# Cohomology of Curves

Goal. Let $X$ be a curve. We compute

$$H^\bullet(X, \mathcal{O}_X^\times) \quad \text{and} \quad H^\bullet(X, \mu_{n,X})$$

under sufficient hypotheses.

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## Higher direct images

Recall $H^i_{et}(X, -) = R^i\Gamma(X, -)$

More generally, let $f: X \longrightarrow Y$ in Sch.

$$\mathrm{Sh}(X_{et}) \longrightarrow \mathrm{Sh}(Y_{et})$$

$$F \longmapsto f_* F$$

$$\begin{array}{c} U \\ \downarrow \\ Y \end{array} \longmapsto F\left(\begin{array}{c} X \times_Y U \\ \downarrow \\ X \end{array}\right)$$

$$\begin{array}{ccc} X \times_Y U & \longrightarrow & U \\ {\scriptstyle et}\downarrow & \Longleftarrow & \downarrow{\scriptstyle et} \\ X & \longrightarrow & Y \end{array}$$

Facts, - $\exists$ left adjoint $f^{-1}$
- $f^{-1}$ is exact

Thus, $f_*$ (injective) = injective

Def. $R^i f_*$ are the right derived functors of

$$f_* : Sh(X_{et}) \longrightarrow Sh(Y_{et})$$

e.g. $X \xrightarrow{f} * = Spec(\bar{k})$

$$f_* : Sh(X_{et}) \longrightarrow Sh(*_{et}) \cong Ab$$

$$F \longmapsto F(X)$$

$f_* = \Gamma$, so $\quad R^i f_* = H^i(X_{et}, -)$

How to compute this?

Prop. $R^i f_* (F)$ is the sheafification of

$$\begin{matrix} U \\ \downarrow \\ Y \end{matrix} \longmapsto H^i(X \times_Y U, F)$$

convention, omit explicit pullbacks.

Pf. let $\varphi(U) = X \times_Y U$

$$\varphi : Y_{ét} \longrightarrow X_{ét}$$

Let $0 \longrightarrow F \longrightarrow I^\bullet$ an injective resolution

$$R^i f_*(F) = H^i(f_* I^\bullet) \quad \text{in } Sh(Y_{ét})$$

$$= shf\left( U \longmapsto \underbrace{\frac{\ker\left(f_* I^n(U) \longrightarrow f_* I^{n+1}(U)\right)}{im\left(f_* I^{n-1}(U) \longrightarrow f_* I^n(U)\right)}} \right)$$

$$\| $$

$$\frac{\ker\left(I^n(\varphi(U)) \longrightarrow I^{n+1}(\varphi(U))\right)}{im\left(I^{n-1}(\varphi(U)) \longrightarrow I^n(\varphi(U))\right)}$$

$$\|$$

$$H^i(\varphi(U), F)$$

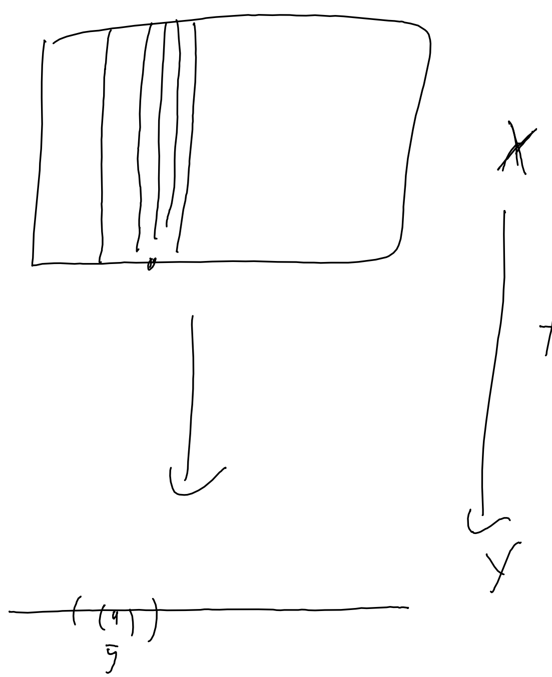
$$\|$$

$$H^i(X \times_Y U, F)$$

$\square$

Cor. $\bar{y}$ a geom pt of $Y$ then

$$(R^i f_* F)_{\bar{y}} = \operatorname*{colim}_{\bar{y} \xrightarrow{et} U} H^i(X \times_Y U, F)$$

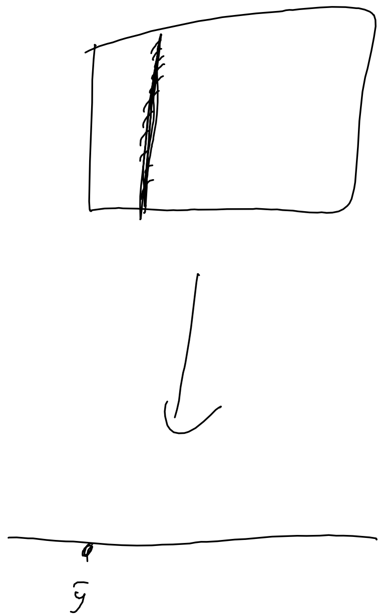



$Y$ parameterizes fibers of $f$

$R^i f_*$, as a sheaf/$Y$, roughly parameterizes the cohomology of these fibers

Indeed, if $F$ qcqs then

$$(R^i f_* F)_{\bar{y}} \cong H^i(X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\bar{y}}), F)$$



Other computational tool

Leray spectral sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$R(f\circ g) = Rf \circ Rg$ in derived category

$E_2$ 1st quadrant SS

$$R^p g_*(R^q f_* F) \Longrightarrow R^{p+q}(g\circ f)_* F$$

# Back to curves

$k = \bar{k}$, $\underline{\text{not}}$ $k^{sep}$.

Recall $\mathcal{O}_X^\times \begin{pmatrix} Y \\ \downarrow \\ X \end{pmatrix} = \Gamma(Y, \mathcal{O}_Y)^\times$

repr by $\operatorname{Spec} \mathbb{Z}[x, x^{-1}] \times X$

$\mu_{n,X} \begin{pmatrix} Y \\ \downarrow \\ X \end{pmatrix} = \mu_n(\Gamma(Y, \mathcal{O}_Y)^\times)$

repr by $\operatorname{Spec} \dfrac{\mathbb{Z}[x]}{(x^n - 1)} \times X$

$0 \to \mu_{n,X} \to \mathcal{O}_X^\times \xrightarrow{n} \mathcal{O}_X^\times$ exact

if $n \in k^\times$, $\mathcal{O}_X^\times \longrightarrow \mathcal{O}_X^\times$ surj.

Heretofore suppose this.

Kummer sequence $0 \to \mu_{n,X} \to \mathcal{O}_X^\times \xrightarrow{n} \mathcal{O}_X^\times \to 0$.

Now let $X$ be proper, smooth, and connected,

$H^0$

$$H^0(X, \mu_{n,X}) = \mu_n(\Gamma(X, \mathcal{O}_X^\times))$$

$$\mathcal{O}_X(X) = \mathrm{Hom}_k(X, \mathbb{A}^1_k)$$

$= k$ as $X$ proper and connected so the image must be a closed point

$$H^0(X, \mu_{n,X}) = \mu_n(k)$$

$$H^0(X, \mathcal{O}_X^\times) = k^\times$$

furthermore, $k^\times \xrightarrow{n} k^\times$ is onto as $k = \overline{k}$.

## $H^1$

Thus, in the Kummer LES, we have

$$0 \to H^1(X, \mu_{n,X}) \to H^1(X, \mathcal{O}_X^\times) \xrightarrow{n} H^1(X, \mathcal{O}_X^\times)$$

$$\underset{\substack{\text{Čech} \\ \text{cocycles}}}{H^1(X, \mathcal{O}_X^\times) =} \mathrm{Pic}_{\acute{e}t}(X) \xleftarrow{\sim} \mathrm{Pic}_{Zar}(X)$$

Thus, $H^1(X, \mu_{n,X}) = \mathrm{Pic}(X)[n]$

$$= \mathrm{Pic}^0(X)[n]$$
$$\cong (\mathbb{Z}/n)^{2g}$$

# $H^3$

Lemma. Let $X$ be a smooth curve.
Then $H^i(X, \mathcal{O}_X^\times) = 0$ for $i \geq 2$

Pf. Resolve $\mathcal{O}_X^\times$ as

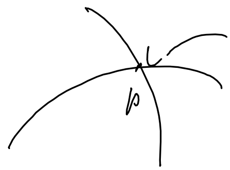
$$0 \to \mathcal{O}_X^\times \to \underbrace{\mathcal{K}_X^\times}_{\text{rat'l fns}} \to \underbrace{\mathcal{D}_X^\times}_{\text{Weil divisors}} \to 0$$

(as $X$ is smooth, Weil Div = Cartier Div $= \mathcal{K}_X^\times / \mathcal{O}_X^\times$)

We claim this is an acyclic resolution, which proves the result.

① $K_X^\times$ acyclic.

Pf. WLOG $X$ irred



local ring min'l primes $\leftrightarrow$ irred components through $p$

Let $\eta = \operatorname{Spec} K$ the generic pt of $X$

and $j : \eta \longrightarrow X$ inclusion.

claim. $R^i j_* \mathcal{O}_\eta^\times = 0 \quad i \geq 1$.

pf. $\operatorname{shf}\left( U \longmapsto H^i(\eta \times_X U, \mathcal{O}_\eta^\times) \right)$

$$\begin{tikzcd}
\eta \times_X U \arrow[r] \arrow[d, "pr"'] & U \arrow[d, "\text{ét}"] \\
\eta \arrow[r, "j"'] & X
\end{tikzcd}$$

$\therefore \eta \times_X U = \coprod_{\text{finite}} \operatorname{Spec} L$, $L/K$ fin sep

So we reduce to showing

$$H^i(\mathrm{Spec}\, L, \mathbb{G}_m) = 0 \quad \forall i \geq 1$$

$$\|$$

$$H^i(G_L, L_{sep}^\times)$$

Thm (Tsen). $K$ fn field of a curve over $\bar{k}$,

c.f. Ch II §3 Serre Galois cohomology

Then $K$ is $C1$, i.e., if $p$ is homog poly deg $N$ and deg $p < N$, $p$ has a nontrivial root in $K$

Fact. Alg exts of $C1$ are $C1$

(or $C_0$ = acf)

So claim that $K$ $C1 \Rightarrow K_{sep}^\times$ cohom trivial / $G_K$.

Indeed, $H^1(G_K, K_{sep}^\times) = 0$ by H 90,

$H^2(G_K, K_{sep}^\times) = Br(K)$.

c.f. §7 Prop 10 Serre Local Fields

Let $D$ be a finite division alg $/K$

w/ $Z(D) = K$.

reduced norm $\to N : D \longrightarrow K$, which is $0$

only iff $0$.

Fix a basis. Then $N$ homogeneous

in $[D:K]$ variables
$\underset{r^2}{\parallel}$

and has degree $r$

Thus, as $K$ is $C1$, $r^2 \le r$

$\therefore r = 1$

$\therefore Br(K) = 0$.

Hence, $K_{sep}^{\times}$ has vanishing cohomology in 2

adjacent degrees so it's cohomologically

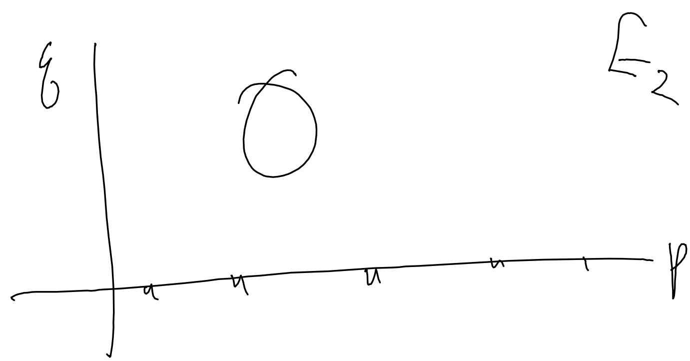
trivial

Serre Local Fields ch. IX §5 Thm 8

Hence, $R^i j_* \mathcal{O}_Y^\times = 0 \quad \forall i \geq 1$

Leray spectral sequence

$$H^p(X, R^q j_*(F)) \Rightarrow H^{p+q}(\operatorname{Spec} K, F)$$




$\therefore d = 0$ so this degenerates

Hence, $H^p(X, i_* F) = H^p(\operatorname{Spec} K, F)$

$F \in Sh(M_{et}) =$ Disc $G_K$-Mod

so $H^p(X, j_* \underbrace{\mathcal{O}_Y^\times}) = 0 \quad \forall p \geq 1$

can replace $\mathcal{O}_Y^\times$ w/ any cohom trivial discrete $G_K$-module, like the additive group or any induced module

$\therefore K_X^\times$ acyclic

(2) $D_X$ acyclic

Pf. $D_X = \bigoplus_{x \in X^1(k)} i_{x*}(\mathbb{Z}_{\operatorname{spec} k})$ as $k = \bar{k}$
so closed pt
$\cup$
$\bar{k}$ pt

$H^i$ commutes w/ $\bigoplus$

By similar LSS reasoning, it suffices to consider the cohomology of $\mathbb{Z}_{\operatorname{spec} k}$

This is acyclic as $k = \bar{k}$.

Thus, $D_X$ is acyclic

$0 \to \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow D_X \longrightarrow 0$ thus shows

that $H^i(X, \mathcal{O}_X^\times) = 0 \quad \forall i \geq 2$

Hence, $H^i(X, \mu_{n,X}) = 0 \quad \forall i \geq 3$

$\underline{H^2}$

$$\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \longrightarrow H^2(X, \mu_{n,X}) \longrightarrow 0$$

$$H^2(X, \mu_{n,X}) = \frac{\mathrm{Pic}(X)}{n\,\mathrm{Pic}(X)}$$

$$0 \to \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \rightleftarrows \mathbb{Z} \to 0$$

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "n"] \arrow[d, equal] & \mathrm{Pic}(X) \\
\mathrm{Pic}^0(X) \oplus \mathbb{Z} \arrow[r, "\begin{pmatrix} n & * \\ 0 & n \end{pmatrix}"] & \mathrm{Pic}^0(X) \oplus \mathbb{Z}
\end{tikzcd}$$

Claim. $\mathrm{Pic}^0(X) \xrightarrow{n} \mathrm{Pic}^0(X)$ surjective

Pf. Its kernel is $(\mathbb{Z}/n)^{2g}$, so finite.

$\mathrm{Pic}^0(X) = \mathrm{Jac}(X)$, an abelian variety

finite maps preserve dimension.

Thus, $\dfrac{\mathrm{Pic}(X)}{n\,\mathrm{Pic}(X)} \cong \mathbb{Z}/n$

Altogether

$$H^i(X, \mathcal{O}_X^\times) = \begin{cases} k^\times & i=0 \\ \mathrm{Pic}(X) & i=1 \\ 0 & i\geq 2 \end{cases}$$

$$H^i(X, \mu_{n,X}) = \begin{cases} \mu_n(k) & i=0 \\ \mathrm{Pic}(X)[n] \simeq (\mathbb{Z}/n)^{2g} & i=1 \\ \simeq \mathbb{Z}/n & i=2 \\ 0 & i\geq 3 \end{cases}$$

Now let $X$ be any curve

Lemma still holds so $H^i(X, \mu_{n,X}) = 0 \ \forall i \geq 3$

$$\begin{tikzcd}
 & & & 0 \\
\mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & H^2(X, \mu_{n,X}) \arrow[r] \arrow[ur] & 0 \\
 & \underbrace{\dfrac{\mathrm{Pic}(X)}{n\,\mathrm{Pic}(X)}}_{\text{finite}} \arrow[ur] & &
\end{tikzcd}$$

$\therefore H^2(X, \mu_{n,X})$ finite

Rmk. If $X$ affine then $\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)$ is onto, as we embed $X \hookrightarrow \bar{X}$ and note that divisors on $X$ extend to degree $0$ divisors on $\bar{X}$, where $\mathrm{Pic}^0(\bar{X}) \xrightarrow{n} \mathrm{Pic}^0(\bar{X})$ is onto.

As for $H^1$, $\mathcal{O}_X(X)^\times = \{f \in k(X)^\times \mid (f) \subseteq \bar{X} - X\}$

claim then that $\mathcal{O}_X(X)$ f.g./$k$, as the divisors supported in $\bar{X} - X$ are fg,

idea: multiplication

$$H^0(\bar{X}, \mathcal{O}(D)) \otimes H^0(\bar{X}, \mathcal{O}(D')) \to H^0(\bar{X}, \mathcal{O}(D+D'))$$

is onto for deg $D$ large, as RR implies linear growth?

If so, $\dfrac{\mathcal{O}_X(X)^\times}{\mathcal{O}_X(X)^{\times n}}$ and $\cancel{\dfrac{k^\times}{k^{\times n}}} = 0$ and the finitely many generators, so it's fg / $\mathbb{Z}$

Also, $\dfrac{\mathcal{O}_X(X)^\times}{\mathcal{O}_X(X)^{\times n}}$ is $n$-torsion $\therefore$ it's a fg $n$-torsion AG grp so finite

Thus, by LES, $\ker(H^1(X, \mu_{n,X}) \to \operatorname{Pic}(X))$ is finite.

Claim also that $\mathrm{Pic}(X)[n]$ is finite.

Indeed, $\ker(\mathrm{Pic}(\bar{X}) \longrightarrow \mathrm{Pic}(X))$ is finitely generated as $\bar{X} - X$ finite, and $\mathrm{Pic}(\bar{X})[n]$
$\shortparallel$
$\mathrm{Pic}^0(\bar{X})[n]$ finite.

so $\ker(H^1(X, \mu_{n,X}) \longrightarrow \mathrm{Pic}(X))$ is finite

and $\ker(\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)) = \mathrm{im}(H^1(X, \mu_{n,X}) \rightarrow \mathrm{Pic}(X))$

is finite

$\therefore H^1(X, \mu_{n,X})$ finite

Altogether. $X$ a sm curve $/ k = \bar{k}$, $n \in k^\times$,

Then $H^i(X, \mu_{n,X})$ finite $\forall i$ and $0$ $\forall i \geq 3$.

Also, $H^2(X, \mu_{n,X}) = 0$ if $X$ affine.