$ s.t.

- L/K fin gal
- normalization of $S_L \longrightarrow S$ étale

then K_S/K Galois and

$G(K_S/K) \cong \pi_1(S, \bar{s})$

Application

$U \subseteq \operatorname{Spec}(\mathcal{O}_K)$ open dense

$\parallel$

$\operatorname{Spec}(\mathcal{O}_K[1/f])$.

$U_L = \operatorname{Spec}((\mathcal{O}_K \otimes L)[1/f]$

$\mathcal{O}_K \otimes L = \Pi$ orders in $\mathcal{O}_L$

$\therefore$ normalization $= \coprod \operatorname{Spec}(\mathcal{O}_L[1/f])$.

$U_v \longrightarrow U$ if v c/K unram outside S

K_S = max'l unramified field ext

$\pi_1(U, \bar{s}) \simeq G(K_S/K)$.

$$\boxed{\text{Further properties}}$$

$S/\mathbb{C}$ f.t., connected

$$\text{Fét}/S \xrightarrow{\sim} \text{Fin Cov}/S^{an}$$

$\bar{s} \in S(\mathbb{C})$

$$\widehat{\pi_1(S^{an}, \bar{s})} = \pi_1(X, \bar{x})$$

Homotopy Seq

$X \longrightarrow S$ flat proper finite presentation

geom fibers conn + red

$\bar{s}$ geom pt

$$\pi_1(X_{\bar{s}}) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

In particular, X qc geom integral $/k$

$\bar{k}/k^{sep}/k$, $X^{sep} = X \times_k k^{sep}$

$\bar{x} \in X^{sep}(\bar{k})$.

$$1 \to \pi_1(X^{sep}, \bar{x}) \to \pi_1(X, \bar{x}) \longrightarrow G(k^{sep}/k) \to 1$$

Basept indep

S connected, $\bar{s}_1, \bar{s}_2$ geom pts

$\exists$ $\mathrm{fib}_{\bar{s}_1} \rightsquigarrow \mathrm{fib}_{\bar{s}_2}$ "a path"

N.B. Same pro-representing system.