§1 Root systems and semisimple Lie algebras
- $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$

§2 Root datum and reductive groups
- SL_2 and PGL_2

$\boxed{\S 1}$ Goal: Classify semisimple Lie algebras and their irreducible representations.

We work $/\mathbb{C}$ in this section.

Recall
- $SL_n = \ker(\det)$
- $d(\det)|_I = \mathrm{tr}$

So $\mathfrak{sl}_n = \ker(\mathrm{tr})$

$\boxed{\mathfrak{sl}_2}$ We have a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Consider first its adjoint representation

$$\mathrm{ad}: \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(\mathfrak{sl}_2)$$

$$\left.\begin{aligned} [h,e] &= 2e \\ [h,f] &= -2f \\ [h,h] &= 0 \end{aligned}\right\}$$

so $\mathrm{ad}(h)$ acts <u>diagonalizably</u>, or semisimply, and e, f, h is an eigenbasis of $\mathrm{ad}(h)$.

$\mathfrak{sl}_2$ is semisimple ($[\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2$), so h is diagonalizable on all f.d. reps (Jordan decomposition)

Let V be a fd irreducible repr. of $\mathfrak{sl}_2$

Then $V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$, $V_\alpha = \{v \in V \mid hv = \alpha v\}$

By $\mathrm{ad}(h)$ calculation, $e: V_\alpha \longrightarrow V_{\alpha+2}$
$f: V_\alpha \longrightarrow V_{\alpha-2}$

Fact. The h-eigenvalues of V are integers. Let $n \in \mathbb{N}$ maximal so that $V_n \neq 0$. Let $v \in V_n - 0$. Then $\{v, fv, \dots, f^n v\}$ is a basis for V.

$$
\begin{tikzcd}
0 \arrow[r, shift left, "e"] & V_{-n} \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & V_{-n+2} \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & \cdots \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] & V_{n-2} \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & V_n \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & 0 \arrow[l, shift left, "f"]
\end{tikzcd}
$$

n is called the highest weight of V

Thm. fd irreps of $\mathfrak{sl}_2$ are uniquely and totally classified by their highest weight n.

Rmk. Existence is as follows. Let $SL_2(\mathbb{C})$ act on $\mathbb{C}[x,y]_n$, the degree n homogeneous polynomials. To the fancy, these are global sections of the line bundle $\mathcal{O}(n)$ on $SL_2(\mathbb{C})/B(\mathbb{C})$ where $B \subseteq SL_2$ is the set of upper triangular matrices

$\boxed{\mathfrak{sl}_3}$

Let $\mathfrak{g} = \mathfrak{sl}_3$. This is, again, semisimple.

$\exists\ 3$ embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$

$$\begin{pmatrix} \cdot & \cdot & \\ \cdot & \cdot & \\ & & \end{pmatrix} \quad h_{12} \quad e_{12} \quad f_{21}$$

$$\begin{pmatrix} \cdot & & \cdot \\ & & \\ \cdot & & \cdot \end{pmatrix} \quad h_{13} \quad e_{13} \quad f_{31}$$

$$\begin{pmatrix} & & \\ & \cdot & \cdot \\ & \cdot & \cdot \end{pmatrix} \quad h_{23} \quad e_{23} \quad f_{32}$$

which $\mathfrak{h}$ do we pick?

Def. $\mathfrak{h} \subseteq \mathfrak{g}$ is the subalgebra of diagonal matrices

Rmk. $\mathfrak{h}$ is a max'l <u>abelian</u> subalgebra

$ad(h_{ij})$ will again be diagonalizable.

$[\mathfrak{h}, \mathfrak{h}] = 0$, so there are simultaneously diagonalizable

Let v be an eigenvector of $\mathfrak{h}$. Then

$$\mathfrak{h} \longrightarrow \mathbb{C}$$
$$h \longmapsto \text{eigenvalue of } h \text{ on } v$$

is $\mathbb{C}$-linear

$\mathfrak{h}^* = \mathbb{C}\langle L_1, L_2, L_3 \rangle / (L_1 + L_2 + L_3 = 0)$

where $L_i(v) = v_{ii}$

we call elements of $\mathfrak{h}$ eigenvalues

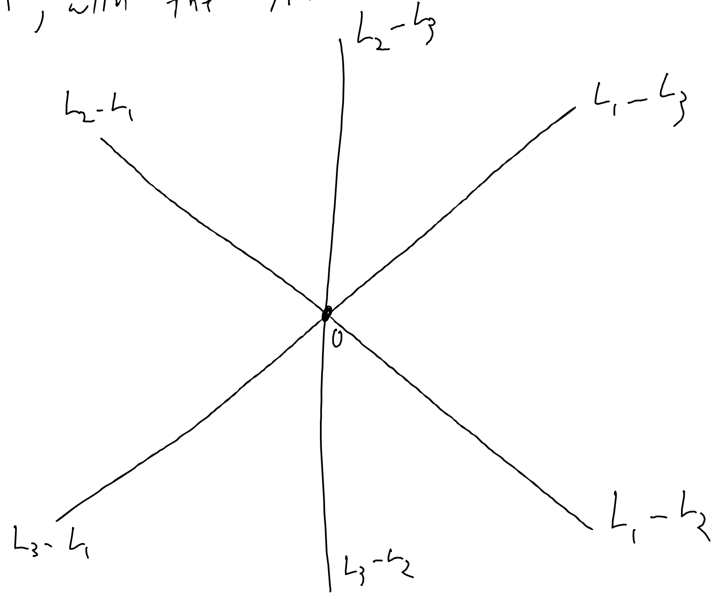
We compute

$$ad(h)(e_{ij}) = (L_i - L_j)(h)\, e_{ij}$$
$$ad(h)(f_{ji}) = (L_j - L_i)(h)\, f_{ji}$$
$$ad(h)(h') = 0 \quad , h' \in \mathfrak{h}$$

Def. The nonzero eigenvalues of $ad(h)$ $R = \{L_i - L_j,\ L_j - L_i\}_{i<j}$ are called the roots of $\mathfrak{g}$ (wrt $\mathfrak{h}$)

Then $\mathfrak{g} = \underset{(=\mathfrak{g}_0)}{\mathfrak{h}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [h, v] = \alpha(h) v \ \forall h \in \mathfrak{h}\}$ the α-eigenspace

$\dim \mathfrak{g}_\alpha = 1$, with the generators e_{ij} and f_{ji} as above



Compute $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

So e.g. e_{13} acts as

e_{12} acts as

e_{23} acts as

<u>Why this drawing?</u>

Def. Define a pairing $K: \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ via
$$x \otimes y \longmapsto tr(ad(x)\, ad(y))$$

Fact. K is nondegenerate (this is equivalent to semisimplicity of $\mathfrak{g}$)

Furthermore, for $h \in \mathfrak{h}$, $K(h,h) = \sum_{\alpha \in R} \alpha(h)^2$

Recall $\alpha(h_\alpha) = 2$. Let $E = \mathbb{R}R$ the $\mathbb{R}$-span of the roots in $\mathfrak{h}^*$.

Then K is positive definite on E and hence yields geometry on E.

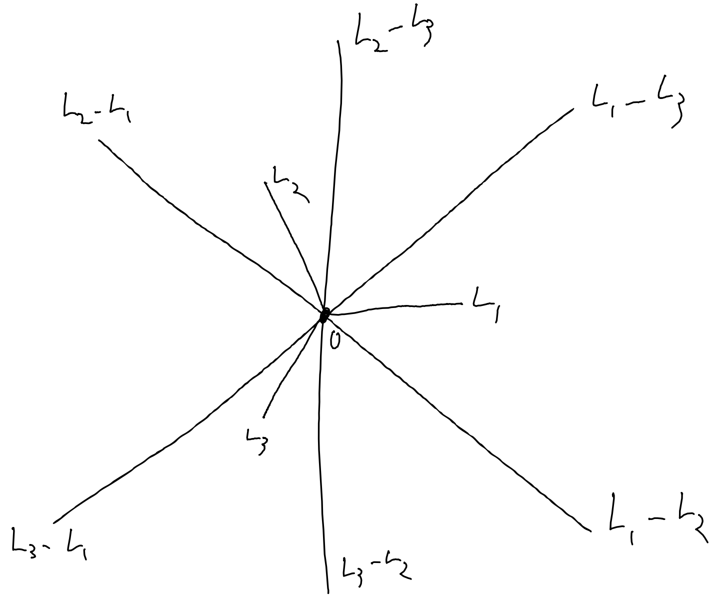
For a root α, let $\Omega_\alpha = span(\alpha)^\perp$

K is chosen s.t. $\Omega_\alpha = \{\beta \in E \mid \beta(h_\alpha) = 0\}$

Def. Let s_α be reflection in E around Ω_α. The Weyl group $W(\mathfrak{g})$ is the group generated by these reflections

Take e.g. $\alpha = L_2 - L_3$, $R_\alpha = \operatorname{span} L_1$

$\langle v, w\rangle = \|v\|\|w\| \cos\theta$



The Weyl group acts on R, and in fact on L_1, L_2, L_3 here.
This is faithful and all transpositions are achieved, so $W(\mathfrak{sl}_3) \cong S_3$.
As K is W-invariant, the angles here must all be 60°

Thm. Take E, R, K as before

i) Let $\alpha \in R$, $n\alpha \in R \iff n = \pm 1$

ii) The reflection s_α preserves roots

iii) Let $\alpha, \beta \in R$. Then $\dfrac{K(\alpha,\beta)}{K(\alpha,\alpha)} \in \frac{1}{2}\mathbb{Z}$ $\left(\text{consider } \dfrac{K(\alpha,\beta)}{K(\alpha,\alpha)} \dfrac{K(\beta,\alpha)}{K(\beta,\beta)} = 4(\cos\theta)^2 \in \mathbb{Z}\right)$

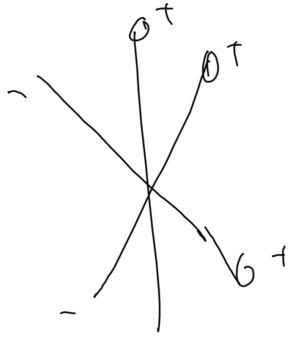
iv) $E = \operatorname{span}(R)$

Def. That's a root system, for E fdvs/$\mathbb{R}$, $R \subseteq E$ finite, and K an inner product on E.

Theorem. This defines an equivalence between semisimple Lie algebras /$\mathbb{C}$ and root systems.

Dynkin diagrams

We need to designate half of our roots as "positive," so that $\forall \alpha \in R$ only one of $\alpha, -\alpha$ are "positive"

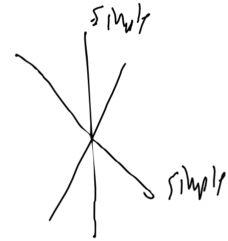


Systematically, these positive ones have e_α as eigenvectors, which lie in $\mathfrak{b}$, the subalgebra of upper triangular matrices, so $\alpha \in R^+ \iff \mathfrak{g}_\alpha \subseteq \mathfrak{b}$

Rmk $\mathfrak{b}$ is called a Borel subalgebra — a maximal solvable subalgebra

Def. $\alpha \in R$ is simple if it's positive and not a sum of 2 positive roots

e.g. $L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3)$



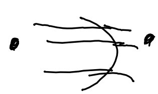
Def. Let (E, R, K) be a root system w/ a subset of positive roots R^+.

The associated Dynkin diagram is an oriented multigraph with vertices the simple roots of R. Let α, β be simple roots. Let θ = angle between α, β.

if $\theta = \frac{\pi}{2}$

if $\theta = \frac{2\pi}{3}$


if $\theta = \frac{3\pi}{4}$, where $\Rrightarrow$ points towards the shorter root


if $\theta = \frac{5\pi}{6}$, " $\Rrightarrow$ "

" " "

Theorem. Root systems are determined by their Dynkin diagrams.

e.g. sl_2 has  A_1

sl_3 has  A_2

Representations of $\mathfrak{sl}_3$

Let V be a fd irrep of $\mathfrak{sl}_3$

Def. A highest weight vector is an $\mathfrak{h}$-eigenvector killed by e_{12}, e_{23}, and e_{13} — the eigenvectors of the positive roots (wrt $\mathfrak{b}$)

Prop. Let λ be the highest weight of V. Then

$$\lambda = \sum a_i L_i$$

w/ $a_i \in \mathbb{N}$, $a_1 \geq a_2 \geq a_3$

Thm. Such weights uniquely and totally determine fd irreps of $\mathfrak{sl}_3$

Existence? $SL_3 / B \cong \{$complete flags in $\mathbb{C}^3\}$

irreps arise via line bundles on this space

§2. Reductive groups and root systems

Above was all for Lie algebras.

s.c. Lie Grp $\longrightarrow$ Lie Alg

is faithful, but s.c. is needed

Consider the double cover $SL_2 \longrightarrow PGL_2$. This is not an iso, and yet $\mathfrak{sl}_2 \xrightarrow{\sim} \mathfrak{pgl}_2$.

Reductive groups will be classified combinatorially over any acf (Chevalley '58) ala s.s. Lie algebras,

Def. Let G be a smooth connected algebraic group over a field k. Its radical $R(G)$ is a maximal smooth connected solvable normal subgroup.

G is called semisimple if $\underbrace{R(G_{\bar{k}})}_{\text{geometric radical}}$ is trivial

Rmk. $R(G)$ exists as solvability is closed under extension and quotient.

Def. An alg. group is unipotent if all nonzero reps have a nonzero fixed vector. Equivalently, all representations V of G have a basis where G acts via $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$.

Def. G smooth, connected alg group / k a field.

Let $R_u(G)$ be a max'l smooth connected unipotent normal subgroup, called the unipotent radical.

G is reductive if $\underbrace{R_u(G_{\bar{k}})}_{\text{geometric unipotent radical}}$ is trivial

unipotent $\Rightarrow$ solvable so semisimple $\Rightarrow$ reductive

Thus, GL_n, SL_n, PGL_n are all reductive.

Let G be SL_2 or PGL_2.

Let $T \subseteq G$ be a maximal torus

SL_2	PGL_2
$T = \left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\}$	$T = \left\{\begin{pmatrix} a & \\ & 1\end{pmatrix}\right\}$, a sub

Def. $X^*(T) = \mathrm{Hom}(T, G_m)$

This is a lattice.

SL_2	PGL_2
$\mathbb{Z} \xrightarrow{\sim} X^*(T)$	$\mathbb{Z} \xrightarrow{\sim} X^*(T)$
$n \longmapsto \left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} \mapsto a^n\right)$	$n \longmapsto \left(\begin{pmatrix} a & \\ & 1\end{pmatrix} \mapsto a^n\right)$

Rmk. Consider $T \hookrightarrow SL_2$. On Lie algebras

this yields $\mathfrak{h} \hookrightarrow \mathfrak{sl}_2$

$$\mathfrak{h} = \mathrm{span}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and $\mathrm{Hom}(T, \mathbb{G}_m) \rightsquigarrow \mathrm{Hom}(\mathfrak{h}, \mathbb{G}_a)$

Def. $R \subseteq X^*(T)$ are the characters corresponding to the roots of $\mathrm{ad}(t)$ on $\mathfrak{g}$

SL_2	PGL_2
$\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \longmapsto \alpha^2$	$\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \longmapsto \alpha$
$\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \longmapsto \alpha^{-2}$	$\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \longmapsto \alpha^{-1}$

Def. $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$

SL_2	PGL_2
$\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, T)$	$\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, T)$
$n \longmapsto \left(\alpha \mapsto \begin{pmatrix} \alpha^n & \\ & \alpha^{-n} \end{pmatrix}\right)$	$n \longmapsto \left(\alpha \mapsto \begin{pmatrix} \alpha^n & \\ & 1 \end{pmatrix}\right)$

Def. $R^\vee \subseteq X_*(T)$ the coroots are the cocharacters arising via

Prop. G reductive, T maximal torus, $\alpha \in R$ a root,

$\exists\ SL_2 \longrightarrow G$ s.t.

$\mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ sends $\underbrace{e}_{\text{a 2-eigenvector}}$ to $\mathfrak{g}_\alpha - \{0\}$

Upon restriction to the (diagonal) maximal torus in SL_2

SL_2	PGL_2
$T \longrightarrow SL_2$	$T \longrightarrow PGL_2$
$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$	$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & \\ & 1 \end{pmatrix}$
or	or
$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$	$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} a^{-2} & \\ & 1 \end{pmatrix}$

There is a pairing

$$\begin{tikzcd} X^*(T) \otimes X_*(T) \arrow[r, "\langle -,- \rangle"] \arrow[d, "\circ"] & \mathbb{Z} \\ \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \arrow[ur, "\sim"'] & \end{tikzcd}$$

Observe that in both cases, there is a bijection $R \xrightarrow{\sim} R^\vee$ written $\alpha \longmapsto \alpha^\vee$

We may verify

- $\langle \alpha, \alpha^\vee \rangle = 2$
- The reflection $s_{\alpha, \alpha^\vee} : X \longrightarrow X$

$$x \longmapsto x - \langle x, \alpha^\vee \rangle \alpha$$

preserves R

- The reflection $s_{\alpha^\vee, \alpha} : X^\vee \longrightarrow X^\vee$

$$y \longmapsto y - \langle y, \alpha \rangle \alpha^\vee$$

preserves $R^\vee$

Rmk. $\mathfrak{g}$ s.s. Lie alg, $\mathfrak{h} \subseteq \mathfrak{g}$ max'l abelian, $\alpha \in R$ a root. Then $\alpha^\vee \in \mathfrak{h}^*$ via $\alpha^\vee = 2 \frac{\alpha}{K(\alpha, \alpha)}$ is the coroot, and $K(\alpha, \alpha^\vee) = 2$, of course.

Def. $(X^*, R, X_*, R^\vee)$ as above is called a root datum

Thm. $\{\text{Reductive groups}\} \xrightarrow{\sim} \{\text{root datum}\}$

Rmk. $\{\text{root datums}\}$ has an involution $(X^*, R, X_*, R^\vee)$

$$\updownarrow$$

$$(X_*, R^\vee, X^*, R)$$

which yields an involution $G \longmapsto G^\vee$ on reductive groups. This is called the Langlands dual.

e.g. $\mathfrak{sl}_2$ had root datum

$$(\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\})$$

and $\mathfrak{pgl}_2$ had root datum

$$(\mathbb{Z}, \{\pm 1\}, \mathbb{Z}, \{\pm 2\})$$

so they are Langlands dual!

Fact: $\mathfrak{gl}_n^\vee = \mathfrak{gl}_n$

3 notions of coroot

$$\alpha \longmapsto \alpha^\vee = 2\frac{\alpha}{K(\alpha,\alpha)}$$

$$R \longrightarrow \mathfrak{h}^*, \quad R \longrightarrow R^\vee$$

by root datum

$$R \longrightarrow \mathfrak{h}$$

and $\alpha \longmapsto h_\alpha$

connection??