Tropical Varieties

We begin a translation from classical algebraic geometry to tropical geometry.

Take K a.c.f.

- "Tropicalize coefficients", i.e. map

$$K \xrightarrow{v} \mathbb{R} \cup \{\infty\}$$

Insist that
- v is a valuation
- v is nontrivial

Then $\mathrm{im}(v)$ is dense.

e.g. $K = \mathbb{C}\{\{t\}\}$ Puiseux series

$K = \overline{\mathbb{Q}_p}$

$K = \mathbb{C}_p$

- Let Γ be the value group. We also insist on a section $\Gamma \longrightarrow K^{\times}$ called $w \longmapsto t^{w}$.
- Let $\Bbbk$ be the residue field.

Thus, polynomials$/K \rightsquigarrow$ tropical polynomials.

In fact, as $a \otimes b = a + b$ is invertible,

Laurent polynomials$/K \rightsquigarrow$ tropical polynomials

$$\sum_{u \in \mathbb{Z}^n} c_u x^u \xmapsto{\text{trop}} \min_{u \in \mathbb{Z}^n} \left(v(c_u) + u \cdot x \right)$$

Let $T = \mathbb{A}^1_K - 0$ the torus.

$K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{O}_{T^n}$, so we will develop:

varieties in T^n $\rightsquigarrow$ "tropical varieties"

Hypersurfaces

Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$\Rightarrow V(f) \subseteq T^n$ a hypersurface.

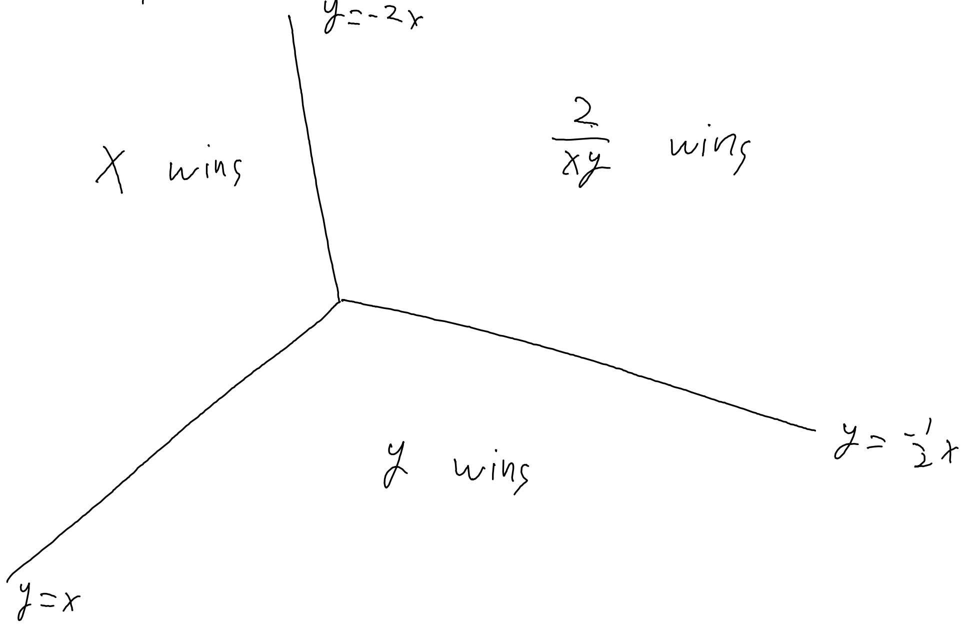
Def. $\operatorname{trop}(V(f)) = \{ w \in \mathbb{R}^n \mid$ the min in $\operatorname{trop}(f)$ is achieved by two of its entries $\}$

This definition makes sense for $\operatorname{trop}(f)$ replaced by any tropical polynomial, and $V(\operatorname{trop}(f)) = \operatorname{trop}(V(f))$.

(Keep these 2 on leftmost board)

e.g. i. $f = \frac{2}{xy} + x + y + 1$

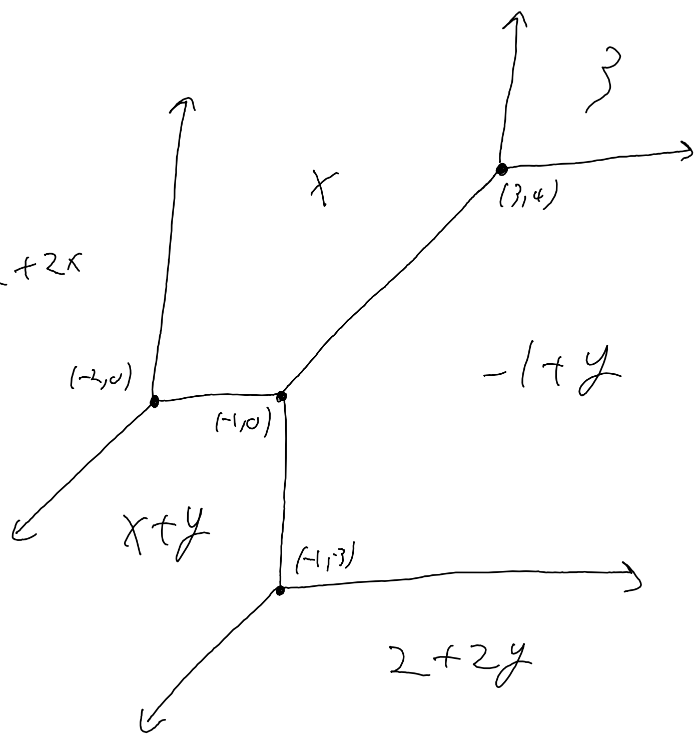
$\operatorname{trop}(f) = \min(-x-y, x, y, 0)$



ii. $f = t^2x^2 + xy + (t^2+t^3)y^2 + (1+t^3)x + t^{-1}y + t^3$

$\operatorname{trop}(f) = \min(2+2x,\ x+y,\ 2+2y,\ x,\ -1+y,\ 3)$

$(3+2y \geq 2+2y)$



- plot trop(f) and look for sharp corners
- go item by item through the min and determine when said item wins

Can reinterpret $\operatorname{trop}(V(f))$ with Gröbner theory.

Def. Let $w \in \mathbb{R}^n$, $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$\operatorname{in}_w(f) = \sum_{\substack{u \in \mathbb{Z}^n \\ v(c_u) + w \cdot u \,=\, \operatorname{trop}(f)(w)}} \overline{t^{-v(c_u)} c_u}\, x^u$$

where $\bar{a}$ is the image in $\Bbbk$, the residue field

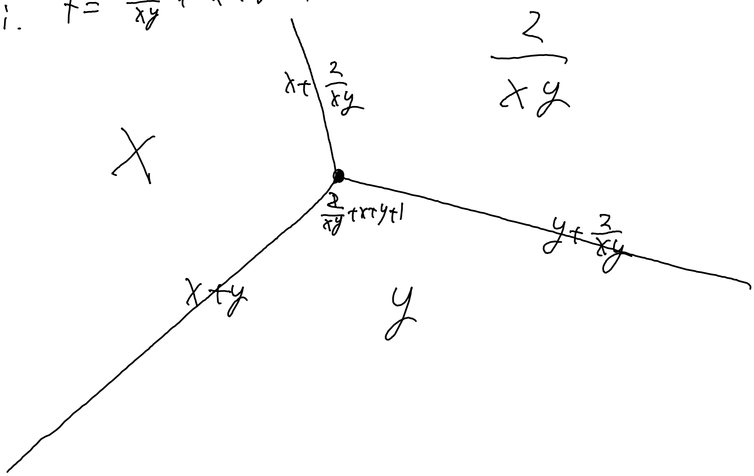
call this the "initial form of f wrt w"

$v(c_u) + w \cdot u$ is the tropicalization of $c_u x^u$

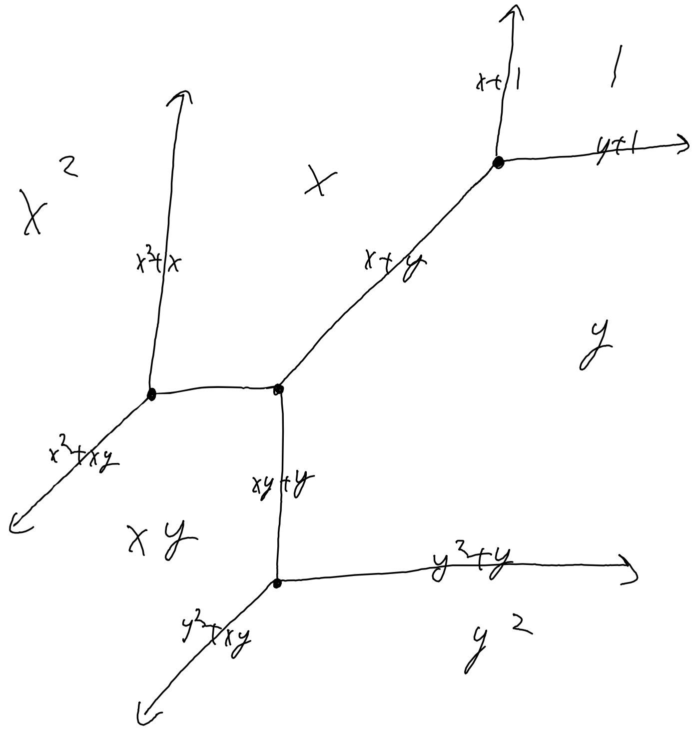
so $\operatorname{in}_w(f)$ labels each point in $\mathbb{R}^n$ with a Laurent polynomial keeping track of which terms in the min are achieved.

e.g.

i. $f = \frac{2}{xy} + x + y + 1$



ii.



$in_w(f)$ is thus a monomial precisely when the min is achieved exactly once

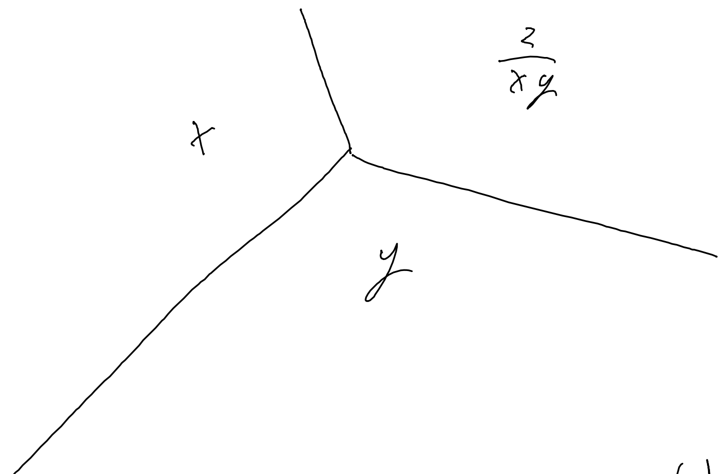
Rmk. $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^\times = \langle k^\times, x_1, \ldots, x_n \rangle$

Prop. $\operatorname{trop}(V(f)) = \{ w \in \mathbb{R}^n \mid in_w(f) \text{ is not a unit} \}$

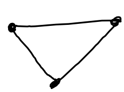
Why introduce the initial form? (do this on the rightboard)

The equivalence is trivial, but connects us to Gröbner theory.

In particular, for hypersurfaces, we have a polyhedral complex w/ maximal cells labeled by monomials.

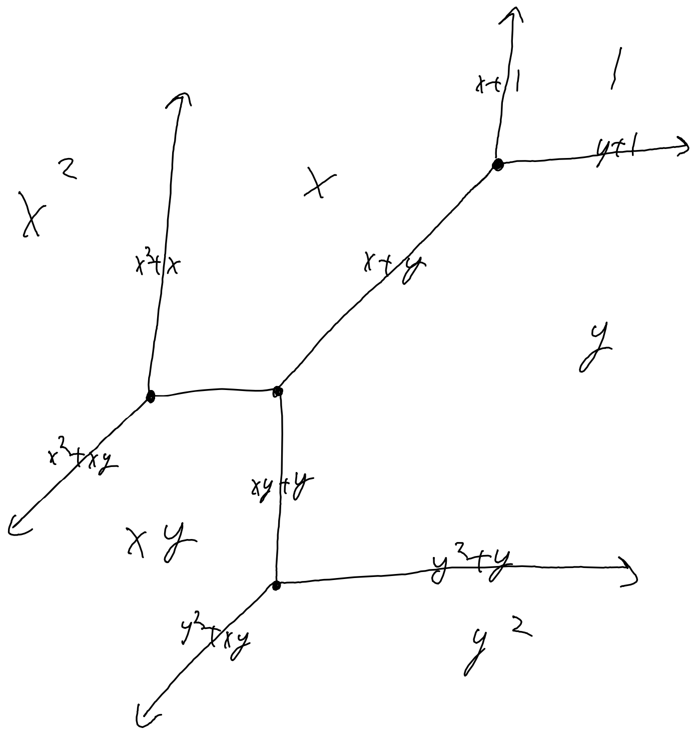


is dual to (draw these polytopes below both examples)

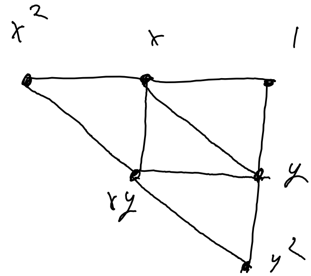


which is the <u>Newton polytope</u> of f

Def. $f = \sum c_u x^u$, $\mathrm{Newt}(f) = \mathrm{conv}\{u \mid c_u \neq 0\}$



is dual to



a <u>subdivision</u> of the Newton polytope of f.

(Prop 3.1.6)

Varieties more generally

Def. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

$X = V(I)$.

We let $\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f))$

We call such an object a tropical variety

Rmk. "balanced polyhedral complex" is a more intrinsic notion of a tropical variety that expands this

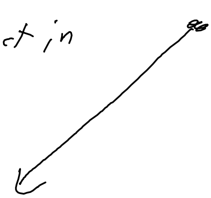
Rmk. To a monomial map $T^n \xrightarrow{\varphi} T^m$ we associate a linear map $\mathbb{Z}^m \to \mathbb{Z}^n$, via $e_i \mapsto y_i$ if $\varphi^* x_i = z^{y_i}$. Dualize to get $\mathrm{trop}(\varphi): \mathbb{Z}^n \to \mathbb{Z}^m$

Rmk. $I = \langle f_1, \dots, f_r \rangle \not\Rightarrow \mathrm{trop}(X) = \bigcap_{i=1}^{r} \mathrm{trop}(V(f_i))$

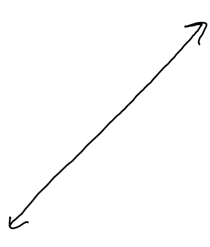
e.g. $I = \langle x+y+1, x+2y \rangle$

$\mathrm{trop}(V(x+y+1)) =$

$\to$ intersect in



$\mathrm{trop}(V(x+2y)) =$



Put $f = (x+y+1) - (x+2y) = 1-y$

has $\operatorname{trop}(f) = \min(0, y)$

and $V(\operatorname{trop}(f)) =$

so $V(X) \subseteq \quad \bullet\ (0,0)$

(in fact, equal)

Def. The initial ideal of I wrt w is $\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) \mid f \in I \rangle$

The previous proposition generalizes to this setting.

Prop. $\operatorname{trop}(X) = \{ w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \neq (1) \}$

Def. $\mathcal{T} \subseteq I$ finite is called a tropical basis if $\operatorname{in}_w(I) = (1) \iff \operatorname{in}_w(\mathcal{T})$ contains a unit

Hence, $\bigcap_{f \in \mathcal{T} \text{ (finite)}} \operatorname{trop}(V(f)) = V(X)$.

Fact. Tropical bases exist.
(Thm 2.6.5)

We saw that X being a point led to $\mathrm{trop}(X)$ being a point.

What is the relationship between points of X and points of $\mathrm{trop}(X)$?

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, dashed, "?"'] & T^n \arrow[d, "v"] \\ \mathrm{trop}(X) \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

Let $a = (a_1, \dots, a_n) \in X$. Claim $v(a) = (v(a_1), \dots, v(a_n)) \in \mathrm{trop}(X)$.

It suffices to show this for $X = V(f)$,

so $f(a) = 0$, so $v(f(a)) = \infty$

Write $f = \sum c_u x^u$. Then for $c_u \neq 0$, $v(f(a)) \geq v(c_u a^u)$.

Recall $v(a) \neq v(b) \Rightarrow v(a+b) = \min(v(a), v(b))$

If the min is achieved only once, $v(f(a))$ would be finite.

Hence, we get a map $X \xrightarrow{v} \mathrm{trop}(X)$.

e.g. $I = \langle x+y+z+1,\ x+y+2z \rangle$

First, we compute $V(I)$.

$$V(I) = \underbrace{(\{x+y+z+1=0\} \cap \{x+y+2z=0\})}_{\shortparallel} \cap T^2$$

$$\left\{ \begin{pmatrix} -2-\alpha \\ \alpha \\ 1 \end{pmatrix} \middle| \alpha \in K \right\}$$

so $V(I) = \left\{ \begin{pmatrix} -2-\alpha \\ \alpha \\ 1 \end{pmatrix} \middle| \alpha \in K - \{0, 2\} \right\}$

$$v\begin{pmatrix} -2-\alpha \\ \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} v(-2-\alpha) \\ v(\alpha) \\ 0 \end{pmatrix}$$

Case 1. $v(\alpha) > 0$. Then $v(-2-\alpha) = 0$, so we get $\begin{pmatrix} 0 \\ v(\alpha) \\ 0 \end{pmatrix}$, $v(\alpha) > 0$

Case 2. $v(\alpha) < 0$. Then $v(-2-\alpha) = v(\alpha)$ so we get $\begin{pmatrix} v(\alpha) \\ v(\alpha) \\ 0 \end{pmatrix}$, $v(\alpha) < 0$

Case 3. $v(\alpha) = 0$. All we can say is

$$v(-2-\alpha) \geq 0$$

so we get $\begin{pmatrix} v(-2-\alpha) \\ 0 \\ 0 \end{pmatrix}$, $v(-2-\alpha) \geq 0$

Hence, $v[X] = \Gamma^{\geq 0}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cup \Gamma^{\leq 0}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cup \Gamma^{\geq 0}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

so $\overline{v[X]} = \mathbb{R}^{\geq 0}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cup \mathbb{R}^{\geq 0}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \cup \mathbb{R}^{\geq 0}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Now, we compute trop(X).

A tropical basis for I is

$$\{x+y+z+1,\ x+y+2z,\ x+y+2,\ z-1\}$$

(c.f. gfan)

<u>$V(\text{trop}(x+y+z+1))$?</u>

$\text{trop}(x+y+z+1) = \min(x,y,z,0)$

so $V(\min(x,y,z,0)) = \left\{ (x,y,z) \;\middle|\; \begin{array}{ll} x=0, & y,z \geq 0 \\ \text{or } y=0, & x,z \geq 0 \\ \text{or } z=0 & x,z \geq 0 \\ \text{or } x=y & z,0 \geq x \\ \text{or } x=z & y,0 \geq x \\ \text{or } y=z & x,0 \geq y \end{array} \right\}$

$$\underline{V(\mathrm{trop}(x+y+2z))} \; ?$$

$$\parallel$$

$$V(\min(x,y,z)) = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x=y, \quad z \geq x \\ \text{or } x=z, \quad y \geq x \\ \text{or } y=z, \quad x \geq y \end{array} \right\}$$

$$\underline{V(\mathrm{trop}(x+y+2))} \; ?$$

$$\parallel$$

$$V(\min(x,y,0)) = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x=0, \quad y \geq 0 \\ \text{or } y=0, \quad x \geq 0 \\ \text{or } x=y, \quad 0 \geq x \end{array} \right\}$$

$$\underline{V(\mathrm{trop}(z-1))} \; ?$$

$$\parallel$$

$$V(\min(z,0)) = \{ z=0 \}$$

Intersect each with $\{z=0\}$ to get

$$\left\{ (x,y,z) \;\middle|\; \begin{array}{l} x=0, \; z=0, \; y \geq 0 \\ \text{or } y=0, \; z=0, \; x \geq 0 \\ \text{or } x=y, \; z=0, \; 0 \geq x \end{array} \right\} = \mathbb{R}^{\geq 0}\begin{pmatrix}1\\0\\0\end{pmatrix} \cup \mathbb{R}^{\geq 0}\begin{pmatrix}0\\1\\0\end{pmatrix} \cup \mathbb{R}^{\geq 0}\begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

So here, $v: X \longrightarrow \mathrm{trop}(X)$ has dense image.

Thm. (Fundamental theorem of tropical geometry)

Let $X \subseteq T^n$ be given by $V(I)$. The following sets agree:

i. $\mathrm{trop}(X)$

ii. $\overline{\{\vec{w} \in \Gamma^n \mid \mathrm{in}_{\vec{w}}(I) \neq (1)\}}$

iii. $\overline{v[X]}$

Rmks.
- (ii). makes it clear that $\mathrm{trop}(X)$ depends only on $\sqrt{I}$, hence on X rather than I
- (iii). affords nice computations if X is well understood, e.g., if parametrized.
- (ii). connects us to the <u>Gröbner complex</u>.

Cor. $\varphi: T^n \longrightarrow T^m$ monomial. Then

$$\mathrm{trop}\left(\overline{\varphi[X]}\right) = \mathrm{trop}(\varphi)[\mathrm{trop}(X)]$$

If φ is rational, $\mathrm{trop}(\varphi)$ can still be defined componentwise, but we only get "$\supseteq$".

Gröbner complexes

shift for now to $K[x_0, \dots, x_n]$ and its homogeneous ideals.

Def. Let $w \in \Gamma^{n+1}$, $I \subseteq K[x_0, \dots, x_n]$ homog.

$$C_I[w] = \{ w' \in \Gamma^{n+1} \mid in_{w'}(I) = in_w(I) \}$$

Prop. 2.5.2 $\overline{C_I[w]}$ is a $\overbrace{\Gamma\text{-rational}}^{\text{of the form } \{Ax \le b,\ A/\mathbb{Q},\ b/\Gamma\}}$ polyhedron containing the line $\mathbb{R}\vec{1}$, $\vec{1} = (1, \dots, 1)$.

Furthermore, if $in_w(I)$ is not monomial then $\overline{C_I[w]}$ is a proper face of $\overline{C_I[w']}$, some w' s.t. $in_{w'}(I)$ is monomial.

we thus work in $\mathbb{R}^{n+1}/\mathbb{R}\vec{1}$, which is a tropical version of modding out by scalar multiplication.

Thm 2.5.3. $\{ \overline{C_I[w]} \mid w \in \Gamma^{n+1} \}$ is a Γ-rational polyhedral complex in $\mathbb{R}^{n+1}/\mathbb{R}\vec{1}$

we call this $\Sigma(I)$, the Gröbner complex.

Back to the tropics.

Given $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we can homogenize wrt x_0. Call this $\hat{f}$.

Def. $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$$I_{proj} = \langle \hat{f} \mid f \in I \rangle$$

We communicate between their initial ideals via

$$\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^{n+1}/\mathbb{R}\vec{1}$$
$$w \longmapsto (0, w)$$

Prop 2.6.2. $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then

$in_w(I) = in_{(0,w)}(I_{proj})$ for $w \in \Gamma^n$.

Hence, $in_w(I) = (1) \iff in_{(0,w)}(I_{proj})$ is monomial

Thus, $\operatorname{trop}(X)$ is a polyhedral complex given by removing the maximal cells from the $\Sigma(I_{proj})$.

The structure theorem (3.3.6) implies that, for X irred dim d that $\mathrm{Trop}(X)$ is pure dimension d, i.e. all non-faces are dim d.

Rmk. We can understand the Gröbner complex via its initial ideals and how they relate to I_{proj}.
A useful tool here is a $\vec{w}$-Gröbner basis, i.e. a finite subset $\mathcal{Y} \subseteq I_{proj}$ so that

$$in_{\vec{w}}(\mathcal{Y}) = in_{\vec{w}}(I_{proj})$$

This can be made uniform,
Def. A universal Gröbner basis is a $\vec{w}$ Gröbner basis $\forall \vec{w} \in \Gamma^n$.
Fact. (2.5.10). These always exist.

Rmk. Classical Gröbner theory is when v is trivial.