Intro to Néron Models

§-1

Refs) Bosch, Lütkebohmert, Raynaud "Néron Models"

Liu - "Algebraic Geometry and Arithmetic Curves"

Cornell, Silverman - "Arithmetic Geometry", particularly Artin's article on Néron models

§0. Setup) Let S be a Dedekind scheme (Noetherian, normal, dim ≤ 1)

Let $K = K(S)$ the ring of rat'l fns on S

e.g. K a number field, $S = \operatorname{Spec} \mathcal{O}_K$

$S = \operatorname{Spec} \mathcal{O}$ a DVR

$S =$ regular curve over a field k (e.g. $\mathbb{P}^1_k$, whence $K = k(t)$)

Let X_K be a scheme $/K$

Def. $X_S \to S$ is an S-model (or an integral model) of X_K if we have a pullback diagram

$$\begin{array}{ccc} X_K & \longrightarrow & X_S \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & S \end{array}$$

e.g. Let $S = \operatorname{Spec} R$ for a Dedekind domain R

Let $X_K = \operatorname{Spec}(A_K)$, $A_K = K[x_1, \dots, x_n]/I_K$.

f.t. affine scheme $/R$.

Clear denominators so that I_K is generated by elements

of $R[x_1, \dots, x_n]$. Then let $A_R = R[x_1, \dots, x_n]/I_R$

for $I_R = I_K \cap R[x_1, \dots, x_n]$.

Then $\operatorname{Spec}(A_R)$ is an R-model of $\operatorname{Spec}(A_K)$

Abstractly, we have $\operatorname{Spec}(A_R) = \overline{\operatorname{Spec}(A_K)}$ via the

inclusion $\mathbb{A}^n_K \hookrightarrow \mathbb{A}^n_R$

$\boxed{\S 1.\text{ Def'n}}$ Let X_K be smooth, separated, and f.t. $/K$. A Néron model of X_K is a smooth, separated, f.t. S-model which is "universal" in the sense that if $Y_S \to S$ is another S-model of X_K then $\forall\, Y_K \to X_K$ we have

$$\begin{tikzcd} Y_S \arrow[r, dashed, "\exists!"] & X_S \\ Y_K \arrow[u] \arrow[r] & X_K \arrow[u] \end{tikzcd}$$

Rmk. X_K is a sheaf on Smooth Sch$/K$ with the smooth topology (jointly surjective finite families of smooth maps).

Let $\iota: \operatorname{Spec} K \longrightarrow S$.

Then $\iota_* X_K$ is a smooth sheaf on SmSch$/S$, and $\iota^* \dashv \iota_*$. A Néron model is a representative of the sheaf $\iota_* X_K$.

Rmk. $X_S \longrightarrow S$ needn't be proper, even if $X_K \to \operatorname{Spec} K$ is. However, if X_K is a group scheme $/K$ then X_S will be a group scheme $/S$.

e.g. let $X_S \longrightarrow S$ an abelian scheme (proper smooth). Then X is the Néron model of $X_K \longrightarrow \operatorname{Spec} K$

Pf. Extensions exist in codim 1 by the valuative criterion of properness. For smooth separated group schemes, one can Weil proved that extend from codimension 1.

(idea: $X \overset{f}{\dashrightarrow} G$ defined is in codim 1 and generic pts, then

$X \times_S X \dashrightarrow G \quad (x,y) \longmapsto f(x) f(y)^{-1}$

is defined on $U \times_S U$ for $U \overset{f}{\dashrightarrow} S$ and is defined on Δ_X)

Thm. A_K an ab. var $/K$, $K = qf(R)$ for a DVR R.

Then A_K has a Néron model over R.

Cor (Serre-Tate). In the above setting A_K has good reduction at $k = R/m$ $\iff$ $T_\ell(A)$ is unramified at m for some $\ell \neq \operatorname{char}(k)$.

§2. Elliptic curves and minimal surfaces

Def. Let $X \to S$ be a regular surface, S a dedekind scheme.

This is minimal if any birational map from a regular surface Y

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

" Y is a regular model of X "

extends to a morphism.

Rmk. $Y_\eta \to X_\eta$ is a birational map of curves and is hence an isomorphism.

This is remarkably close to the def'n of a Néron model, but w/out assumption of smoothness.

Fact. $X \to S$ is minimal $\iff K_X$ is nef$/S$.

(Recall Castelnuovo's thm which says (-1) curves may be contracted).
(smooth complex projective)

Fact. Minimal models of surfaces exist.
(Enriques-Kodaira)

Thm. Let E be an elliptic curve over $K = k(S)$ and $\mathcal{E} \dashrightarrow E$ a minimal regular model $/S$. Let $N \subseteq \mathcal{E}$ be the smooth points of $\mathcal{E} \to S$, then N is the Néron model of E over S.

idea: $N(S) \to \mathcal{E}(S) \to E(K)$ are bijective, which allows extension of $Y_\eta \to E$ in codimension 1.

$N(S) \xrightarrow{\sim} \mathcal{E}(S)$ as $\mathcal{E}$ is regular so its S points are in the smooth locus,

$\mathcal{E}(S) \xrightarrow{\sim} E(K)$ by the valuative criterion

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{E} \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow \\ \operatorname{Spec} \mathcal{O}_{S,s} & \longrightarrow & S \end{array}$$
] via a choice of point in $E(K)$

$\operatorname{Spec} \mathcal{O}_{S,s} \to \mathcal{E}$, these glue uniquely by separatedness

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_{S,s} & \to & \mathcal{E} \\ \cup & \nearrow & \\ U & & \end{array},$$

Let $X_\eta \xrightarrow{f} E$, equivalently $X \dashrightarrow N$.

Let $\xi \in X$ be codim 1, $T = \operatorname{Spec} \mathcal{O}_{X,\xi}$.

Then $N(T) = E_{k(\eta)}(K(T)) = E(K(X))$, so f is defined in codim 1. $\square$

Examples

1. Consider the elliptic curve $E = \{x^3+y^3+z^3 = txyz\} \subseteq \mathbb{P}^3_{\mathbb{C}(t)}$.

This yields a rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ "Hesse pencil"

$$[xyz : x^3+y^3+z^3]$$

whose generic fiber is indeed E.

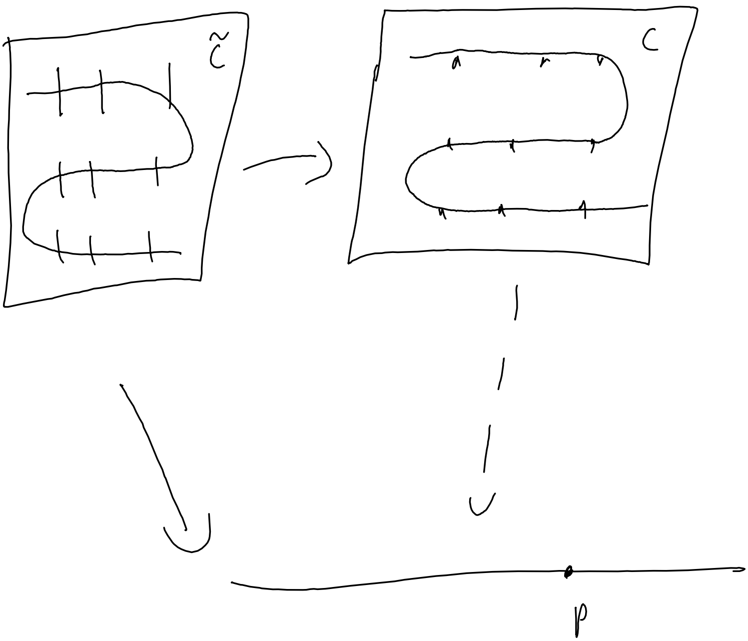
$\mathbb{P}^3_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ is a pencil of plane cubics w/ indeterminacy along the base locus $B = \{xyz = 0,\ x^3+y^3+z^3 = 0\}$ which is 9 distinct points.

Let

$$\begin{tikzcd} \widetilde{\mathbb{P}^2} = Bl_B \mathbb{P}^2 \arrow[r] \arrow[dr, "\pi"'] & \mathbb{P}^3 \arrow[d, dashed] \\ & \mathbb{P}^1 \end{tikzcd}$$

(Rmk: $\widetilde{\mathbb{P}^2}$ is the elliptic modular surface of level 3, a la Shioda)

Then $\widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^1$ is minimal $/\mathbb{P}^1$!



$C \subseteq \mathbb{P}^2$ is 3ℓ

$K_{\mathbb{P}^2}$ is -3ℓ

$\tilde{C} \subseteq \widetilde{\mathbb{P}^2}$ is $3\ell - \sum_{i=1}^{9} E_i$

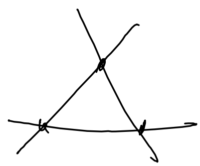
OTOH, $K_{\widetilde{\mathbb{P}^2}} = -3\ell + \sum_{i=1}^{9} E_i = -\tilde{C}$

Compute $K_{\widetilde{\mathbb{P}^2}} \cdot \tilde{C} = 9\ell^2 - \sum_{i=1}^{9} E_i^2 = 0$

$\widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^1$ has 4 singular fibers

$$\underbrace{\infty}_{xyz=0}, \quad \underbrace{-3, \quad -3\zeta_3, \quad -3\zeta_3^2}_{\text{product of 3 distinct linear forms}}$$

So all fibers are



Thus, $N = \widetilde{\mathbb{P}^2} - \{12 \text{ vertices}\}$ is the Néron model.

2. $E/\mathbb{Q}$ via $y^2+y = x^3+1$,

W the closure of E in $\mathbb{P}^2_{\mathbb{Z}}$

$\Delta = -3^3 \cdot 5^2$, so the singular fibers of W are at $p=3$ and $p=5$.

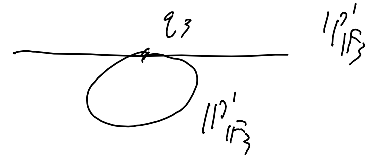
W_5 has a unique singular point $\xi_5 = [0;2;1]$, regular in W

ξ_5

type II

W_3 has a unique singular point $[1;1;1]$ which is singular in W.

Let $\mathcal{E} \longrightarrow W$ be the blowup at $[1;1;1]$ in W_3.

$\mathcal{E}$ is regular and $\mathcal{E}_3$ = 

type III

Then $\mathcal{N} = \mathcal{E} \setminus \{q_3, q_5\}$ is the Néron model of E over $\mathbb{Z}$.