Setup

$S$ a Dedekind scheme of dimension 1, $K = k(S)$ the function field.

$X_K$ a scheme smooth, sep, fil. / $K$

Def. A Néron Model of $X_K$ is a scheme which is smooth, sep, fil. / $S$ fil.

- The generic fiber of $X_K$ is $X_K$
- A $Y_S \to S$ smooth we have

$$\text{Hom}_S(X_S, X_S) \overset{\sim}{\longrightarrow} \text{Hom}_K(X_K, X_K)$$

Rmk. i) Let $Y = S$. The above then says

$$X_S(S) \overset{\sim}{\longrightarrow} X_K(K)$$

This is a restricted valuative criterion as $X_S \to S$ need not be proper.

ii) If $X_K$ is a $K$-group then letting $Y = X_S \times_S X_S$, there is a unique lift $M : X_K \times_S X_K \to X_K$, so $X_S$ is a $S$-group.
§1. Néron Models of Elliptic Curves

Def. Let $C$ be a curve $/K$

An $\mathcal{P}$-model of $C$ over $S$ is an $S$-scheme $X \to S$ at dimension 2

with generic fiber $C$, satisfying properties $\mathcal{P}$ (regular, proper, etc.)

A morphism of $\mathcal{P}$-models is a map

\[
\begin{array}{c}
X \\
\downarrow \phi \\
S
\end{array} \to
\begin{array}{c}
Y \\
\downarrow \psi \\
S
\end{array}
\]

which is an isomorphism on the generic fiber, i.e., $\psi \circ \phi = 1_S$, $\phi \circ \psi = 1_S$.

Def. An $\mathcal{P}$-model $X \to S$ is minimal if it only dominates itself, i.e., any $X \to Y$ of models isomorphic.

Thm. Let $C$ be a smooth curve of genus at least 1,

Then a minimal regular proper model exists and it dominates all regular proper models.

Recipe. Prep. Take any proper model $X \to S$ of $C$; e.g., a projective closure.

Step 1. Normalize $X$, a finitely many changes

(should use a curve)

Step 2. Resolve the singularities of $X$ by blowup, only finitely many fibers are singular, and only finitely many points in each fiber are singular.
Step 3: Normalize again. Now we have a regular proper model of $C$.

Step 4: Contract exceptional curves. This terminology as, say, the (Krishnamoorthy) relative Picard number disaster at each step.

Now, we may form $N\ell$-torus models.

Let $E$ be an elliptic curve over $K$ (with a point in $E(K)$).
Let $\Sigma \to S$ be its minimal regular proper model.
Let $N \to S$ be the open subscheme at point smooth over $S$.

Thm: $N \to S$ is the $N\ell$-torus model of $E/K$.

Ps. Lemma 1: $N(S) \to \Sigma(S) \to E(K)$ are all bijective.

Ps. 1: Clearly injective. Let $s \to \Sigma$ a section. By

regularity of $\Sigma$, the image of $s$ lands in the smooth locus of $\Sigma$, which is $N$.

(2) This is injective by separability,

let $s \to E$ a section. Let $s \in S$. By properties of $E$, $x$ extends to $S$. Thus this

also extends to $U \to \Sigma$ for some $U \subseteq S$. But $S$ is open, then

there uniquely glue to a section $s \to \Sigma$. Hence this

is onto.

\qed
For any nilpotent group, the nilpotent series is a series of subgroups where each subgroup is a normal subgroup of the next. Let $N$ be the nilpotent group, and let $S$ be a normal series of subgroups of $N$. Then $S$ is a nilpotent group.

Next, let $f : S \rightarrow S'$ be a homomorphism. This is an automorphism, thereby extending to an automorphism.

Furthermore, a homomorphism is defined as a map $f : S \rightarrow S'$ such that $f$ is a homomorphism. Therefore, the projection from $f$ to $N$ is a homomorphism, and the claim holds.

Indeed, the homomorphism $f : S \rightarrow S'$ is an automorphism.
Now we translate $U$ around $N$. Consider $\Sigma x_5 t_x(u) \xrightarrow{t'} \Sigma x_5 t_x(u)$

$$(a, t_x(b)) \xrightarrow{} (t_x \circ t')(t(a, b))$$

$$(q, t_x(b)) \xrightarrow{} (x \circ q + 1, x)$$

$t\big|_{\Sigma x_5 k} = t\big|_{\Sigma x_5 k} \iff t\big|_{\Sigma x_5 (u \circ t_x(b))} = t\big|_{\Sigma x_5 (u \circ t_x(b))}$

Thus, via $t'$, $t$ extends to $\Sigma x_5 t_x(u)$.
- Let $Z_s \in N_s$, $y_s \in U_s$.

- By construction, $Z_s$, $y_s$ lift to $Z, y \in N(s)$.

- Let $x_s = b_y^{-1}(Z)$, so $b_x(y) = b_y(x) = Z$, whence $Z_s = b_x(y_s) \in b_y(y)$ as desired.

- Finally, \[ N \rightarrow N \]

\[ \begin{array}{ccc}
\cap & \cap \\
\downarrow & \\
\Sigma x_s N & \rightarrow & \Sigma Z_s N
\end{array} \]

With these lemmas, let $Y \rightarrow S$ be smooth and let $f: Y \rightarrow N = E$.

This is a rational map $Y \dashrightarrow N$. Take a codim 1 point $\lambda \in Y_s$ and let $\overline{\lambda} = \text{Spec } O_{Y, \lambda}$. As in lemma 2, $\Sigma x_s \rightarrow \overline{\lambda}$ is minimal with smooth locus $N_{\overline{\lambda}}$.

The by lemma 1, \[ N(\overline{\lambda}) = N_{\overline{\lambda}}(\overline{\lambda}) \rightarrow E_{k(\overline{\lambda})}(k(\overline{\lambda})) = E_f(k(\overline{\lambda})) \]

so if $f(\overline{\lambda}) = \lambda$, then $\theta(\lambda) \leq 1$.
Lemma 7. Let \( G \rightarrow S \) be a smooth \( \mathbb{G}_m \)-equivariant group scheme over \( S \) normal Noetherian.

Let \( \pi : X \rightarrow S \) smooth and \( f : X \rightarrow G \) be defined in codimension \( 1 \).

Then \( f \) is defined everywhere.

**pf.** Let \( f \) be defined on \( U \), \( X \cap U \rightarrow G \)

\[(g_1, g_2) \mapsto f(g_1)f(g_2)^{-1}\]

be defined on \( V \), so \( U \cap V \subseteq V \),

Show that \( \Delta \subseteq U \cap V = U \).

Then, normality constrains things to codim \( 2 \), as \( A = \bigcup_{i=1}^{n} \text{ht}(I_i) \)

\[x \mapsto \text{Sm}(A) \] is undefined in \( \text{ht}(I_i) \) codim \( 2 \).

\[\alpha \]

Apply \( f \) to our group \( N \) to extend.
§ 2 Fibers

Kodaira classified singular fibers of minimal elliptic surfaces over \( \mathbb{C} \).

Néron did the same classification for bad reduction of \( E/K \) over a finite field.

Tate constructed an algorithm to deduce the structure of a fiber in a Weierstrass model.

Tate's paper includes a table listing, in particular, given data.

For any fiber \( \mathfrak{g} = N \), we have that \( G_{ij} \in E, \) \( \mathfrak{g} \), or \( \mathfrak{g} m \) over the residue field. (Rmk. \( J_1 \) for \( n \geq 2 \) will be a non-split torus, i.e., a torus in a quadratic extension of \( K \).)

\( \mathfrak{g}_{ij} \) will be finite abelian, and \( G_{ij} \) an extension of \( \mathfrak{g} \) by \( \mathfrak{g}_{ij} \).

Wikipedia ("Elliptic Surfaces") has a table of monodromy groups such as Néron-Ogg-Shafarevich, archimedean and arithmetic.