p-adic and Λ-adic modular forms

References. Hida "Elementary theory ..."

Hida 80 "Iwasawa ..."

Lafferty

Notes by Samuel Marks "p-adic modular forms à la Serre"

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p-adic interpolation

Given

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C} \\ \mathbb{Z} \arrow[u, hook] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "?"] & \overline{\mathbb{Q}} \arrow[u, hook] \arrow[d, hook] \\ \mathbb{C}_p \arrow[r] & \mathbb{C}_p \end{tikzcd}$$

better for congruences

Can we interpret these integral points p-adically?, i.e., First, Eisenstein Series

Recall $E_k(z) = \frac{1}{2}\zeta(1-k) + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$ for $k \geq 2$

Consider $\sigma_{k-1}^{(p)}(n) = \sum_{\substack{d \mid n \\ p \nmid d}} d^{k-1}$

Then $d \in (\mathbb{Z}/p^\alpha)^\times$ $\forall d$, so as $|(\mathbb{Z}/p^\alpha)^\times| = \varphi(p^\alpha) = p^{\alpha-1}(p-1)$,

we deduce

$$k \equiv k' \bmod p^{\alpha-1}(p-1) \Rightarrow \sigma_{k-1}^{(p)}(n) \equiv \sigma_{k'-1}^{(p)}(n) \bmod p^\alpha$$

This is p-adic continuity, so we expect to take a limit E_{k_i} with k_i converging in $\mathbb{Z}_p^\times$.

Indeed, let $\underset{\in \mathbb{Z}}{k_i} \longrightarrow k \in \mathbb{Z}_p^\times$

Then $\sigma_{k_i} \longrightarrow \sigma_k$ and we write

$$E(k) = \sum a_n q^n \in \mathbb{Z}_p[[q]]$$

where $a_0 = \lim_{i\to\infty} \frac{1}{2}\zeta(1-k_i)$, $a_n = \lim_{i\to\infty} \sigma_{k_i-1}(n)$, $n \geq 1$.

Def (Serre). A p-adic modular form (for $SL_2\mathbb{Z}$) is a formal power series in $\mathbb{Q}_p[[q]]$ s.t. $\exists f_i$ rational modular forms s.t. $f_i \longrightarrow f$ in the sense of Fourier coefficients,

If f_i has weight k_i, then $k = \lim_{i\to\infty} k_i \in \mathbb{Z}_p^\times$ converges and we say f has weight k.

Rmk. ~ we put a norm on $M(\mathbb{Q})$ via

$$\left| \sum a_n q^n \right| = \sup |a_n|_p$$

so that $M(\mathbb{Q}_p)$ is the completion.

- The weight now ranges in $\mathbb{Z}_p^\times$, which is no longer discrete.

What about the constant coefficient?

Prop. Let $f_i \longrightarrow f$ in $M(\mathbb{Q}_p)$ so that f_i has weight k_i.

If i. $a_n(f_i) \longrightarrow a_n(f)$ uniformly

ii. $k_i \longrightarrow k \neq 0$

Then $a_0(f_i) \longrightarrow a_0(f)$

Apply this to $f(k)$ to conclude that

$$k \longmapsto \lim_{i\to\infty} \varphi(1-k_i) \overset{(\text{def})}{=} \varphi_p(1-k)$$

is continuous.

Observe also that $f(k) = E_k(z) - p^{k-1}E_k(pz)$

so that $\varphi_p(1-k) = (1-p^{k-1})\varphi(1-k)$

for $k \in \mathbb{Z}$.

Thus, we reach the Kubota-Leopoldt L-function.

More generally we follow Hida 86.

Setup:

$K/\mathbb{Q}_p$ finite

$K_0/\mathbb{Q}$ finite s.t. $K_0 \hookrightarrow K$ dense

Fix $\mathbb{C}_p$

First, consider $\Delta = \Gamma_1(N)$ or $\Gamma(N)$

$$M_k(\Delta; K_0) = \{ f \in M_k(\Delta) \mid a_n(f) \in K_0 \ \forall n \in \tfrac{1}{N}\mathbb{Z} \}$$

$$M_k(\Delta; K) = M_k(\Delta; K_0) \otimes K$$

Observe that $M_k(\Delta; K_0) \hookrightarrow \mathbb{C}_p[[q^{1/N}]]$ which is equipped with the norm $\left| \sum a_n q^{n/N} \right| = \sup |a_n|$,

whence $M_k(\Delta; K) = \overline{M_k(\Delta; K_0)} \hookrightarrow \mathbb{C}_p[[q^{1/N}]]$.

Now consider $\Phi = \Gamma_0(N)$ or $\Gamma_1(N) \cap \Gamma_0(p^r)$

ψ a Dirichlet character mod N or p^r with values in K_0

$$M_k(\Phi, \psi; K) = M_k(\Phi, \psi; K_0) \otimes K$$

we replace the K coefficients with $\mathcal{O}_K$ if $|f| \le 1$, i.e. the Fourier coefficients lie in $\mathcal{O}_K$.

we similarly define cusp forms, denoted with $\mathcal{S}$.

Let $A = K$ or $\mathcal{O}_K$. Write

$$\mathcal{M}(\Delta', A) = \bigcup_{j \ge 1} \underbrace{\bigoplus_{k \ge 0}^{j} \mathcal{M}_k(\Delta', A)}_{\subseteq A[[q^{1/N}]]}$$

and let $\overline{\mathcal{M}}(\Delta', A)$ be its completion (closure in $A[[q^{1/N}]]$)

This is the space of p-adic modular forms.

Do the same for Φ, Ψ and for $\mathcal{S}$.

Katz's geometric approach

Roughly: weight k modular forms are sections of ω^k on the modular curve

Thus, they take pairs (E, ω)
elliptic curve, holo 1-form

to complex numbers

p-adic modular forms have a similar moduli approach, but E is now defined over an algebra $R/\mathcal{O}_K$ for $K/\mathbb{Q}_p$ finite s.t.
- p nilpotent in R
- E is supersingular, i.e. $E_{p-1}(E, \omega)$ invertible
- these take values in R rather than $\mathbb{C}$.

Hida §6 defines Hecke operators via

Katz's approach.

Instead, we restrict to Lafferty's definition

Def (Lafferty). χ a Dirichlet char mod N. $\mathbb{Z}[\chi] \subseteq A \subseteq \mathbb{C}_p$.

$$M_k(N, \chi; A) = M_k(N, \chi; \mathbb{Z}[\chi]) \otimes A$$

$$= A\text{-span of } M_k(N, \chi; \mathbb{Z}[\chi])$$

in $A[[q]]$.

Now let $f = \sum a_n(f) q^n \in M_k(N, \chi; A)$

$$f T_m = \sum a_n(f T_m) q^n$$

where $$a_n(f T_m) = \sum_{d \mid \gcd(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f)$$

$$\mathcal{H}_k(N, \chi; A) \hookrightarrow \mathrm{End}(M_k(N, \chi; A))$$

$$h_k(N, \chi; A) \hookrightarrow \mathrm{End}(S_k(N, \chi; A))$$

We have, as before, perfect pairings $\mathcal{H} \otimes M \to A$, $h \otimes \mathcal{S} \to A$.

Λ-adic forms Hida, "Elementary . . ."

We are seeking a notion of a family parametrized by f_k of p-adic modular forms the weight k.

Notation

$$\ell = \begin{cases} 4 & p = 3 \\ p & p \text{ odd} \end{cases}$$

$\omega : (\mathbb{Z}/\ell)^\times \longrightarrow \mathbb{Z}_p^\times$ the Teichmüller character

$\mathcal{O} = \mathcal{O}_K$ some finite $K/\mathbb{Q}_p$. $\Lambda = \mathcal{O}[[X]]$.

$u = 1+\ell$. a topological generator of $\mathbb{Z}_p^\times$.

Def. A p-adic analytic family of character ω^a s.t. is $\{f_k\}$ of p-adic modular forms

- $f_k \in M_k(\Gamma_0(N), \psi\omega^{-k})$, $k \in \mathbb{N}^{\geq M}$, $\psi = \omega^a$.
- $a_n(f_k) \in \overline{\mathbb{Q}}$
- $\exists A_n(X) \in \mathcal{O}[[X]]$ $\forall n \geq 0$ s.t.

$$a_n(f_k) = A_n(u^k - 1)$$

Rmk. - $u^k - 1 = (1+\ell)^k - 1$, which is divisible by ℓ.

so $|u^k - 1| < 1$.

- $f = px + \binom{p}{2}x^2 + \dots + x^p$

then $f \in \mathrm{End}(F)$ for F the multiplicative formal group law

and $u^k - 1 = [k]_F$ commutes with f.

Def. $F \in \Lambda[[q]]$ is a Λ-adic form of character ψ if $F(u^k - 1)$ is the q-expansion of a p-adic modular form in

$$M_k(\Gamma_0(\ell), \psi\omega^{-k}; \mathcal{O})$$

for a.e. $k \in \mathbb{N}$.

This is cuspidal if it's a.e. cuspidal at specialization.

This is a p-adic family if $F(u^k - 1)$ is a classical modular form for a.e. k.

Back to Eisenstein.

We want a Λ-adic form parametrizing E_k

First, we find Φ s.t. $\Phi(u^k-1) = a^k$

Let $s(z) = \dfrac{\log(z)}{\log(u)}$, $\log : \mathbb{Z}_p \xrightarrow{\sim} 1+l\mathbb{Z}_p$

If $d \equiv 1 \mod q$ then $d = u^{s(d)}$, so consider

$$A_d(X) = d^{-1}(1+X)^{s(d)}$$

Then $A_d(u^k-1) = d^{-1}u^{s(d)k} = d^{k-1}$

for $d \not\equiv 1 \mod q$, recall $\mathbb{Z}_p^\times = \mu \times (1+l\mathbb{Z}_p)$

$$\rightsquigarrow \begin{array}{c} x \longmapsto \omega(x)^{-1}x \\ \mathbb{Z}_p^\times \longrightarrow 1+l\mathbb{Z}_p) \end{array}$$

Then let $A_d(X) = d^{-1}(1+X)^{s(\omega(d)^{-1}d)}$ for $p \nmid d$.

$A_d(u^k-1) = d^{-1}\,\cancel{\$}u^{s(\omega(d)^{-1}d)} = \omega(d)^{-k}d^{k-1}$, which is d^{k-1} for $k \equiv 0 \mod \varphi(l)$.

Thus, for $\psi = \omega^{-a}$ even, let

$$A_n^{\psi}(X) = \sum_{\substack{d \mid n \\ p \nmid d}} \psi(d) A_d(X)$$

so that $A_n^{\psi}(u^k - 1) = \sigma_{k-1}^{(p)}(n)$ is $k \equiv a \mod \varphi(\ell)$.

What about a_0?

Consider $L(s, \xi) = \sum \xi(n) n^{-s}$ → §7.6 Hida "Elementary..." on (formal) group schemes

$\exists\ \Phi_{\psi}(X) \in \mathbb{Z}_p[[X]]$ for $\psi : (\mathbb{Z}/\ell)^{\times} \to \mathbb{C}^{\times}$, $\psi(-1) = 1$

s.t.
$$\Phi_{\psi}(u^k - 1) = \begin{cases} (1 - \psi\omega^{-k}(p) p^{k-1}) L(1-k, \psi\omega^{-k}) & \psi \neq id \\ (u^k - 1)(1 - \omega^{-k}(p) p^{k-1}) L(1-k, \omega^{-k}) & \psi = id \end{cases}$$

Then let $E(\psi)(X) = \sum A_n^{\psi}(X) q^n$

$$A_0^{\psi}(X) = \begin{cases} \Phi_{\psi}(X)/2 & \psi \neq id \\ \Phi_{id}(X)/2X & \psi = id \end{cases}$$

Thus, a Λ-adic form parametrizing $E(k)$.