Ref. FK 1.11

Milne I, 21 (only nonsingular)

1. Analytification of schemes
2. Analytification of sheaves
3. Comparing cohomology

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1. Def. A complex analytic variety is a locally ringed space $(X, \mathcal{O})$ which is locally modeled on the following

$U \subseteq \mathbb{C}^n$ open

$Y = \{f_1 = \dots = f_k = 0\}$, f_i hol.

$(U \cap Y, \mathcal{O}_U/(f_1, \dots, f_k))$

e.g. Complex mflds

affine alg varieties

$$\begin{tikzcd} \mathrm{ft}\,\mathrm{Sch}/\mathbb{C} \arrow[rr] \arrow[dr, hook] & & \mathbb{C}_{an}\text{-Var} \arrow[dl, hook] \\ & \mathrm{LRS} & \end{tikzcd}$$

$$X \longmapsto X_{an}$$

defined via $X = \bigcup U_\alpha$ affine open

and gluing via the same datum.

Facts. - This is well defined and functorial.

- X sep $\Longleftrightarrow$ X_{an} Hausdorff
- X proper $\Longleftrightarrow$ X_{an} compact

c.f. Hartshorne appendix B transcendental methods

(or GAGA)

Rmk. In GAGA coherent sheaves are easily made analytic. But here, our sheaves are étale.

2. Analytification of sheaves

Def. Let Z, U be $\mathbb{C}$an-Var.

$U \longrightarrow Z$ is étale if it's a local biholomorphism

($\Leftrightarrow$ derivative nonsingular for smth pts)

Analytification hence yields $\text{Ét}(X) \longrightarrow \text{Ét}(X_{an})$

Prop. This is fully faithful

Pf. $Y, Y' \longrightarrow X$ ét

$\mathrm{Hom}_X(Y, Y') = \mathrm{Hom}_Y(Y, Y \times_X Y')$

$\leftrightarrow$ conn comp of $Y \times_X Y'$ mapping isomorphically to Y

Fact. $X \mapsto X_{an}$ preserves connected components

This is H^0 of our coming comparison thm.

For X smth curve, use RR. Then induct on dimension

See Milne (Riemann Existence Theorem, Thm 21.3)

$$\Leftrightarrow \pi_0\left(Y \times_X Y'\right) = \pi_0\left(\left(Y \times_Y Y'\right)_{an}\right)$$

$$= \pi_0\left(Y_{an} \times_{X_{an}} Y'_{an}\right)$$

$$= \mathrm{Hom}_{Y_{an}}\left(X_{an}, Y_{an} \times_{X_{an}} Y'_{an}\right)$$

$$= \mathrm{Hom}_{X_{an}}\left(X_{an}, Y'_{an}\right)$$

as desired. $\square$

Now, let Z in $\mathbb{C}_{an}$-Var and $F \in Sh(Z)$ an analytic sheaf.

We view $F : \acute{E}t(Z) \longrightarrow Set$ via

$$F\left(\begin{array}{c} Y \\ \downarrow p \\ Z \end{array}\right) = \left(p^* F\right)(Y)$$

Equivalently, let $Y \longrightarrow Z$. Define $\bar{Y}(U) = \mathrm{Hom}_Z(U, Y)$.

Then $(p^*F)(Y) \xrightarrow{\sim} \mathrm{Hom}(\bar{Y}, F)$. p^*F is the pullback in the analytic topology, and p is a local biholomorphism, so sections of $(p^*F)(Y)$ are given by open covers $Y = \bigcup U_\alpha$ on which $p|_{U_\alpha}$ is a biholomorphism and sections of $F(p[U_\alpha])$, which is $\mathrm{Hom}(\overline{p[U_\alpha]}, F)$ by Yoneda.

So we freely evaluate analytic sheaves on Z at étale maps $U \to Z$.

Def. $Sh(X_{an}) \longrightarrow Sh(X_{et})$

$$F \longmapsto F_{al}$$

where $F_{al}\left(\begin{array}{c} Y \\ \downarrow \\ X \end{array}\right) = F\left(\begin{array}{c} U_{an} \\ \downarrow \\ X_{an} \end{array}\right)$

Prop/Def. al has a left adjoint an

$$Sh(X_{et}) \xrightarrow{an} Sh(X_{an})$$

$$F \longmapsto F_{an}$$

Pf. In the case of $F = \overline{Y}$ for $Y \to X$ étale,

$$Hom(\overline{Y}, F_{al}) = F_{al}(Y) = F(Y_{an})$$

$$= Hom(\overline{Y_{an}}, F)$$

so let $(\overline{Y})_{an} \underset{(def)}{=} \overline{Y_{an}}$.

Now let F be any étale sheaf

(#) Then $F = \operatorname{colim} \widetilde{Y_\alpha}$

and we define $F_{an} \overset{(\text{def})}{=} \operatorname{colim} \widetilde{Y_{\alpha,an}}$

(#) Recall for $G: C \to \text{Set}$ the category of elements $El(G) = \left\{ \begin{array}{l} (c,x) \mid c \in C,\ x \in G(c) \\ (c,x) \to (c',x') \text{ given by } \\ c \xrightarrow{u} c' \text{ s.t. } \\ G(u)(x) = x' \end{array} \right.$

$G \in \text{Psh}(\text{Set})$ then $G = \operatorname{colim}(El(G) \to C \xrightarrow{\alpha} \text{Psh}(C))$

Properties.

1. There is a map

$$\begin{array}{ccc} F(U) & \longrightarrow & F_{an}(U_{an}) \\ \| & & \| \\ \operatorname{Hom}(\hat{U}, F) & \longrightarrow & \operatorname{Hom}(\widetilde{U_{an}}, F_{an}) \\ s & \longmapsto & s_{an} \end{array}$$

via

2. an is cocont (it's a left adjoint)

3. $f_{an}^*(F_{an}) = (f^*F)_{an}$

use adjoint, $f^* \dashv f_*$ and that al commutes with f_*.

4. $x \in X(\mathbb{C})$ then $F_x = (F_{an})_x$

use commutativity w/ f^* to reduce to $X = *$, where $\hat{Et}(*) \cong \hat{Et}(*_{an})$.

Prop. F, G sheaves on X,

$$\mathrm{Hom}(F, G) \longrightarrow \mathrm{Hom}(F_{an}, G_{an}) \text{ is injective,}$$

and is bijective for G constructible.

Pf. For injectivity, let $\varphi : F \longrightarrow G$.

Then $\varphi_{an} = 0 \iff$ it is zero at all stalks
$\iff \varphi$ is zero at all stalks
$\iff \varphi = 0$

For bijectivity, let F be represented by U, where we seek to show $G(U) \cong G_{an}(U_{an})$.
Proceed by Nötherian induction, using that G is representable on a Zariski dense open.

Now, an is exact so we have

a map $D(X) \longrightarrow D(X_{an})$

étale sheaves $\qquad$ analytic sheaves

$$A^\bullet \longmapsto A_{an}^\bullet$$

Let $f: X \longrightarrow S$ in $Sch/\mathbb{C}$,

$\mathcal{F} \in Sh(X_{et})$.

we have

$$(f_* \mathcal{F})(U) = \mathcal{F}(U \times_S X) \to \mathcal{F}_{an}((U \times_S X)_{an})$$

$$\| $$

$$\mathcal{F}_{an}(U_{an} \times_{S_{an}} X_{an})$$

$$\|$$

$$(f_{an*} \mathcal{F}_{an})(U_{an})$$

$$\|$$

$$(f_{an*} \mathcal{F}_{an})_{a*}(U)$$

so $f_* \mathcal{F} \longrightarrow (f_{an*} \mathcal{F}_{an})_{a*}$

$\rightsquigarrow (f_* \mathcal{F})_{an} \longrightarrow f_{an*} \mathcal{F}_{an}$

$$\begin{tikzcd}
Sh(X_{\acute{e}t}) \arrow[r, "f_*"] \arrow[d, "au"'] & Sh(S_{\acute{e}t}) \arrow[d, "au"] \arrow[dl, phantom, "\Swarrow"] \\
Sh(X_{au}) \arrow[r, "f_{*au}"'] & Sh(S_{au})
\end{tikzcd}$$

$A^\bullet \in D_+(X)$, $\quad A^\bullet \longrightarrow I(A^\bullet)$ inj res

$A^\bullet_{au} \longrightarrow I(A^\bullet_{au})$ inj res

by exactness, $A^\bullet_{au} \longrightarrow (I(A^\bullet))_{au}$ is quasi-iso,

whence

$$\begin{tikzcd}
A^\bullet_{au} \arrow[r] \arrow[dr] & I(A^\bullet_{au}) \arrow[d] \\
& (I(A^\bullet))_{au}
\end{tikzcd}$$

so

$$\begin{tikzcd}
(Rf_*(A^\bullet))_{au} \arrow[r] \arrow[d, phantom, "\shortparallel"] & Rf_{au*}(A^\bullet_{au}) \arrow[d, phantom, "\shortparallel"] \\
(f_* I(A^\bullet))_{au} \arrow[r] & f_{au*}(I A^\bullet_{au})
\end{tikzcd}$$

$$
\begin{tikzcd}
D(X_{\acute{e}t}) \arrow[r, "Rf_*"] \arrow[d, "an"'] & D(S_{\acute{e}t}) \arrow[d, "an"] \arrow[dl, phantom, "\Swarrow"] \\
D(X_{an}) \arrow[r, "Rf_{*an}"'] & D(S_{an})
\end{tikzcd}
$$

If $S = \operatorname{Spec} \mathbb{C}$, this includes $H^i(X, F) \longrightarrow H^i(X_{an}, F_{an})$

Let's make it compactly supported.

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

By stalks, $(j_! F)_{an} = j_{an!}(F_{an})$, the extension by 0 along $\bar{X}_{an} - X_{an}$

"known" that $R^v \bar{f}_{an*}(j_{an!}(F_{an})) \cong R^v f_{an!}(F_{an})$

(Lemma 8.5 FK)

$$\begin{tikzcd} D_+(X_{et}, \mathrm{tor}) \arrow[r, "Rf_!"] \arrow[d, "an"'] & D_+(S_{et}, \mathrm{tor}) \arrow[d, "an"] \arrow[dl, Rightarrow] \\ D_+(X_{an}, \mathrm{tor}) \arrow[r, "Rf_{an!}"'] & D_+(S_{an}, \mathrm{tor}) \end{tikzcd}$$

$S = \mathrm{Spec}\ \mathbb{C}$

$$H_c^v(X, F) \longrightarrow H_c^v(X_{an}, F_{an})$$

Thm. 1. $\Rightarrow$ is an iso in the compactly supported case. ($f_!$)

2. $\Rightarrow$ is an iso in the f_* case.

"Pf." 2. using Hironaka's res of sing

1. By the reduction to curves in 8.8 to compute $R^{i}f_!$, we prove this for X smooth, irreducible, and projective and $\mathcal{F} = (\mathbb{Z}/n)_X = \mu_{n,X}$

Let $U \to X$ ét.

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{U}}^{*}(U) & \to & \mathcal{O}_{U_{an}}^{*}(U_{an}) \\ \shortparallel & & \shortparallel \\ \mathcal{O}_X^{*}(U) & & \mathcal{O}_{X_{an}}^{*}(U_{an}) \\ & & \shortparallel \\ & & (\mathcal{O}_{X_{an}}^{*})_{al}(U) \end{array}$$

so $\mathcal{O}_X^* \longrightarrow (\mathcal{O}_{X_{an}}^*)_{al}$, where $(\mathcal{O}_X^*)_{an} \longrightarrow \mathcal{O}_{X_{an}}^*$.

Thus,

$$\begin{tikzcd} H^1(X_{et}, \mathcal{O}_X^*) \arrow[r] \arrow[d] & H^1(X_{an}, \mathcal{O}_{X_{an}}^*) \\ H^1(X_{an}, (\mathcal{O}_X^*)_{an}) \arrow[ur] & \end{tikzcd}$$

This is the map $\mathrm{Pic}(X_{et}) \longrightarrow \mathrm{Pic}(X_{an})$, where $\mathrm{Pic}(X) = \mathrm{Pic}(X_{et})$,

by Čech cohomology, which is an

isomorphism by GAGA - $\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X_{an})$, $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X_{an})$ $\square$

Now apply the Kummer sequence.