References, Hida "Elementary theory ..."
Hida 80 "It was away ..."
Lafferty
Note by Samuel Marks "p-adic modular forms @ a Semi"

\( p \)-adic interpolation

Given

\( \mathbb{C} \overset{J}{\rightarrow} \mathbb{Z} \leftarrow \mathbb{Q} \)

Can we interpret these integral point products? i.e.
First, Eisenstein Series

Recall \( E_k(z) = \frac{1}{2\pi i} \sum_{n \neq 1} \frac{\zeta(1-k)}{n^k} \) for \( k > 2 \)

Consider \( \sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1} \)

Then \( d \in (\mathbb{Z}/p^e)^\times \) and, so as \( |(\mathbb{Z}/p^e)^\times| = \mathcal{E}(p^e) = p^{\nu_e}(p-1) \)

we deduce

\[ k \equiv k' \mod p^{k-1}(p-1) \Rightarrow \sigma_{k-1}(n) \equiv \sigma_{k'-1}(n) \mod p^e \]

This is \( p \)-adic continuity, so we expect to take a limit \( E_k \) with \( K = \lim_{i \to \infty} E_{k_i} \) converging in \( \mathbb{Z}_p^\times \).

Indeed, let \( K = \lim_{i \to \infty} E_{k_i} \in \mathbb{Z} \)

Then \( \sigma_{k_i} \rightarrow \sigma_k \) and we write

\[ E(k) = \sum a_n e^n \in \mathbb{Z}_p[[ e]] \]

where \( a_0 = \lim_{i \to \infty} \frac{1}{2\pi i} \zeta(1-k_i) \), \( a_n = \lim_{i \to \infty} \sigma_{k_i-1}(n) \), \( n \geq 1 \).
Def (Serre): A $p$-adic modular form (for $\Gamma_1(\mathbb{Z})$) is a formal power series in $\mathcal{O}_p[[q]]$

s.t. $\exists \tilde{f}_i$ rational modular forms

s.t. $\tilde{f}_i \rightarrow f$ in the sense of Fourier coefficients.

If $f_i$ has weight $k_i$, then $k = \lim_{i \to \infty} k_i \in \mathbb{Z}_p^+$

converges and we say $f$ has weight $k$.

Remk.: We put a norm on $M(G)$ via

$$ | E_{n} \tilde{e}_{\gamma} | = \sup_{|b|_p < 1} | \gamma b | $$

so that $M(G, \mathcal{O}_p)$ is the completion.

The weight now ranges in $\mathbb{Z}_p^+$, which is no longer discrete.

What about the constant coefficient?

Prop. Let $f_i \rightarrow f$ in $M(G, \mathcal{O}_p)$ so that $f_i$ has weight $k_i$.

If i. $q_n(f_i) \rightarrow q_n(f)$ uniformly

ii. $k_i \rightarrow k \in \mathbb{Z}_p^+$

Then $a_0(f_i) \rightarrow a_0(f)$.
Apply this to $E(k)$ to conclude that

$$\lim_{i \to \infty} \varphi(1-k) = \varphi(1-k)^{(\text{def})}$$

is continuous.

Observe also that $E(k) = \varphi_{p}(1) - p^{k-1} \varphi_{p}(1+p)$, so that $\varphi_{p}(1-k) = (1-p^{k-1}) \varphi(1-k)$ for $k \in \mathbb{Z}$.

Thus, we reach the Kubota-Leopoldt $L$-function.
More generally we follow Hida, et.

Setup:

\[ K/\mathcal{O}_p \text{ finite} \]
\[ \mathcal{O}_0/\mathcal{O} \text{ finite so } K_0 \subset K \text{ dense} \]
\[ \text{Fix } \varepsilon_{0} \]

First, consider $\Delta = \mathcal{P}_1(N)$ or $\mathcal{P}(N)$

\[ \mathcal{M}_K (\Delta, K_0) = \{ f \in \mathcal{M}_K (\Delta) \mid \sigma_n(f) \in K_0 \forall n \in \mathbb{Z} \} \]

\[ \mathcal{M}_K (\Delta, K) = \mathcal{M}_K (\Delta, K_0) \otimes K \]

Observe that $\mathcal{M}_K (\Delta, K_0) \rightarrow \mathcal{C}_p [ [ \xi^{1/\nu} ] ]$ which is equipped with the norm $\| \xi^{an_{\nu}} \| = \sup |an_{\nu}|$,

whence $\mathcal{M}_K (\Delta, K) = \mathcal{M}_K (\Delta, K_0) \rightarrow \mathcal{C}_p [ [ \xi^{1/\nu} ] ]$

Now consider $\Phi = \mathcal{P}_0(N)$ or $\mathcal{P}_1(N) \otimes_{\mathcal{O}} \mathbb{Q}_p$

4 a Dirichlet character mod $N$ or $\mathbb{Q}_p$ with values in $K_0$

\[ \mathcal{M}_K (\Phi, K) = \mathcal{M}_K (\Phi, K_0) \otimes K \]
we replace the $K$ coefficients with $\Omega_k$ if $|f| \leq 1$, i.e., the Fourier coefficients lip in $\Omega_k$.

we similarly define cusp forms, denoted with $\mathcal{S}$.

Let $\Lambda = K \otimes \Omega_k$. Write

$$M(\Delta', A) = \bigcup_{|\Delta'|} \bigoplus_{\Delta \leq \Lambda} \text{Mod}(\Delta', A)$$

and let $\overline{M}(\Delta', A)$ be its completion (closure in $\Lambda[[q]]$)

This is the space of $p$-adic modular forms.

Do the same for $\mathcal{G}$, and for $\mathcal{S}$. 

Roughly, weight \( k \) modular forms are sections of \( \mathcal{W}^k \) on the modular curve.

Thus, they take pairs \((E, \omega)\)

elliptic curve, holomorphic 1-form

\( E \) complex numbers

\( \mathcal{W} \) modular forms, have a similar

\( \mathcal{W} \) modular approach, but \( E \) is now defined

over an algebra \( \mathcal{R} / \mathcal{O}_L \) for \( \mathcal{C} \) a finite

\( \mathfrak{p} \) prime ideal in \( \mathcal{O}_L \)

- \( \mathfrak{p} \) nilpotent in \( \mathcal{R} \)

- \( E \) is supercuspidal, i.e., \( \mathcal{E}_{p-1}(E, \omega) \) invertible

- these take values in \( \mathcal{R} \) rather than \( \mathbb{C} \).
Hida defines Hecke operators via Katz’s approach.

Instead, we restrict to Lafferty’s definition.

Def (Lafferty), \( \chi \) a Dirichlet character mod \( N, \mathbb{Z} \chi \subseteq \mathbb{C} \).

\[
M_\chi(N, \chi, A) = M_\chi(N, \chi; \mathbb{Z}[x]) \otimes_A
\]

is \( A \)-span of \( M_\chi(N, \chi; \mathbb{Q}(x)) \) in \( A[[Q]] \).

Now let \( f = \sum_{n \leq X} \psi(n) a_n \in M_\chi(N, \chi, A) \)

\[
f f \pi_m = \sum_{n \leq X} q_\pi(f \pi_m) \psi(n) \psi(n/m) \quad (f)
\]

where \( q_\pi(f \pi_m) = \sum_{d | \gcd(d, N/m \chi)} \psi(d) \psi^{k-1}(m/m \chi)(f) \).

\[
H_k(N, \chi; A) \hookrightarrow \text{End}(M_k(N, \chi, A))
\]

\[
\text{Hom}(\mathbb{C}, N, \chi; A) \hookrightarrow \text{End}(S_k(N, \chi, A))
\]

We have, as before, perfect pairings \( \text{Hom} \rightarrow A, \text{Hom} \ightarrow A \).
$\Lambda$-adic forms

Hida, "Elementary..."

We are seeking a notion of a family of $p$-adic modular forms parameterized by the weight $K$.

Notation

$\mathfrak{L} = \bigcap_{p \text{ odd}} \mathfrak{L}_p$

$\psi: (\mathbb{Z}/\mathfrak{L})^\times \to \mathbb{Z}_p^\times$ the Teichmüller character

$\mathfrak{O} = \mathfrak{O}_K$ our finite $K(\mathfrak{O}_p, \mathcal{L} = \mathcal{O}[\mathcal{L}])$

$u = 1 + \mathfrak{L}$, a topological generator of $\mathbb{Z}_p^\times$

Des. A $p$-adic quasifinite family of character $\psi$ on

$s$ $\mathfrak{fr}_k$ of $p$-adic modular forms \( s \cdot \mathfrak{fr}_k \)

$s, \mathfrak{fr}_k \in \mathcal{M}_\psi(P_g(N), \psi w^{-k} \mathfrak{L}), p \in \mathfrak{N}^{\geq 2}, \psi = w^a$

$\mathfrak{a}_n(\mathfrak{fr}_k) \in \mathfrak{O}$

$s, \mathfrak{fr}_k \in \mathfrak{O}[[X]]$

$\mathfrak{a}_n(\mathfrak{fr}_k) = A_n(u^{k-1})$
\[ u^{r-1} - 1 = (1 - d)^{r-1}, \text{ which is divisible by } r, \]

so \( u^r - 1 \) is divisible by \( r \).

\[ f = p x (x + \cdots + x^p) \]

then \( f \in \text{End}(F) \) for \( F \) the multiplicative formal group law.

and \( u^r - 1 \) is divisible by \( f \).

\[ \text{Des. } F \in \Lambda[[X]] \text{ is a } \Lambda\text{-adic form of} \]

character 4 is \( f(u^r - 1) \) is the \( q \)-expansion of a \( \Lambda\text{-adic modular form in} \]
\[ \text{Res}(\text{so}(1), 4w^{-1}, 0) \]

for \( q \neq p, \Lambda \in \mathbb{N}. \)

This is cuspidal if it's \( q \neq p, \) cuspidal at specialization

This is a \( \Lambda\text{-adic family of } F(u^r - 1) \) is a classical modular form for \( q \neq p, \Lambda. \)
Back to Eisenstein.

We want a \( \Gamma \)-adic formal parametrization \( \mathbb{E}_h \).

First, we find \( \Phi \) s.t., \( \Phi(u^{k-1}) = q^k \).

Let \( s(z) = \frac{\log(z)}{\log(q)} \), \( \log: \mathbb{Z}_p \to \mathbb{F}_p \).

If \( d \equiv 1 \pmod{q} \) then \( d = u^{s(d)} \), so consider

\[ A_d(x) = d^{-1}(1+x) s(d) \]

Then \( A_d(u^{k-1}) = d^{-1} u^{s(d)k} = d^{k-1} \).

For \( d \equiv 1 \pmod{q} \), recall \( \mathbb{Z}_p \times = \mathbb{Z}_p \times \mathbb{L}_p \)

\[ \mathbb{Z}_p^\times \to \mathbb{L}_p \]

Then let \( A_d(x) = d^{-1} (1+x) s(d) \) for \( p \nmid d \).

\[ A_d(u^{k-1}) = d^{-1} s(u^{(d-1)k}) = u^{(d-1)k} d^{k-1} \] for \( k \equiv 0 \pmod{q(d)} \).
Thus, for \( y = \omega^u \mod \) let

\[ A^4_n(x) = \sum_{d|n} \varphi(d) A_d(x) \]

So that \( A^4_n(u^{k-1}) = \sigma_{n-1}(u) \) is \( k \equiv a \mod \varphi(d) \).

What about \( \varphi_0 \)?

Consider \( L(\zeta, \zeta) = \sum \zeta(n) n^{-s} \to \delta_{\varphi_0} \) Yildin "Elements..." on (formal) group schemes

\[ \Phi_4(x) \in \mathbb{Z}[\zeta][x] \] for \( \Phi_4(\xi x) \to \zeta x \)

\[ \Phi_4(u^{k-1}) = \left\{ \begin{array}{ll}
(1 - 4\omega^{-k} v (1 - v^{-k})) L(1 - v, 4\omega^{-k}) & u \neq i d \\
(1^{k-1}) (1 - v^{-k} (1 - v^{-k})) L(1 - v, \omega^{-k}) & u = i d
\end{array} \right. \]

Thus let \( E(y)(x) = \sum A^4_n(x) y^n \)

\[ A^4_\sigma(x) = \left\{ \begin{array}{ll}
\Phi_4(x)/2 & \sigma \neq \text{id} \\
\Phi_{1x}(x)/2 & \sigma = \text{id}
\end{array} \right. \]

Thus, a \( \Lambda \)-order form parametrizing \( E(y) \).