Let $S$ be a Dedekind scheme

($\text{Noetherian, normal, dim} \leq 1$)

Let $K = K(S)$ the ring of rational fun on $S$

E.g., $K$ a number field, $S = \text{Spec} \, \mathcal{O}$

$S = \text{Spec} \, \mathcal{O}$ a DVR

$S$ = regular curve over a field $K$

t.e., $\mathbb{P}^1_K$, where $K = K(t)$

Let $X_K$ be a scheme / $K$

Def. $X_S \to S$ is an $S$-model (or an integral model) of $X_K$ if we have a null homomorphism $X_K \to X_S$
Let $S = \text{Spec } R$ for a Dedekind domain $R$.

Let $Y_K = \text{Spec } (A_K)$, $A_K = K[x_1, ..., x_n]/I_K$.

Set $T_K = \text{affine scheme } (R, \text{denomination } \text{so that } \text{I}_K)$.

Then let $X_K = R[x_1, ..., x_n]/I_K$.

Then $\text{Spec } (A_K)$ is an $R$-model of $\text{Spec } (A_K)$.

Abstractly, we have $\text{Spec } (A_K) = \text{Spec } (A_K)$ via the inclusion $A_K^n \rightarrow A_K^n$.

Let $X_K$ be smooth, separated, and f.i. $/K$. A Néron model of $X_K$ is a smooth, separated, f.i. $S$-model which is "universal" in the sense that if $Y_S \rightarrow S$ is another $S$-model of $X_K$ then a $Y_K \rightarrow X_K$ we have $Y_S \rightarrow Y_K$.

[Diagram]

\[ Y_K \rightarrow X_K \]
Rmk. \( X_K \) is a sheaf on \( \text{Sm}_S/\text{K} \) with the smooth topology (jointly surjective finite families of smooth maps).

Let \( \mathcal{L} \), spec \( K \rightarrow S \),

Then \( X_K \) is a smooth sheaf on \( \text{Sm}_S/\text{S} \), and \( T^* X_K \). \( A \) N\&\( \ddot{e} \)ron model is a representation of the sheaf \( \mathcal{L} \).

Rmk. \( X_S \rightarrow S \) need not be proper, even if \( X_K \rightarrow \text{Spec} K \) is.

However, if \( X_K \) is a group scheme / \( K \) then \( X_S \) will be a group scheme / \( S \).

Rmk. Let \( X_S \rightarrow S \) an abelian scheme, then \( X \) is the N\&\( \ddot{e} \)ron (proper smooth)

model of \( X_K \rightarrow \text{Spec} K \).

Pf. Existence exist in codimensional \( 1 \) by the valuative criterion of properness.

We will prove first for smooth separated group schemes. One can obtain from codimension \( 1 \),
\[
\begin{cases}
\text{ideal} \ X \longrightarrow G \text{ defines } S \text{ in codim1 and generic pt. then} \\
X_S \longrightarrow S, \quad (x,y) \longmapsto f(x+y)
\end{cases}
\]

is defined on \( u \) and is defined on \( A_X \).
Thus, $A_k$ on abelian $K$ over $\text{URR}$, $K = \text{GF}(R)$ for a DVR $R$.

Then $A_k$ has a $\overline{\text{Neron}}$ model over $R$.

(From Teil), in the above setting $A_k$ has good reduction at $R$ $\iff R \otimes \mathbb{Z}_p(A)$ is unramified at $p$ for some $p \neq \text{char}(R)$.

§2. Elliptic curves and minimal surfaces

Def. Let $X \to S$ be a regular surface, so denote $\text{Schw}$

This is minimal if an birational map from a regular surface $Y$

$Y \to X \leftarrow \text{Yirg regular model of } X$.

$\to S$.

Extends to a morphism $Y_{\text{Yirg}} \to Y_{\text{Xreg}}$ by birational maps of curves, and is hence an isomorphism.

This is remarkable close to the def' of a Neron model, but with assumption of smoothness.
Fact. $X \to S$ is minimal $\iff$ $K_X$ is nef on $S$.

(Remember Castelnuovo's thm which says (-1) curves may be contracted)

Fact. Minimal models of 1-surfaces exist. (Enrique - Kodaira)

Thm. Let $E$ be an elliptic curve over $K = k(s)$ and $\Sigma \to E$ a minimal regular model / $S$.

Let $N \subset \Sigma$ be the smooth points of $\Sigma \to S$. Then $N$ is the Néron model of $E$ over $S$.

Idem: $N(s) \to \Sigma(s) \to E(k)$ are bijective, when allowed extension of $k_N \to E$ in codimension 1.

\[ N(s) \overset{\sim}{\to} \Sigma(s) \text{ as } \Sigma \text{ is regular so \# S-points} \]

Am in the smooth locus,

\[ \Sigma(s) \overset{\sim}{\to} E(k) \text{ by the valuative criterion} \]

\[ \text{spec } K \overset{\sim}{\to} \Sigma \text{ via a choice of point in } E(k) \]

\[ \text{spec } O_{S,s} \overset{\sim}{\to} S \]

\[ \text{spec } O_{\Sigma,s} \overset{\sim}{\to} \Sigma \text{ these two uniquely by separability} \]
Let $X \rightarrow E$, equivalently $\chi \rightarrow \mathcal{N}$,

Let $\chi \in k$ be cyclic, $\chi \simeq \text{Aut} \otimes \mathbb{Q}$.

Then $\mathcal{N}(\chi) = E(K(\chi)) = E(\chi(\chi))$, so if $\chi$ is defined in $\text{cyclic}$.

Example

1. Consider the elliptic curve $E = \mathbb{C} \times \frac{\{1, 4, 9, 16\}}{\mathbb{Q}}$.

This yields a relevant map $\mathbb{P}^2 \rightarrow \mathbb{P}^1$.

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[xy^2, y^3 + y + z^3]
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where generic fiber is indeed $E$.

$\mathbb{P}^2 \rightarrow \mathbb{P}^1$ is a pencil of plane curves w/ indeterminacy

along the locus $\Gamma \propto \mathbb{Z} x y z = 0$, $\chi^3 x y z = 0$, which is a distinct fiber point.

Let $\hat{\mathbb{P}^2} = \mathbb{P}^2 \mathbb{P}^2 \rightarrow \mathbb{P}^3$.

$\mathbb{P}^2$ is the elliptic modular surface at level 3, aka Shioda.
Then $\Pi^2 \rightarrow \Pi^1$ is minimal. 

\[ \exists \gamma \left( \gamma^2 = 3 \right) \]
\[ K_{\gamma^2} = -3 \gamma \]
\[ \bar{c} \leq \Pi^2 \text{ is } 3 \gamma - \sum_{i=1}^{g} \xi_i \]

So that, $K_{\gamma^2} = -3 \gamma + \sum_{i=1}^{g} \xi_i = -\bar{c}$

Compute $K_{\gamma^2}, \quad \bar{c} = 9 \gamma^2 - \sum_{i=1}^{g} \xi_i \gamma^2 = 0$
\[ \mathbb{P}^2 \rightarrow \mathbb{P}^1 \text{ has 4 singular fibers} \]
\[ 0, \ -3, \ -3y, \ -3y^2 \]
\[ \text{product of 3 distinct lines} \]
\[ \text{new four} \]

so all fibers are \[ \begin{array}{c}
\text{lines}
\end{array} \]

Thus, \[ N = \mathbb{P}^2 - 3 \text{ vertices} \] is the Nevan model.

2. \[ E/\Theta \text{ via } y^2 + y = x^3 + 1 \]

\[ W \text{ the closure of } E \text{ in } \mathbb{P}^2 \]
\[ A = -3, 5, 5, \text{ so the singular fibers of } W \text{ are at } p = 2 \text{ and } p = 5. \]

\[ W_p \text{ has a unique singular point } \tilde{E}_p = [0:1:2:1], \text{ results in } W \text{ type II} \]

\[ W_3 \text{ has a unique singular point } [1:1:1] \text{ which is singular in } W. \]

Let \[ E \rightarrow W \text{ the blow up at } C_1(1:1:1) \text{ in } W_3. \]
$\Sigma$ is regular and $\Sigma_2 = \overset{\text{type III}}{\Sigma_2}$

Then $N = \Sigma - \Sigma_2, \Sigma_2)$ is the Néron model of $\mathcal{C}$ over $\mathbb{Z}$. 