Kodaira's classification + Tate's algorithm

§ Intersection theory

We recall some facts about intersection theory on a regular fibered surface $X \xrightarrow{f} S$.

Rmk. X isn't necessarily complete, so this is wonky as linear equivalence can send an intersection to ∞ where it disappears. This issue is resolved for vertical divisors.

e.g. R a DVR w/ uniformizer $\pi \neq 0$.

$\Gamma_1 = \{x=0\}$, $\Gamma_2 = \{x + \pi^n y\} = 0$ in $\mathbb{P}^1_R$.

Γ_1 and Γ_2 intersect @ $x = \{x = \pi = 0\}$ on the special fiber.

$(\Gamma_1 . \Gamma_2)_x = \dim_k R[x]_{(x)} / (x, x + \pi^n) = n$

Let $\Gamma_3 = \{x=0\} = \Gamma_2 + \operatorname{div}\left(\frac{x + \pi^n y}{y}\right)$. $\Gamma_1 \cap \Gamma_3 = \phi$

but $\Gamma_2 \sim \Gamma_3$. Thus, $\Gamma_1 . \Gamma_2 \neq \Gamma_1 . \Gamma_3$!

Let $S \subseteq S$ closed, let $\mathrm{Div}_S(X)$ be the divisors in X vertical$/S$.

Thm. $\exists!$ $\mathrm{Div}(X) \otimes \mathrm{Div}_S(X) \longrightarrow \mathbb{Z}$ s.t.

i) If D, E have no common components then

$$D.E = \sum_{\substack{x \in X \\ \text{closed}}} i_x(D,E)\,[k(x):k(s)]$$

ii) Symmetric

iii) well def on linear equivalence

iv) $0 < E \leq X_s \Rightarrow D \cdot E = \deg_{k(s)} \mathcal{O}_X(D)|_E$

Thm (Adjunction formula).

Let $0 < E \leq X_s$. Then

$$(K_{X/S} + E) \cdot E = \deg(K_E)$$

Cor. Let $X \xrightarrow{f} S$ minimal, i.e. $K_{X/S} \equiv 0$. Then for

any $\Gamma < X_s$ irreducible, $h^1(\mathcal{O}_\Gamma) = 0$ and $\Gamma^2 = [k' : k(s)]$

where $k' = H^0(\mathcal{O}_\Gamma) = k(\Gamma) \cap \bar{k}$.

Pf idea. By adjunction, $2g(\Gamma) - 2 = \deg(K_\Gamma) = \Gamma^2$, so it suffices

+RR

to show $\Gamma^2 < 0$. If $X_s = \Gamma + \Gamma'$, then $\Gamma \cdot X_s = 0$ and $\Gamma \cdot \Gamma' \geq 0$, so $\Gamma \cdot \Gamma' \leq 0$.

Prop. Let $E \in \mathrm{Div}_s(X)$. $E \cdot X_s = 0$

Pf. Replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, so $X_s = f^*\{x=0\}$ a

principal divisor, hence trivial on $\mathrm{Div}_s(X)$.

§ Fibers

Let R be a DVR w/ fraction field K and residue field k.

Let E/K be an elliptic curve.

Let $\mathcal{E} \longrightarrow \operatorname{Spec} R$ be the <u>minimal</u> (under domination) regular proper model of E over R.

Recall that $\mathcal{N}$ = smooth points of $\mathcal{E}/R$ is the Néron model

Let $\overline{\mathcal{E}} = \mathcal{E} \times_R k$, the special fiber, a group variety$/k$.

Write $\overline{\mathcal{E}} = \sum_{i=1}^{n} d_i \Gamma_i$, $\Gamma_i \subseteq \overline{\mathcal{E}}$ integral curves.

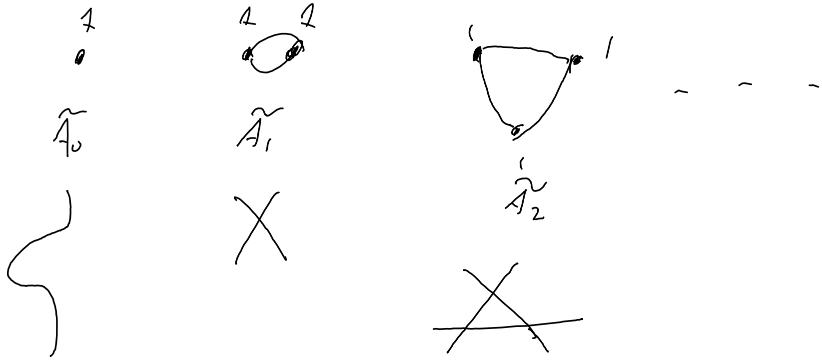
Def. The dual graph of $\overline{\mathcal{E}}$ is an undirected graph $G = G(\overline{\mathcal{E}})$ with vertices $\{\Gamma_1, \ldots, \Gamma_n\}$ s.t. there are $\Gamma_i \cdot \Gamma_j$ edges between any $i \neq j$.

We may additionally weight the vertex Γ_i w/ m_i.

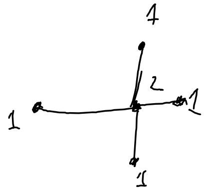
Rmk. $\overline{\mathcal{E}}^2 = 0$, If $r \geq 2$, $\Gamma_i^2 = -2\,[k_i : k]$ for $k_i = H^0(\Gamma_i, \mathcal{O}_{\Gamma_i})$ and Γ_i is a conic$/k_i$.

One can show that the intersection form on $\mathbb{R}\langle \Gamma_1, \ldots, \Gamma_n \rangle$ is negative semidefinite w/ a 1d kernel $\mathbb{R}\langle \overline{\mathcal{E}} \rangle$. By some combinatorial bashing, this forces the structure of the dual graph G to be an extended Dynkin diagram!

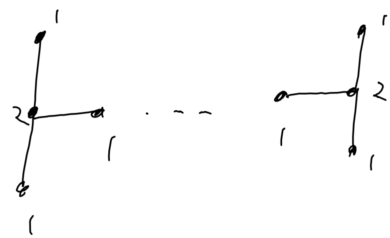
$\tilde{A}_n$ = cycle of $n+1$ vertices with weight 1 per vertex



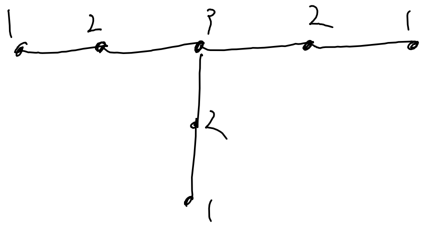
$\tilde{D}_4 =$



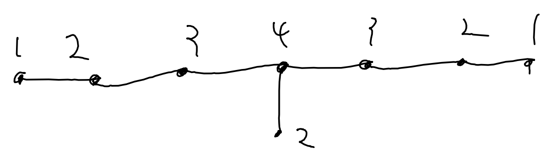
, $\tilde{D}_n =$



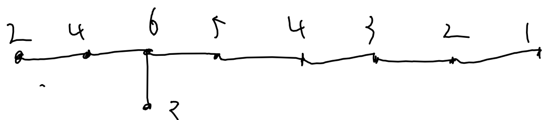
$\tilde{E}_6 =$



$\tilde{E}_7 =$



$\tilde{E}_8 =$



One can use this to deduce the structure of $\bar{\Sigma}$ for $k = \bar{k}$.

Let k be perfect.

Case I. $n=1$, so $\bar{\Sigma}$ is irreducible. By properness, $\Sigma(R) = E(K)$, so Σ has an R point P. Thus, $\tilde{\Sigma}$ has a k point $\tilde{p}$ coming from P. We claim that $\tilde{\Sigma}$ is nonsingular at $\tilde{p}$. Indeed, $p: \operatorname{Spec} R \longrightarrow \Sigma$ splits $\pi: \Sigma \longrightarrow \operatorname{Spec} R$, so we may write

$$m = (\pi \circ p)^*(m) = p^* \pi^* m$$

If $\tilde{\Sigma}$ were singular at $\tilde{p}$, then $\pi^* m \subseteq m^2_{\tilde{\Sigma}, \tilde{p}}$. Then

$$p^* \pi^* m \subseteq p^*(m_{\tilde{\Sigma}, \tilde{p}})^2 = m^2$$

so $m \subseteq m^2$, but R is a DVR $\times$.

Thus, $\tilde{\Sigma}$ has a nonsingular point, so $d_1 = 1$, so $\tilde{\Sigma}$ is integral.

By flatness the arithmetic genus of $\tilde{\Sigma}$ is 1. In the $k = \bar{k}$ case, this simply means that $\tilde{\Sigma}$ is an elliptic curve (I_0), a node (I_1) and cusp (II). More generally,

$C \xrightarrow{f} \tilde{\Sigma}$ be the normalization. Consider

$$0 \to \mathcal{O}_{\tilde{\Sigma}} \to f_* \mathcal{O}_C \to S \to 0,$$

when S is now a skyscraper sheaf supported at the singular locus of Γ_1.

Then by LES,

$$0 \to \underbrace{k}_{\substack{\tilde{\Sigma} \text{ has} \\ \text{a } k\text{-point} \\ \text{as above}}} \to k \to H^0(S) \to H^1(\tilde{\Sigma}) \to H^1(\mathcal{O}_C) \to 0$$

$h^1(\tilde{\Sigma}) = 1$ and $S \neq 0$, so $h^1(\mathcal{O}_C) = 0$, thus $C \cong \mathbb{P}^1_k$

and $h^0(S) = 1$, so $\exists!$ singular point $q \in \tilde{\Sigma}(k)$.

$f^{-1}[q] = C_q$ on underlying spaces.

$$C_q = \operatorname{Spec}((f_* \mathcal{O}_C)_q \otimes k(q))$$

and by the SES at stalk q, $\dfrac{(f_* \mathcal{O}_C)_q}{\mathcal{O}_{\tilde{\Sigma},q}} \cong H^0(S)$,

a 1-dim'l k, so $h^0(C_q) = 2$ $(/k)$.

$\therefore C_q =$ 2 k-points $\hookrightarrow \tilde{\Sigma}$ a node w/ split multiplicative reduction, I_1

= a quadratic point $\hookrightarrow \tilde{\Sigma}_{k(q)}$ a node, so non-split multiplicative reduction, $I_{1,2}$

= double k-point $\hookrightarrow \tilde{\Sigma}$ a cusp w/ additive reduction II, $\square$

Now, suppose $n \geq 2$.

The argument in case 1 shows that whatever irreducible component Γ_i contains said k-point will be integral and have $k_i = k$. WLOG let this be Γ_1, so

$$\Sigma = \Gamma_1 + \sum_{i=2}^{n} d_i \Gamma_i$$

Let $r_i = [k_i : k]$.

Lemma. $2 r_i d_i = \sum_{\substack{1 \leq j \leq n \\ i \neq j}} d_j \Gamma_i . \Gamma_j$

Pf. $0 = \Gamma_i . \Sigma = \sum_{1 \leq j \leq n} d_j \Gamma_i . \Gamma_j$

$$= d_i \Gamma_i^2 + \sum_{\substack{j \\ j \neq i}} d_j \Gamma_i . \Gamma_j$$

and $\Gamma_i^2 = -2 r_i$. □

As discussed, $d_1 = r_1 = 1$, so we also have

$$2 = \sum_{i=2}^{n} d_i \Gamma_1 . \Gamma_i \qquad (+)$$

So Γ_1 must intersect the remaining components in at least 2 points.

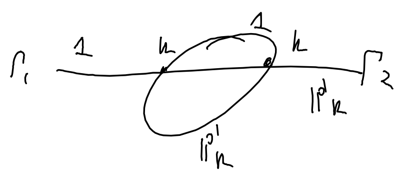
Case 2. Γ_1 meets the remaining Γ_i in exactly two distinct points.

Then these points are k-rational and the intersections are transverse.

If $n=2$, $Z = \Gamma_1 + d_2 \Gamma_2$. Hence, $k_2 = k$, so

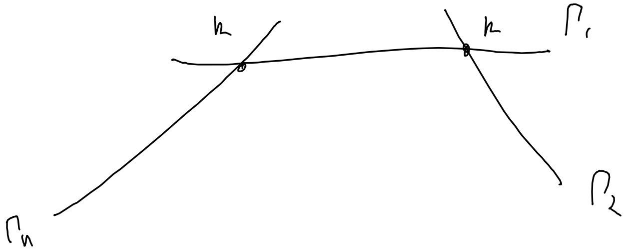
	Γ_1	Γ_2
Γ_1	-2	2
Γ_2	2	-2

By the lemma, $d_2 = 1$



I_2

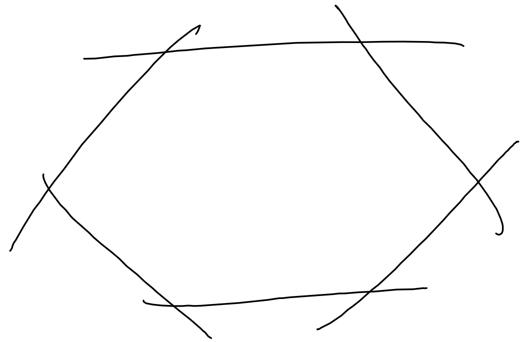
For $n \geq 3$, so Γ_1 intersects Γ_2 and Γ_n.



By $(*)$, $2 = d_2 + d_n$ so $d_2 = d_n = 1$.

Repeat this argument replacing Γ_1 by Γ_2, which intersects precisely the other Γ_i, when $i>2$.

Thus, we end up with



I_4

$\tilde{A}_4$

- all $\Gamma_i \cong \mathbb{P}^1_k$
- $\Gamma_i.\Gamma_j = \begin{cases} 1 & i\neq j \\ -2 & i=j \end{cases}$
- $d_i = 1 \ \forall i$
- Σ is split multiplicative with $\pi_0(\Sigma)(k) \cong \mathbb{Z}/4\mathbb{Z}$

Case 3. Γ_1 intersects the other components at a quadratic point p, i.e. $[k(p):k]=2$. WLOG it meets Γ_2 at p.

Then $\Gamma_1\Gamma_2 = 2$, so by $(*)$, $\Gamma_1.\Gamma_i = 0 \ \forall i\geq 3$ and $d_2 = 1$. Furthermore, $k_2 \subseteq k(p)$.

3.i) $k_2 = k(p)$. Then $\Gamma_2 \cong \mathbb{P}^1_{k(p)}$ and by the lemma

$$4 = \Gamma_1.\Gamma_2 + \sum_{i\geq 3} d_i \Gamma_2.\Gamma_i$$

so as $\Gamma_1\Gamma_2 = 2$, $2 = \sum_{i\geq 3} d_i \Gamma_2\Gamma_i$. This can only happen for $d_3=1$, $\Gamma_2\Gamma_3 = 2$, $\Gamma_2\Gamma_i = 0 \ \forall i\geq 4$. That is, Γ_2 intersects Γ_3

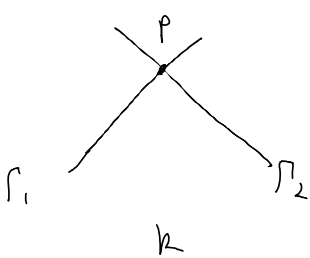
at a unique regular point q with $k(q)=k(p)$.

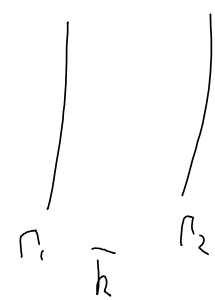
Repeat this for Γ_2 replaced with Γ_3, and so on.

So $\Gamma_i \cong \mathbb{P}^1_{k(p)}$ $\forall\, 1 \leq i \leq n$ and Γ_i, Γ_{i+1} intersect in a unique point defined over $k(p)$. And also $d_i = 1$.

As for Γ_n, it's a conic $/k$ intersecting only Γ_{n-1} at a unique $k(p)$-point. If Γ_n is singular, we call this $I_{2n-1,2}$. Else we call it $I_{2n-2,2}$. Indeed, over $\bar{k}$ these double points split in 2 and all but Γ_1 also split in 2. $\bar{\Sigma}$ is then non-split multiplicative with $\pi_0(\bar{\Sigma})(\bar{k}) \cong \mathbb{Z}/2n-2$ or $\mathbb{Z}/2n-1$.

3,ii) If $k_2 = k$ then $r_2 = 1$, so by the lemma we have that $\Gamma_2 . \Gamma_i = 0$ $\forall\, i \geq 3$. Thus, $\Gamma_1 + \Gamma_2$ is a connected component of $\bar{\Sigma}$. But by Zariski's main theorem, $\bar{\Sigma}$ is connected. Thus, $n = 2$.





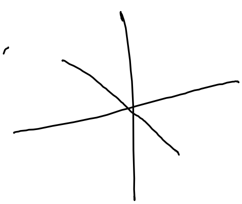
(gap! what is this type? If we assume geometric connectedness, it doesn't occur)

Case 4. Γ_1 meets the others at a unique k-point P.

4.i) Say Γ_1 meets two components at P, say Γ_2, Γ_3.

$$2 = d_2 \Gamma_1 \Gamma_2 + d_3 \Gamma_1 \Gamma_2 + \sum_{i \geq 4} d_i \Gamma_1 \Gamma_i$$

so Γ_1 only intersects Γ_2 and Γ_3, and $d_2 = d_3 = 1$.

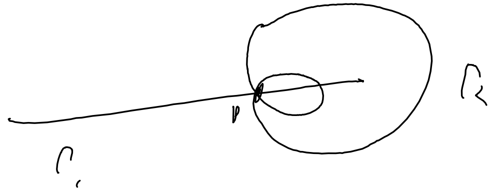


type IV split additive reduction.

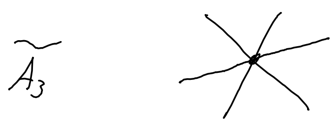
$$\tilde{A}_2$$

4.ii) Γ_1 meets only Γ_2

$2 = d_2 \Gamma_1 \Gamma_2$. If $\Gamma_1\Gamma_2 = 2$, $d_2 = 1$ and Γ_2 is a conic $/k$. If Γ_2 is singular, we call this type IV_2.



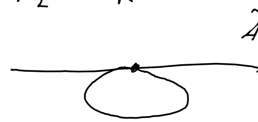
Singularity means the conic degenerates, so in $\bar{k}$ it becomes 2 lines through p. Whence,



over $\bar{k}$, which is type IV.

Here, $\mathcal{E}$ has non-split additive reduction and $\pi_0(\mathcal{E})(\bar{k}) \cong \mathbb{Z}/3\mathbb{Z}$.

If Γ_2 is smooth, $\Gamma_2 \cong \mathbb{P}^1_k$ and $\Gamma_1 \cap \Gamma_2 = \{p\}$ is k-rational. As $\Gamma_1\Gamma_2 = 2$, we get



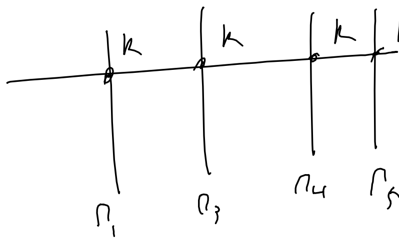
, called type III.

This is split additive reduction and $\pi_0(\mathcal{E})(k) \cong \mathbb{Z}/2\mathbb{Z}$.

Now suppose $\Gamma_1\Gamma_2 = 1$ and $d_2 = 2$. Then $\Gamma_2 \cong \mathbb{P}^1_k$ and $3 = \sum_{i \geq 3} d_i \Gamma_2 \Gamma_i$. So there's at most $\Gamma_3, \Gamma_4, \Gamma_5$ left, so $3 = d_3\Gamma_2\Gamma_3 + d_4\Gamma_2\Gamma_4 + d_5\Gamma_2\Gamma_5$.

– all $d_i = 1$ Then the cases are

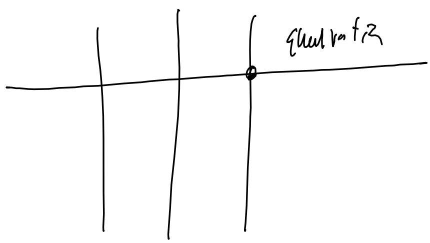
$\tilde{D}_4$



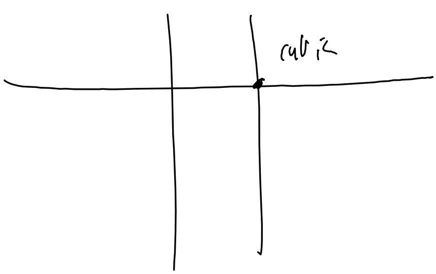
I_0^*

split additive

$\pi_0(\mathcal{E})(k) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$



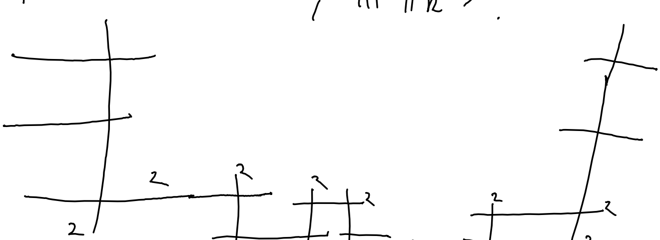
$I^*_{0,2}$ non-split



$I^*_{0,3}$ non-split

– $d_3 = 2$, all intersections rational, all $\mathbb{P}^1$'s.

$\tilde{D}_{n-1}$ I^*_{n-5}

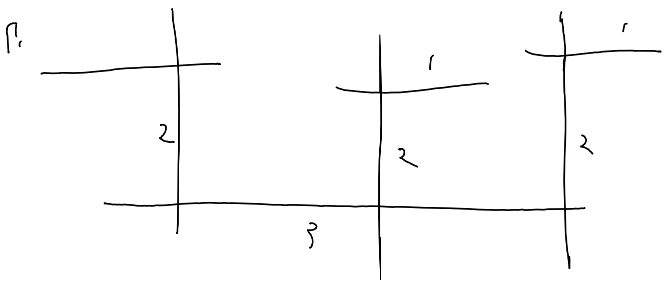


- $d_3 = 2$

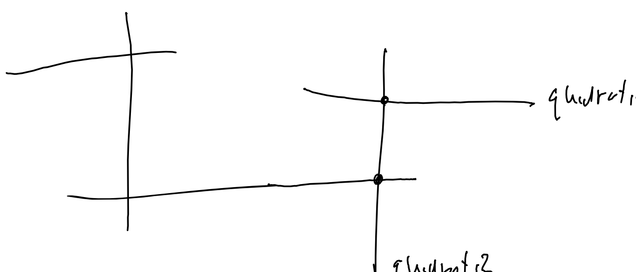
$I^*_{n=4,2}$

same multiplicities

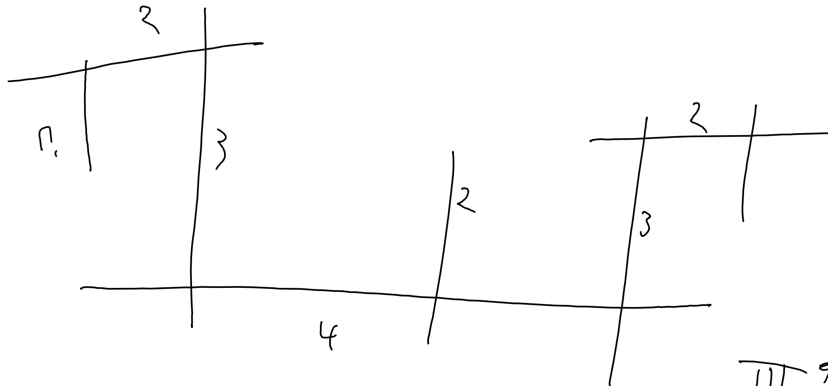
- $d_3 = 3$, so Γ_2 only meets Γ_1, Γ_3



$IV^* \sim E_6$

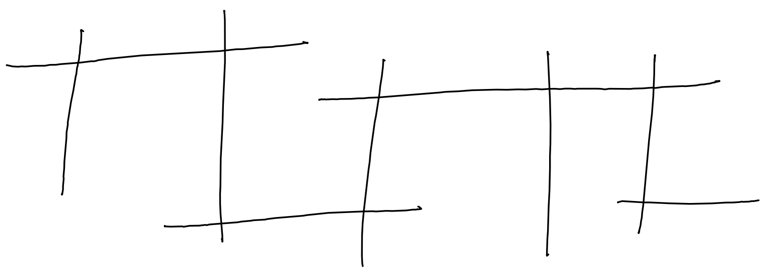


IV_2^*



$\tilde{E}_7$

III^*



$\tilde{E}_8$

II^*

Tate's algorithm

Input E/K in Weierstrass form

Output - Kodaira symbol of reduction type in special fibre

- $\#$ $\bar{k}$ components of $\bar{\mathcal{E}}$
- $v(\Delta_{E/K})$, the valuation of the minimal discriminant
- $f(E/K)$ the exponent of the conductor

$$\shortparallel$$

$$\dim_{\mathbb{Q}_\ell}\left(V_\ell(E)/V_\ell(E)^{I}\right) - \sum_{i=1}^{\infty} \frac{g_i(L/K)}{g_0(L/K)} \dim_{\mathbb{F}_\ell}\left(\frac{E[\ell]}{E[\ell]^{g_i(L/K)}}\right)$$

Rmk, if $\mathrm{char}(k) \neq 2,3$ there is no wild inertia so the latter sum is 0.

- $c(E/K) = |\pi_0(\bar{\mathcal{E}})(k)|$

- minimal Weierstrass model

The method is to keep getting larger valuations on the coefficients, exiting to Kodaira symbols when possible, and iterating by a change of variable, to get a smaller discriminant.

Rmks. i) Recall via formal groups that

$$\mathfrak{m} \xrightarrow{\sim} E_1(K)$$

and $\omega = dz(1+O(z))$

Hence, $\int_{E_1(K)} |\omega| = \int_{\mathfrak{m}} |dz|$

$$(\text{Haar}) = \frac{\int_R |dz|}{[R:\mathfrak{m}]}$$

$$= 1/q, \quad q = |k|$$

also, $L_v(1) = q/N_v$, $N_v = |\tilde{\mathcal{E}}^0(k)|$

(seems told Tate this?)

$N_v = [E_0(K) : E_1(K)]$, so

$$\int_{E_0(K)} |\omega| = \frac{1}{L_v(1)}, \quad \text{thus}$$

p-adic period $\rightarrow$ $\int_{E(K)} |\omega| = \dfrac{[E(K) : E_0(K)]}{L_v(1)}$ $\leftarrow$ BSD fudge factor

$L_v(1)$ $\leftarrow$ special value

ii) Ogg's formula $f(E/K) = v(\Delta_{E/K}) - m(E/K) + 1$

Proven via case analysis on the Kodaira types

iii) The proof of Tate's algorithm shows that if W is the minimal Weierstrass model of E and $W^0 \subseteq W$ the smooth points / R, then W^0 is the connected component at the identity of $\mathcal{N}$.