Φ Finishing Touches and Applications

Refs. Milne, FK
Deligne '74 Weil I

Recall

$$\begin{tikzcd} \tilde{X}_0 \arrow[r, "\pi_0"] \arrow[d, "f_0"] & X_0 \\ S_0 \subseteq D_0 & \end{tikzcd}$$

$U_0 = D_0 - S_0$

$\dim X_0 = n+1 = 2m+2$ even

Let $u \in U$. $E =$ span of vanishing cycles in $H^n(X_u, \mathbb{Q}_\ell)$.

This is an ℓ-adic $\pi_1(U, u)$ representation, and hence yields $\mathcal{E}$, a locally constant sheaf on U s.t. $\mathcal{E} \subseteq R^n f_* \mathbb{Q}_\ell$

This arises from $\mathcal{E}_0$

The missing piece to RH was rationality of the polynomial

$$\det(1 - F_x t, \; \mathcal{E}_0 / \mathcal{E}_0 \cap \mathcal{E}_0^{\perp})$$

for $x \in |U_0|$ (closed points)

Lemma 1. Let $\mathcal{F}_0$ be a constant $\mathbb{Q}_\ell$ sheaf $/U_0$ s.t. $\mathcal{F}/U$ constant.

$$\det(1 - F_x t, \mathcal{F}_0) = \prod_i (1 - \alpha_i^{\deg(x)} t)$$

for $x \in |U_0|$ and α_i ℓ-adic units in $\overline{\mathbb{Q}_\ell}$

we have

$$Z(X_x, t) = \prod_i \det(1 - F_x t, \underbrace{R^i f_* \mathbb{Q}_\ell}_{H^i(X_x, \mathbb{Q}_\ell)})^{(-1)^{i+1}}$$

At $i = n$, $\det(1 - F_x t, R^n f_* \mathbb{Q}_\ell) = \det(1 - F_x t, R^n f_* \mathbb{Q}_\ell / \mathcal{E}_0)$
$\cdot \det(1 - F_x t, \mathcal{E}_0 \cap \mathcal{E}_0^{\perp})$
$\cdot \det(1 - F_x t, \mathcal{E}_0 / \mathcal{E}_0 \cap \mathcal{E}_0^{\perp})$

By Lemma 1, $\underbrace{Z(X_x, t)}_{\mathbb{Q}(t)} = \underbrace{\dfrac{\prod (1 - \alpha_i^{\deg(x)} t)}{\prod (1 - \beta_j^{\deg(x)} t)}}_{\text{focus here}} \underbrace{\det(1 - F_x t, \mathcal{E}_0 / \mathcal{E}_0 \cap \mathcal{E}_0^{\perp})}_{?\, \mathbb{Q}(t)\, ?}$

WLOG $\alpha_i \neq \beta_j$

We want to show that $\dfrac{\prod(1-\alpha_i t)}{\prod(1-\beta_j t)}$

are rational, so we show the α_i, β_j are defined/$\mathbb{Q}$

Prop 1. Let $\{\gamma_i\}$, $\{\delta_j\}$ be disjoint finite sets in $\overline{\mathbb{Q}_\ell}^\times$

Let K be a large finite set of integers $\neq 1$

and L a large subset of $|U_0|$ w/ density 0.

If $x \in |U_0| - L$ has $k \nmid \deg(x)$ for $k \in K$

then the denominator of the reduced form of

$$\frac{\det(1-F_x t, \mathcal{E}_0/\mathcal{E}_0 \cap \mathcal{E}_0^+)\prod(1-\gamma_i^{\deg(x)} t)}{\prod(1-\delta_j^{\deg(x)} t)}$$

is $\prod(1-\delta_j^{\deg(x)} t)$.

Rmk. Per Deligne, this is en route to intrinsically characterizing the δ_j via the family of rational function $Z(X_x, t)$, $x \in |U_0|$.

The next step along these lines is

Lemma 2. Let K as before and $\{\delta_j\}_{j=1}^{q}$, $\{\varepsilon_j\}_{j=1}^{q}$ be subsets of a field.

for $n > 0$ be coprime to K,

if $\{\delta_j^n\} = \{\varepsilon_j^n\}$ then $\{\delta_j\} = \{\varepsilon_j\}$

Prop 2. Let $\{\gamma_i\}$, $\{\delta_j\}$ be finite sets in $\overline{\mathbb{Q}_\ell}^\times$.

$$R(t) = \prod (1-\gamma_i t)$$
$$S(t) = \prod (1-\delta_j t)$$

If for $x \in |U_0|$,

$$\prod (1-\delta_j^{\deg(x)} t) \Big| \prod (1-\gamma_i^{\deg(x)} t) \det(1-F_x t, \mathcal{E}_0 / \mathcal{E}_0 \cap \mathcal{E}_0^{+})$$

Then $S | R$

Pf. Prop 1. (?)

This forms our intrinsic characterization, and hence rationality (?)

Pf. of Prop. 1.

We have the extension

$$0 \to \pi_1(U, u) \to \pi_1(U_0, u) \to \hat{\mathbb{Z}} \to 0$$

Let $\rho : \pi_1(U_0, u) \to \mathrm{Symp}(\mathcal{F}_u)$, $\mathcal{F}_0 = \mathcal{E}_0 / \mathcal{E}_0 \cap \mathcal{E}_0^{\perp}$,

$\rightsquigarrow$ $\pi_1(U_0, u) \to \hat{\mathbb{Z}} \times \mathcal{F}_u$

$\searrow$ $\{ \xi^{-n} = \mu(g) \} = H_1$

μ = "symplectic multiplier"

Fact. The image in H_1 is open

Fact. For $\delta \in \overline{\mathbb{Q}}_\ell^{\times}$, the set $Z = \{ (n, g) \in H_1 \mid \delta^n \text{ an eigenvalue of } g \}$ has Haar measure 0 and is closed

Fix i, j. The $n \mid \lambda_i^n = \delta_j^n$ is the set of multiples of a fixed n_{ij}. We assume $n_{ij} \neq 1$.

By Čebotarev density, the $x \in |U_0|$ s.t. $\lambda_j^{\deg(x)}$ is an eigenvalue of F_x on $\mathcal{F}_0$ is density 0.

$K = \{ n_{ij} \}$, $L = \{ \text{such } x \}$.

Applications

Refs. - Milne, FK

- Deligne '74 Weil I
- Deligne '74 Formes Modulaires et représentations ℓ-adiques

- Serre '69 Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)

Counting points

Thm. Let $X_0 \subseteq \mathbb{P}_0^{n+r}$ be a smooth (8.1 Weil I) complete intersection $/\mathbb{F}_q$ with dimension n and multidegree $(d_1, \dots, d_r)$.

Let $b = \begin{cases} b_n(X) & n \text{ odd} \\ b_n(X) - 1 & n \text{ even} \end{cases}$

Then $\left| \# X_0(\mathbb{F}_q) - \# \mathbb{P}^n(\mathbb{F}_q) \right| \leq b\, q^{n/2}$

Rmk. $\# X_0(\mathbb{F}_q) = O(q^n)$ by Nöther normalization, so this is quite precise.

Cohomology of complete intersections

First, hypersurfaces.

$X_0 \subseteq \mathbb{P}_0^{n+1}$ a smooth hypersurface.

$U_0 = \mathbb{P}_0^{n+1} - X_0$

By Gysin,

$$H^{r+1}(U, F(1)) \to H^r(X, F) \to H^{r+2}(\mathbb{P}^{n+1}, F(1)) \to H^{r+2}(U, F(1))$$

for F a locally constant sheaf of Λ-modules

U_0 is affine, so $cd(U) \leq n+1$

Hence, for $r > n+1$ we have

$$\begin{tikzcd} H^r(X, F) \arrow[r, "\sim"] & H^r(\mathbb{P}^{n+1}, F(1)) \\ & H^r(\mathbb{P}^n, F) \arrow[u, "\sim"] \end{tikzcd}$$

(another Gysin)

By Poincaré duality, we are hence only missing $r = n$, whence Gysin yields

$$H^r(X, F) \to H(\mathbb{P}^{n+1}, F(1)) \to 0$$

Let its kernel be $H^r(X, F)'$, which Deligne calls the "primitive part"

(de la partie primitive)

Hence, $H^*(X, F) \cong H^*(\mathbb{P}^n, F) \oplus H^m(X, F)'$

as graded modules

For a general complete intersection, do this with $(\mathbb{P}_0^{n+1}, X_0)$ replaced by (X_0, Y_0) for Y_0 a smooth hyperplane section of X_0.

$X_0 - Y_0$ still affine

Thus, for $X_0 \subseteq \mathbb{P}_0^{n+r}$ as in the theorem,

$$H^*(X, F) \cong H^*(\mathbb{P}^n, F) \oplus \underbrace{H^n(X, F)'}_{\text{a Frob-invariant subspace of } H^n(X,F)}$$

Pf. of thm. By the Lefschetz fixed point theorem

$$\# X_0(\mathbb{F}_q) = \sum_{i=0}^{n} q^i + (-1)^n \sum_j \alpha_j$$

with $|\alpha_j| = q^{n/2}$ by the Riemann hypothesis.

There are at most b many α_j.

$$\begin{aligned}|\# X_0(\mathbb{F}_q) - \# \mathbb{P}_0^n(\mathbb{F}_q)| &= \left|\sum \alpha_j\right| \\ &\leq \sum |\alpha_j| \\ &\leq b\, q^{n/2}\end{aligned}$$

$\square$

Corollary. E an elliptic curve $/\mathbb{F}_q$.

$$|E(\mathbb{F}_q) - (q+1)| \le 2\sqrt{q}$$

Corollary. E/K elliptic / # field.

The L-series

$$L_{E/K}(s) = \prod_{v \text{ good}} Z(E_v/k_v, |k_v|^{-s})^{-1}$$

converges for $Re(s) > \frac{3}{2}$.

Hasse-Weil L functions

Serre '69

n-lab café 2010 July 28, Baez

wikipedia

Let X/K be a variety / # field

v unramified $\rightsquigarrow$ Euler factor $Z(X_v, q_v^{-s})$

v ramified ?

Recall $Z(X_v, T) = \prod P_i(T)^{(-1)^{i+1}}$

P_i = char poly of Frobenius on H^i, a Galois representation

For v ramified, do the same for the inertia-fixed subrepresentation of H^i.

Per Serre, this converges on $Re(s) > 1 + \frac{k}{2}$,

and has a functional equation about

$s \longleftrightarrow m+1-s$

Ramanujan-Petersson

Deligne '74 "Formes..."

Thm. $N \geq 1$, $\varepsilon : (\mathbb{Z}/N)^\times \longrightarrow \mathbb{C}^\times$, $k \geq 2$,

f a modular form / $\Gamma_0(N)$ with

weight k and character ε.

If f is also a cuspidal newform, it's a

Hecke eigenform away from the level,

Then $|a_p| \leq 2p^{\frac{k-1}{2}}$

i.e. the roots of $T^2 - a_p T + \varepsilon(p) p^{k-1}$ have

absolute value $p^{\frac{k-1}{2}}$

Eichler-Shimura realizes a_p as an eigenvalue of Frobenius on H^{k-1}.