§0. Setup

S a Dedekind scheme of dimension 1,
$K = k(S)$ the function field,

X_K a scheme smooth, sep, f.t. $/K$

Def. A Néron model of X_K is a scheme which is
smooth, sep, f.t. $/S$ s.t.

- The generic fiber of X_S is X_K
- $\forall\, Y_S \to S$ smooth we have

$$\mathrm{Hom}_S(Y_S, X_S) \xrightarrow{\sim} \mathrm{Hom}_K(Y_K, X_K)$$

Rmk. i) Let $Y = S$. The above then says

$$X_S(S) \xrightarrow{\sim} X_K(K)$$

This is a restricted valuative criterion, as $X_S \to S$ need not be proper.

ii) If X_K is a K-group then letting $Y = X_S \times_S X_S$, there is a unique lift of $m: X_K \times_K X_K \longrightarrow X_K$, so X_S is an S-group.

§1. Néron Models of Elliptic Curves

Def. Let C be a curve$/K$

A P-model of C over S is an S-scheme $X \longrightarrow S$ of dimension 2

w/ generic fiber C, satisfying properties P (regular, proper, etc.)

A morphism of P-models is a map

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

which is an isomorphism on the generic fiber, we say X dominates Y.

Def. A P-model $X \longrightarrow S$ is minimal if it only dominates

itself, i.e. any

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of models is an iso.

Thm. Let C be a smooth curve of genus at least 1,

Then a minimal regular proper model exists and is dominated

by all regular proper models,

Recipe. Prep. Take some proper model $X \longrightarrow S$ of C, e.g. a projective closure.

Step 1. Normalize X, a finite change

(normal coverings) Step 2. Resolve the singularities of X by blowup. Only finitely many fibers are singular and only finitely many points in each fiber are singular.

Step 3. Normalize again. Now we have a regular proper model of C

Step 4. Contract exceptional curves. This terminates as, say, the relative Picard number ($K_{X_{/S}}$-negative) drops at each step.

Now, we may form Néron models.

Let E be an elliptic curve over K (w/ a point in $E(K)$).

Let $\mathcal{E} \longrightarrow S$ be its minimal regular proper model.

Let $N \longrightarrow S$ be the open subscheme of points smooth over S.

Thm. $N \longrightarrow S$ is the Néron model of E/K.

Pf. Lemma 1. $N(S) \xrightarrow{①} \mathcal{E}(S) \xrightarrow{②} E(K)$ are all bijective.

Pf. ① Clearly injective. Let $S \longrightarrow \mathcal{E}$ a section. By regularity of $\mathcal{E}$, the image of σ lands in the smooth locus of $\mathcal{E}$, which is N.

② This is injective by separatedness.

Let $\operatorname{Spec} K \longrightarrow E$ a section. Let $s \in S$. By properness of $\mathcal{E}$, σ extends to $\operatorname{Spec} \mathcal{O}_{S,s} \longrightarrow \mathcal{E}$. Then this extends to $U \longrightarrow \mathcal{E}$ for a nbhd $s \in U \subseteq S$. By separatedness, these uniquely glue to a section $S \longrightarrow \mathcal{E}$. Hence, this is onto. $\square$

Lemma 2. N/S is a group.

Pf. Step 1. Let $x \in \Sigma(S)$. Then $x_K : E \xrightarrow{\sim} E$ by translation extends to $t_x : \Sigma \xrightarrow{\sim} \Sigma$. Indeed, x_K yields $\Sigma \dashrightarrow \Sigma$ which extends by minimality.

Step 2. $E \times_K E \xrightarrow{\sim} E \times_K E$ extends to an automorphism
$(a,b) \longmapsto (a+b, b)$

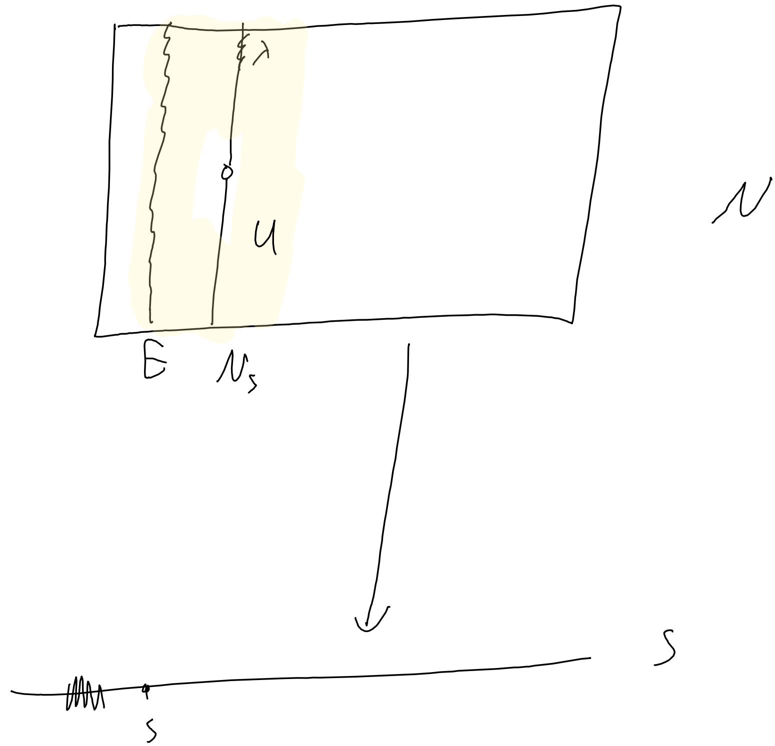
$\Sigma \times_S N \longrightarrow \Sigma \times_S N$

Indeed, have $\Sigma \times_S N \overset{t}{\dashrightarrow} \Sigma \times_S N$, which we extend to a morphism.

- This is local in S, so take $S = \mathrm{Spec}(R)$.

  Furthermore, a morphism f is defined everywhere iff the projection from the graph $\Gamma_f \longrightarrow \mathrm{dom}(f)$ is an isomorphism. This can be checked via faithfully flat descent, so WLOG R is a complete DVR w/ sep closed residue field, via completion and étale base change.

- Let N_s be the special fiber and λ a generic point of N_s. Let $T = \mathrm{Spec}\, \mathcal{O}_{N,\lambda}$. Then $\Sigma \times_S T \longrightarrow T$ is a minimal regular proper surface, so as before t extends to $\Sigma \times_S T \underset{\sim}{\xrightarrow{t}} \Sigma \times_S T$, so t extends to $\Sigma \times_S U$ for some nbhd $\lambda \in U$ s.t. $E \subseteq U$.



- Now we translate U around N

Consider $\Sigma \gamma_s\, t_\gamma(U) \xrightarrow{t'} \Sigma \gamma_s\, t_\gamma(U)$

$(a, t_x(b)) \longmapsto (t_\gamma \wedge t_\gamma)(t(a,b))$

" $(a, x+b) \longmapsto (x+a+b, x)$ "

$t'|_{\text{space}} = t|_{\text{space}} \;\therefore\; t'|_{\Sigma\gamma_s(U \cap t_\gamma(U))} = t|_{\Sigma\gamma_s(U \cap t_\gamma(U))}$

Thus via t', t extends to $\Sigma \gamma_s\, t_\gamma(U)$

- claim $\bigcup_x t_x(U) = N$

Let $z_s \in N_s$, $y \in U_s$.

By completeness, z_s, y_s lift to $z, y \in N(S)$

let $x = t_y^{-1}(z)$, so $t_x(y) = t_y(x) = z$, whence

$z_s = t_x(y_s) \in t_x(U)$ as desired

- Finally

$$\begin{tikzcd} N \times_S N \arrow[r] \arrow[d, hook] & N \times_S N \arrow[d, hook] \\ \Sigma \times_S N \arrow[r] & \Sigma \times_S N \end{tikzcd}$$

$\square$

With these lemmas, let $Y \to S$ be smooth and

let $f: Y_\eta \to N_\eta = E$

This is a rational map $Y \dashrightarrow N$. Take a codim 1

point $\lambda \in Y_s$ and let $T = \operatorname{Spec} \mathcal{O}_{Y,\lambda}$. As in lemma

2, $\Sigma \times_S T \to T$ is minimal with smooth locus $N \times_S T$.

Thus by lemma 1, $N(T) = N_T(T) \xrightarrow{\sim} E_{K(T)}(K(T)) = E(K(Y))$,

(defined at λ $\longleftrightarrow$ f)

so f extends to λ.

Lemma ?. Let $G \longrightarrow S$ a smooth separated group scheme over S normal Noetherian.

Let $Y \longrightarrow S$ smooth and $f : Y \dashrightarrow G$ be defined in codimension 1.

Then f is defined everywhere.

Pf. Let f be defined on U, $Y \times_S Y \dashrightarrow G$

$$(y_1, y_2) \longmapsto f(y_1) f(y_2)^{-1}$$

be defined on V, so $U \times_S U \subseteq V$.

Show that $\Delta \subseteq U \times_S U = V$.

Idea. Normality constrains things to codim 2, as $A = \bigcap_{ht(p)=1} A_p$

$(X$ normal$)$

so a map $X' \dashrightarrow \operatorname{Spec}(A)$ is undefined in pure codim ≤ 1. $\square$

Apply this to our group N to extend. $\square$

§2 Fibers

Kodaira classified singular fibers of minimal elliptic surfaces / $\mathbb{C}$

Néron did the same classification for bad reduction of E/K # field,

Tate constructed an algorithm to deduce the structure of a fiber in a Weierstrass model.

Tate's paper includes a table listing, in particular, group laws,

For any fiber $G = \mathcal{N}_s$, we have that G^0 is E, $\mathbb{G}_a$, or $\mathbb{G}_m$ over the residue field. (Rmk. $I_{n,2}$ for $n \geq 1$ will be a nonsplit torus, i.e. a torus in a quadratic ext'n of $k(s)$) $\pi_0(G)$ will be finite abelian, and G is an extension of G^0 by $\pi_0(G)$

Wikipedia ("Elliptic Surfaces") has a table of monodromy

C.f. Néron - Ogg - Shafarevich, archimedean and arithmetic.