Unramified

Geometric

Langlands

Correspondence

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§3. Geometrizing automorphy

§4. The Correspondence

Refs

- Frenkel "Lectures on the Langlands Program and Conformal Field Theory"

- Raphaël's notes

- Bump, Cogdell, Gaitsgory, de Shalit, Kowalski, Kudla "An Introduction to the Langlands Program"

(Gaitsgory's article on Geometric Langlands)

§1. Intro

Class field theory yields an isomorphism

$$\pi_0 C_K \xrightarrow{\sim} G_K^{ab}$$

where $C_K = K^\times \backslash \mathbb{A}_K^\times$ is the idèle class group

and where K is a number field

Under this correspondence,

$$\pi_0\left(F^\times \backslash \mathbb{A}_F^\times / \mathcal{O}_v^\times\right) \xrightarrow{\sim} G_K^{ab, ur\, v}$$

$$(1, \dots, 1, \pi_v, 1, \dots) \longmapsto Fr_v \quad \text{geometric Frobenius}$$

(Recall in local CFT that $\mathcal{O}_v^\times \xrightarrow{\sim} I_v$)

This isomorphism is unique!

This motivates a higher dimensional passage.

In the case of K a function field, we have

$$(*)\quad \left\{\begin{array}{c}\text{representations} \\ G_K \longrightarrow GL_n(F) \\ \text{(w/ finite order determinant)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{automorphic} \\ \text{representations of} \\ GL_n \\ \text{(with finite order central character}\end{array}\right\}$$

those appearing in

$(F = \overline{\mathbb{Q}}_\ell, \mathbb{C})$ $\qquad \mathcal{A}^0(GL_n(K)\backslash GL_n(\mathbb{A}_K) \xrightarrow{\chi} F)$

such that if

$$\rho \longleftrightarrow \pi$$

then the Frobenius eigenvalues of ρ at unramified places

equal the Hecke eigenvalues of π

(equivalently, an equality of associated L functions)

Proven for $n=2$ by Drinfeld

$n \geq 3$ by Lafforgue

Aside: Reminder on Hecke algebras

For $x \in X$ ($F = k(X)$), let

$$\mathcal{H}_x = \left\{ GL_n(F_x) \xrightarrow{f} \overline{\mathbb{Q}}_\ell \;\middle|\; \begin{array}{l} f \text{ compactly supported} \\ f \ GL_n(\mathcal{O}_x) \text{ biinvariant} \end{array} \right\}$$

Let $H_{i,x} = \chi_{M_n^i(\mathcal{O}_x)}$

$$M_n^i(\mathcal{O}_x) = GL_n(\mathcal{O}_x) \begin{pmatrix} t_x & & & & \\ & \ddots & & & \\ & & t_x & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} GL_n(\mathcal{O}_x)$$

So $\mathcal{H}_x \cong \mathbb{C}[H_{1,x}, \ldots, H_{n,x}, H_{n,x}^{\pm}]$
$\cong \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{S_n}$

$\{ z_1(\pi_x), \ldots, z_n(\pi_x) \}$ is the set of Hecke eigenvalues at x of $\pi = \bigotimes_y' \pi_y$ with π_x unramified ($\pi_x^{GL_n(\mathcal{O}_x)} \neq 0$)

For $K = k(X)$ a function field,

we are afforded geometric means,

Recall $\{\text{sm proj curves}/k\} \simeq \{\text{fn fields of tr deg } 1/k\}$)

Our goal now is to reformulate all aspects of this correspondence into geometric terms

§2. Geometrizing Galois

We work now in the unramified setting.

$\boxed{\text{Notation}}$ Let X be a smooth projective curve / k a perfect field.

Let $K = k(X)$

Let $G_K = G(K^{sep}/K)$

$G_K^{ur} = G(K^{ur}/K)$.

Let F be an algebraically closed field of char 0 ($\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$)

Recall for $\bar{x} \in X$ a geometric point, the étale fundamental group $\pi_1^{et}(X, \bar{x})$ classifying finite étale covers of X.

Thm. $\pi_1^{et}(X, \bar{x}) \cong G_K^{ur}$ as profinite groups.

Thus,

$$\left\{ \begin{array}{c} \text{Unramified} \\ \text{Galois reps} \\ G_K^{ur} \to GL_n(F) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{representations} \\ \pi_1(X, \bar{x}) \to GL_n(F) \end{array} \right\}$$

Def. A local system $\mathcal{L}$ on X of F-vector spaces is a locally constant sheaf of F-vector spaces.

All stalks $\mathcal{L}_x$ are thus isomorphic to a fixed F-vector space L.

In the cases
- $k = F = \mathbb{C}$
- $F = \overline{\mathbb{Q}_\ell}$

There is a natural monodromy action
- $\pi_1^{top}(X, x) \to Aut(L)$
- $\pi_1^{et}(X, x) \to Aut(L)$

Thm. In these cases, this map

$$\left\{ \begin{array}{c} \text{local systems of} \\ \text{rank } n \text{ on } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} n\text{-dim representations} \\ \text{of } \pi_1(X) \end{array} \right\}$$

is an equivalence of categories.

Def. Let $\mathcal{E}$ be a vector bundle on X, a complex manifold. A (Flat) connection on $\mathcal{E}$ is a map of sheaves

$$\begin{array}{ccc} TX & \longrightarrow & End(\mathcal{E}) \\ \xi & \longmapsto & \nabla_\xi \end{array}$$

s.t.

- $\nabla_\xi (f s) = f \nabla_\xi (s) + (\xi . f) s$
- $\nabla_{f\xi} = f \nabla_\xi$
- $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi, \eta]}$

Locally on X a complex curve, these are of the form

$$\frac{\partial}{\partial z} + A(z)$$

for $A: U \longrightarrow M_{n\times n}(\mathbb{C})$ holomorphic

Thm. (Riemann-Hilbert correspondence). For X a smooth complex projective variety

The functor

$$\text{Sol}: \{\text{vbs w/ flat connections}\} \longrightarrow \{\text{local systems}\}$$

$$(\mathcal{E}, \nabla) \longmapsto \text{sheaf of horizontal sections,}$$

i.e., $\nabla s = 0$

is an equivalence of categories

So in the complex setting we have

$$\left\{ \text{n-dim Galois reps} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{rk } n \text{ vector bundles with} \\ \text{a flat connection} \end{array} \right\}$$

§3. Geometrizing automorphy

The automorphic side consists of GL_n representations in $\mathcal{A}^0(GL_n(k)\backslash GL_n(\mathbb{A}_k), \mathbb{F})$

Thm. Let π be an irreducible automorphic representation. Then

$$\pi = \bigotimes_{x\in X}{}' \pi_x$$

with π_x an irrep of $GL_n(K_x)$ with all but finitely many π_x <u>unramified</u>, in the sense that $\pi_x^{GL_n(\mathcal{O}_x)} \neq 0$ (and is hence 1d). Fix $v_x \in \pi_x^{GL_n(\mathcal{O}_x)} - \{0\}$.

Suppose π is (everywhere) unramified.

Then let $v = \bigotimes_{x \in X} v_x \in \bigotimes_{x \in X}' \pi_x$, inducing

a function $f_\pi : GL_n(K) \backslash GL_n(\mathbb{A}_K) \longrightarrow F$

which is $GL_n(\mathcal{O})$- invariant $\left(\mathcal{O} = \prod \mathcal{O}_x \right)$

so f_π descends to

$$GL_n(K) \backslash GL_n(\mathbb{A}_K) / GL_n(\mathcal{O}) \longrightarrow F$$

Furthermore, $\forall x \in X$, $H_x \curvearrowright \pi_x^{GL_n(\mathcal{O}_x)}$, which

is one-dimensional, so f_π is a <u>Hecke eigenfunction</u>.

Indeed, $H_{i,x} * f_\pi = q_x^{i(n-i)/2} s_i(z_1(\pi_x), \ldots, z_n(\pi_x)) f_\pi$, s_i the i^{th} sym poly.

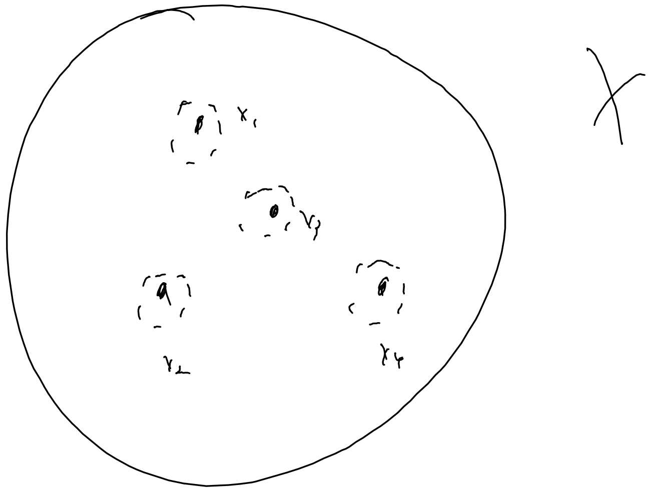
Lemma. $GL_n(K) \backslash GL_n(\mathbb{A}_K) / GL_n(\mathcal{O}) \cong \left\{ \begin{array}{l} \text{rk } n \text{ vector bundles} \\ \text{on } X \end{array} \right\} / \cong$

Pf. Let $(g_x)_{x \in X}$ represent an element of the LHS.

Let $x_1, \ldots, x_n \in X$ be s.t. $g_x \in GL_n(\mathcal{O}_x)$ for $x \neq x_i$.

Then defined a vector bundle $\mathcal{E}$ on X

which is trivial on $X - \{x_1, \ldots, x_n\}$



For each x_i, $g_{x_i} \in GL_n(K_{x_i})$ determines

a transition function on the punctured formal

disks at x_i, $\operatorname{Spec} R((t_{x_i}))$

<u>Choices</u>

- trivialization near x_i
- trivialization on $X - \{x_i\}$
- choice of $\{x_i\}$

- changing g_{x_i} by $GL_n(\mathcal{O}_{x_i})$ on the right yields an isomorphic vector bundle. (interpret transition from disk → big open)

- changing the trivialization on $X - \{x_i\}$ by $GL_n(\mathcal{O}(X - \{x_i\}))$ does, too

- take a colimit under all finite subsets. ☐

Thus, unramified automorphic reps of GL_n

$\updownarrow$

functions on $\{$rk n vbs on $X\}/\cong$

Let $k = \mathbb{F}_q$.

Let Bun_n be the moduli stack of vector bundles of rank n on X.

By Grothendieck's sheaf-function dictionary, we can geometrize functions on $\mathrm{Bun}_n(\mathbb{F}_q) = \{\text{rk } n \text{ vbs on } X\}/\cong$ via perverse sheaves on Bun_n.

Recall, for $K^\bullet \in D^b_c(V, \overline{\mathbb{Q}_\ell})$, V variety $/\mathbb{F}_q$, the function

$$f_{K^\bullet} : V(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}_\ell}$$
$$x \longmapsto \sum_i (-1)^i \operatorname{Tr}(\mathrm{Fr}_{\bar{x}} \circlearrowright H^i_{\bar{x}}(K^\bullet))$$

$$\mathrm{Perv}(V) \longrightarrow \mathrm{Fun}(V(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) \text{ sends}$$

$\otimes \longmapsto$ mult

$\varphi^* \longmapsto$ pullback

$\varphi_! \longmapsto$ integration along fibers

Summary

{unramified automorphic reps of GL_n}

$$\updownarrow$$

{functions on $GL_n(k)\backslash GL_n(\mathbb{A}_k)/GL_n(\mathcal{O})$}

$$\|$$

{fns on $Bun_n(\mathbb{F}_q)$}

$$\updownarrow \ /\mathbb{F}_q$$

$$\boxed{Perv(Bun_n)}$$

$$\sim\updownarrow \ /\mathbb{C}$$

$\mathcal{D}$-modules

Geometrizing Hecke

If $\mathrm{Perv}(\mathrm{Bun}_n)$ is our replacement for automorphic reps, we need a Hecke action on $\mathrm{Perv}(\mathrm{Bun}_n)$.

Let $k = \mathbb{F}_q, \mathbb{C}$

Def. $\underline{\mathrm{Hecke}}_i$ is the moduli stack of

$$(\mathcal{M}, \mathcal{M}', x, \mathcal{M}' \overset{\beta}{\hookrightarrow} \mathcal{M})$$

with

- $\mathcal{M}', \mathcal{M} \in \mathrm{Bun}_n$
- $x \in X$
- $\beta : \mathcal{M}' \hookrightarrow \mathcal{M}$ s.t.

  $\mathrm{coker}(\beta) \cong \mathcal{O}_x^{\oplus i}$, a skyscraper sheaf

Consider

$$\begin{tikzcd} & \underline{\mathrm{Hecke}}_i \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "\mathrm{supp} \times h^{\rightarrow}"] & \\ \mathrm{Bun}_n & & X \times \mathrm{Bun}_n \end{tikzcd}$$

where

$$(\mathcal{M}, \mathcal{M}', x, \beta) \begin{array}{l} \xmapsto{h^{\leftarrow}} \mathcal{M} \\ \xmapsto{h^{\rightarrow}} \mathcal{M}' \\ \xmapsto{\mathrm{supp}} x \end{array}$$

Let $\underline{\mathrm{Hecke}}_{i,x} = \mathrm{supp}^{-1}(x)$, yielding

$$\begin{tikzcd} & \underline{\mathrm{Hecke}}_{i,x} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

For $k = \mathbb{F}_q$, apply the sheaf-fn dictionary

and consider the operator

$$T_{i,x}(f) = h_*^{\rightarrow}\left(h^{\leftarrow *}(f)\right)$$

acting on functions on $\mathrm{Bun}_n(\mathbb{F}_q) = GL_n(K) \backslash GL_n(\mathbb{A}_K) / GL_n(\mathcal{O})$

$h_*^{\rightarrow}$ is integration along the fiber

Fix $(g_y)_{y \in X} \in GL_n(K) \backslash GL_n(\mathbb{A}_K) / GL_n(\mathcal{O})$

The fiber under $(g_y)_{y \in X}$ under $h^{\rightarrow}$

corresponds to all $M' \subseteq M$ w/ $M/M' \cong \mathcal{O}_x^{\oplus i}$

so $M'_y = M_y \ \forall y \neq x$

Thus if we represent M' by $(h_y)_{y \in X}$, we have

$h_y = g_y \ \forall y \neq x$.

At x, we are choosing an i-plane in $M_x^\vee$ dual to M'

so $h_x = g_x \alpha_x$, $\alpha_\epsilon \in GL_n(\mathcal{O}_x) \begin{pmatrix} t_x & & & & \\ & \ddots & & & \\ & & t_x & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} GL_n(\mathcal{O}_x) = M_n^i(\mathcal{O}_x)$

$H_{i,x} * f$ is the function

$$g \longmapsto \int H_{i,x}(z)\,(z \cdot f)\,(g)\,dz$$

$\in$
$(g_a)_{a \in X}$

$$= \int_{M_{n^i}(x)} f(\underbrace{gz}_{\text{ranges over } h^{\to} \text{ of the fiber over } g})\,dz$$

$$= T_{i,x}(f)$$

so $T_{i,x}(f) = H_{i,x} * f$, justifying the usage of "Hecke".

Def. $\operatorname{Perv}(\operatorname{Bun}_n) \xrightarrow{H_i} \operatorname{Perv}(X \times \operatorname{Bun}_n)$

$$K^\bullet \longmapsto R(\operatorname{supp} \times h^{\to})_* \left(h^{\leftarrow *}(K^\bullet)\right)$$

Def. Let E be a local system of rank n on X.
$K^\bullet \in \operatorname{Perv}(\operatorname{Bun}_n)$ is a Hecke eigensheaf w/ eigenvalue E if
$\exists$ isos $H_i(K^\bullet) \xrightarrow{\sim} \Lambda^i E \boxtimes K^\bullet[-i(n-i)]$

For an irreducible cuspidal unramified automorphic rep π, we had eigenvalues

$$H_{i,x} * f_\pi = S_i(z_1(\pi_x), \ldots, z_n(\pi_x)) \; q^{i(n-i)/2} \; f_\pi$$

? $\updownarrow$ $\qquad$ $\updownarrow$ Weil

$$H_i(K^\circ) = \Lambda^i E \boxtimes K^\circ[-i(n-i)]$$

? Suppose π comes from a Galois rep ρ

$$\mathrm{Tr}(\mathrm{Fr}_{\bar{x}} \circlearrowright E_{\bar{x}}) = \sum_{i=1}^{n} z_i(\rho_x)$$

$$\mathrm{Tr}(\mathrm{Fr}_{\bar{x}} \circlearrowright \Lambda^i E_{\bar{x}}) = S_i(z_1(\rho_x), \ldots, z_n(\rho_x))$$

as both sides are the i^{th} coeff of the char poly of $\mathrm{Fr}_{\bar{x}} \circlearrowright \Lambda^i E_{\bar{x}}$

Rmk. H_i is defined globally not pointwise.

§4. The Correspondence

Unramified Galois reps $\quad\quad$ Unramified automorphic reps

$\updownarrow ? \quad\quad\quad\quad \updownarrow ?$

Local systems on X $\quad\quad$ $\mathrm{Perv}(\mathrm{Bun}_n)$

Thm (Drinfeld, Laumon). Let $k = \mathbb{F}_q$ or $\mathbb{C}$.

Let E be an irreducible rank n local system on X.

$\exists$ a perverse sheaf Aut_E on Bun_n which is a Hecke eigensheaf of eigenvalue E.

Aut_E is irreducible on the connected components Bun_n^d of Bun_n.

Deligne proved this for GL_1

For $n=1$, $Bun_1 = Pic$

Let E be a local system of rank 1 on X.

We seek a perverse sheaf Aut_E on Pic

so that $h^{\leftarrow *}(Aut_E) \cong E \boxtimes Aut\, E$

(note: $h^{\rightarrow}$ is 1-1 for $n=1$)

here, $h^{\leftarrow *}: X \times Pic \longrightarrow Pic$

$(x, L) \longmapsto L(x)$

Let $S^d X = X^d / S_d$

Let $\pi_d: S^d X \longrightarrow Pic^d$

$D \longmapsto \mathcal{O}(D)$

For $d > 2g-2$, this is a projective bundle with fibers $\mathbb{P} H^0(X, L)$

Let $\mathrm{Sym}^d : X^d \longrightarrow S^d X$

Let $E^{(d)} = \left(\mathrm{Sym}^d_* E^{\boxtimes n}\right)^{S_d}$

Let $\tilde{h}^{\leftarrow} : S^d X \times X \longrightarrow S^{d+1} X$

$(D, x) \longmapsto D + x$

Then $\tilde{h}^{\leftarrow *}(E^{(d+1)}) \cong E \boxtimes E^{(d)}$

much like our Hecke eigensheaf condition.

Descend from $S^d X$ to $\mathrm{Pic}^d X$.

As π_d is a projective bundle, its fibers are simply connected, so $E^{(d)}$ is a constant local system along fibers.

So it descends to Pic^d, call it Aut^d_E

The Hecke eigensheaf property then lets us induct down to $d \leq 2g-1$.

Rmks. 1) $k = \mathbb{C}$.

We can define Aut^0_E via:

E is given by a map

$$\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[d] & \mathbb{C}^\times \\
\pi_1(X)^{ab} \arrow[ur] & \\
H_1(X;\mathbb{Z}) \arrow[u, phantom, "\cong" rotate=90] & \\
H^1(X;\mathbb{Z}) \arrow[u, phantom, "\cong" rotate=90] & \\
\pi_1(\mathrm{Pic}^0(X)) \arrow[u, phantom, "\cong" rotate=90] \arrow[uuuur] &
\end{tikzcd}$$

The map $\pi_1(\mathrm{Pic}^0(X)) \longrightarrow \mathbb{C}^\times$ determines system Aut^0_E.

More generally,
$$S^d X \longrightarrow \mathrm{Pic}^0 X$$
$$D \longmapsto \mathcal{O}(D - dx_0)$$

induces an iso $H_1(S^d X;\mathbb{Z}) \xrightarrow{\sim} H_1(\mathrm{Pic}^0(X);\mathbb{Z})$

$\therefore E^{(d)}$ constant along fibers of π_d.

2) $k = \mathbb{F}_q$

Apply sheaf - fn dictionary

$E \longleftrightarrow \rho$ ℓ-adic Galois rep

$\wr$

$\mathrm{Aut}_E \longleftrightarrow$ function f_ρ on $F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times$

$$\overset{\shortparallel}{(\mathrm{Pic}\, X)(\mathbb{F}_q)}$$

satisfying $f_\rho(L(x)) = \rho(\mathrm{Fr}_{\bar{x}}) f_\rho(L)$

We can construct f_ρ ala our construction of Aut_E.

Let $f_\rho'\left(\sum n_i x_i\right) = \prod \rho(\mathrm{Fr}_{\bar{x}_i})^{n_i}$

f_ρ' is defined on $(S^d X)(\mathbb{F}_q)$, and we seek to descend to $(\mathrm{Pic}^d X)(\mathbb{F}_q)$

Equivalently to show that if $\sum n_i x_i = (\varphi)$, some $\varphi \in k(X)$, then $\prod \rho(\mathrm{Fr}_{x_i})^{n_i} = 1$. That is, the Artin map is trivial on principal divisors!