Motivic Cohomology of Fields

§1. Introducing the players

§2. Computations for fields

§1.

We are interested in

- algebraic K-theory
- Milnor K-theory
- Motivic cohomology
- Étale cohomology

"Definitions"

Algebraic K-theory

Grothendieck defined, for a scheme X,

$$K_0(X) = \mathbb{Z}\langle \mathrm{Coh}(X) \mid A + C = B \text{ if } 0 \to A \to B \to C \to 0 \rangle$$

This is the universal additive invariant of coherent sheaves on X.

For X smooth quasiprojective $/k$ a field, we can replace $\mathrm{Coh}(X)$ with $\mathrm{LocFree}(X)$.

e.g. $\chi : K_0(X) \longrightarrow \mathbb{Z}$ for X proper $/k$

$f_! : K_0(X) \longrightarrow K_0(Y)$ for f proper

$F \longmapsto \sum (-1)^i R^i f_* F$

Quillen developed higher K-theory by associating a space ΩBQC to any exact category C.

For $C = \mathrm{Coh}(X)$, $K(X) = \Omega BQC$ a $K_i(X) = \pi_i(K(X))$.

This recovers $K_0(X)$ as usual.

$K_1(\operatorname{Spec} A) = GL(A)^{ab}$

$$K_2(\operatorname{Spec} F) = \frac{F^\times \otimes F^\times}{\langle a \otimes (1-a) \rangle}$$ by Matsumoto.

Quillen's construction has the property that if $B \subseteq A$ is an abelian subcategory closed under subs, quotients, and extensions

then

$$K(B) \to K(A) \longrightarrow K(A/B)$$

is a homotopy fiber sequence, inducing a LES on homotopy groups.

For $Z \xrightarrow[\text{closed}]{i} X \xleftarrow[\text{open}]{j} U$

take $A = \operatorname{Coh}(X)$, $B = \operatorname{Coh}_Z(X) = \{F|_U = 0\}$.

Then $A/B = \operatorname{Coh}(U)$. Hence, we get a localization sequence

$$\cdots \longrightarrow K_n(Z) \to K_n(X) \to K_n(U) \to K_{n-1}(Z) \to \cdots$$

If $X = U \cup V$, $X - U \hookrightarrow X$ and $V - (X-U) \hookrightarrow V$ yield Mayer-Vietoris.

Millnor K-theory

$$K^M(F) = \frac{T(F)}{\langle a \otimes (1-a) \mid a \neq 0,1 \rangle}$$

$$= \frac{\bigoplus_{n \geq 0} F^{\times \otimes n}}{\langle a \otimes (1-a) \mid a \neq 0,1 \rangle}$$

$$K_0^M(F) = \mathbb{Z}$$

$$K_1^M(F) = F^\times$$

$$K_2^M(F) = \frac{F^\times \otimes F^\times}{\langle a \otimes (1-a) \rangle}$$

Motivic Cohomology

For $X \to \operatorname{Spec} k$ smooth, f.t.,

we define

$$H^i(X, \mathbb{Z}(j)) := CH^j(X, 2j-i)$$

Rmk. In $X \to \operatorname{Spec} k$ is merely f.t.,

we have

$$H_{2i}(X, \mathbb{Z}(i)) = CH_i(X)$$

In the smooth setting, Poincaré duality

would imply then

$$H^{2i}(X, \mathbb{Z}(i)) = CH^i(X)$$

Étale cohomology

μ_m is the étale sheaf

$$U \longmapsto \mu_m(H^0(U, \mathcal{O}_U^*))$$

We will consider the cohomology

$$H^i_{\acute{e}t}(X; \mu_m^{\otimes j})$$

For $X = \operatorname{Spec} F$, note that

$$H^i_{\acute{e}t}(\operatorname{Spec} F; \mu_m^{\otimes j}) \cong H^i(G_F; \mu_m^{\otimes j})$$

We write both as $H^i_{\acute{e}t}(F; \mu_m^{\otimes j})$.

Relationships between our players

$$
\begin{tikzcd}
K^M \arrow[r, "①"] \arrow[d, "②"'] & K \arrow[dl, shift right, "③\ \text{chern character}"'] \\
H^i_{mot} \arrow[ur, shift right, "④\ \text{motivic ss}"'] \arrow[r, "\text{cycle class map}\ ⑤"'] & H^i_{et}
\end{tikzcd}
$$

(1) Let F be a field.

We have a map

$$F^\times \xrightarrow{\sim} K_1(\operatorname{Spec} F) \subseteq \bigoplus K_n(\operatorname{Spec} F))$$

As $K_2(\operatorname{Spec} F) = \frac{F^\times \otimes F^\times}{\langle a \otimes (1-a) \rangle}$ by Matsumoto, this

extends to

$$K^M_\bullet(F) \longrightarrow K_\bullet(\operatorname{Spec} F)$$

which is an isomorphism for degrees 0, 1, 2,

(2) By Nesterenko-Suslin, Totaro, we have isomorphisms

$$K_n^M(F) \cong H^n(F, \mathbb{Z}(n))$$

The RHS is $CH^n(F, n)$, which is generated by codim n cycles in $(\mathbb{A}^1_F - \{0,1\})^n$, similar to the symbols for $K_n^M(F)$.

$K_\cdot^M(F) \longrightarrow CH^\cdot(F, \cdot)$ show Steinberg holds in $CH^2(F,2)$

$$CH^\cdot(F, \cdot) \longrightarrow K_\cdot^M(F)$$

$$p \longmapsto N_{k(p)/F}\{x_1, \dots, x_n\}$$

$$\parallel$$

$$(x_1, \dots, x_n)$$

well definition uses Suslin reciprocity.

For a smooth curve C, $\sum_{\substack{w \in C \\ \text{closed}}} N_{k(w)/F}\, \partial_w x = 0$ in $K_n^M(F)$

③ Recall for a smooth variety X

$$K_0(X)\otimes\mathbb{Q} \xrightarrow{\sim} CH^\bullet(X)\otimes\mathbb{Q}$$

Bloch extends this to an isomorphism

$$K_m(X)\otimes\mathbb{Q} \xrightarrow{\sim} CH^\bullet(X,m)\otimes\mathbb{Q}$$
$$\shortparallel$$
$$H^{2\bullet-n}(X;\mathbb{Z}(n))\otimes\mathbb{Q}$$

via a projective bundle formula for higher Chow groups to define a Chern character, and a multiplication by the Todd class

$$\begin{tikzcd}
K_n(X) \arrow[r, "ch"] \arrow[rr, bend right] & CH^\bullet(X,m)\otimes\mathbb{Q} \arrow[r, "\cdot Td(X)"] & CH^\bullet(X,m)\otimes\mathbb{Q})
\end{tikzcd}$$

④ In topology, we have the Atiyah-Hirzebruch spectral sequence

$$E_2^{pq}(X) = H^p(X; K^q(*))$$

$$\Downarrow$$

$$K^{p+q}(X)$$

where K is complex topological K-theory.

Tensoring by $\mathbb{Q}$ makes all these differentials 0, yielding

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\sim} H^{2\bullet - n}(X; \mathbb{Q})$$

There is a spectral sequence

$$E_2^{pq} = H^{p-q}(X; \mathbb{Z}(-q))$$

$$\Downarrow$$

$$K_{-p-q}(X)$$

For X a field, by Bloch–Lichtenbaum

For X smth/field, by Friedlander–Suslin

For X smth/Dedekind scheme, by Levine

(5) There is a cycle class map, for m invertible.

$$H^i(X, \mathbb{Z}/m\mathbb{Z}(j)) \longrightarrow H^i_{et}(X, \mu_m^{\otimes j})$$

For $X = \operatorname{Spec} F$ and $i=j$ this is

$$K_i^M(F)/m \longrightarrow H^i_{et}(F, \mu_m^{\otimes i})$$

the norm residue map, defined as follows:

$$i=1. \quad F^\times/F^{\times m} \xrightarrow[\chi]{\sim} H^1_{et}(F, \mu_m)$$

and extending via cup products

If $\mu_m \subseteq F$, on $i=2$ we get

$$K_2^M(F)/m \longrightarrow H^2_{et}(F, \mu_m^{\otimes 2}) = Br(F)[m]$$

If F is a local nonarchimedean field, this is $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$, recovering the Hilbert symbol.

Thm (Norm residue isomorphism)

F a field, m invertible in F,

$$H^i(F; \mathbb{Z}/m\mathbb{Z}(j)) \xrightarrow{\sim} H^i_{ét}(F; \mu_m^{\otimes j})$$

for $i \leq j$.

Rmk. For $j < i$, $H^i(F; \mathbb{Z}/m\mathbb{Z}(j)) = CH^j(F, 2j-i) = 0$

More generally, let $X \longrightarrow \operatorname{Spec} F$ smooth variety, m invertible in F.

$$H^i(F; \mathbb{Z}/m\mathbb{Z}(j)) \xrightarrow{\sim} H^i_{ét}(X; \mu_m^{\otimes j})$$

for $i \leq j$.

Suppose $F \supseteq \mu_m$, $m \in F^\times$,

Then $H^0(F, \mathbb{Z}/m\mathbb{Z}(1)) = \mu_m(F)$.

Choosing $\varphi \in \mu_m(F)$ primitive yields $u \in H^0(F, \mathbb{Z}/m\mathbb{Z}(1))$.

Hence, an element $u \in H^0(X; \mathbb{Z}/m\mathbb{Z}(1))$ for any

$X \longrightarrow \operatorname{Spec} F$ smooth variety,

Thus, we have multiplication maps

$$H^i(X; \mathbb{Z}/m\mathbb{Z}(i)) \longrightarrow H^i(X; \mathbb{Z}/m\mathbb{Z}(i+1))$$

$$\begin{tikzcd}
H^{2i}(X; \mathbb{Z}/m\mathbb{Z}(i)) \arrow[r] \arrow[d, equal] & H^{2i}(X; \mathbb{Z}/m\mathbb{Z}(i+1)) \arrow[r] & \cdots \arrow[r] & H^{2i}(X; \mathbb{Z}/m\mathbb{Z}(2i)) \arrow[d, equal, "\text{(via } \varphi)"] \\
CH^i(X)/\ell \arrow[rrr, "\text{cycle class map}"'] & & & H^{2i}_{et}(X; \mathbb{Z}/\ell\mathbb{Z})
\end{tikzcd}$$

For fields, we get $0 \to 0 \to \cdots \to \underset{j=i}{H^i_{et}(F, \mathbb{Z}/\ell\mathbb{Z})} \xrightarrow{\sim} H^i_{et}(F; \mathbb{Z}/\ell\mathbb{Z}) \to \cdots$

§2. Computations for fields

The Norm residue isomorphism theorem tells us motivic cohomology of fields with finite coefficients via Galois cohomology.

Consider a finite field $\mathbb{F}_q$ and some $m \equiv 1 \bmod q$.

The Galois group of $\mathbb{F}_q$ is $\hat{\mathbb{Z}}$, which has cohomological dimension 1, as any extension

$$0 \to A \to B \to \hat{\mathbb{Z}} \to 0$$

w/ A finite splits, so $H^2(\hat{\mathbb{Z}}, A) = 0$. Then use dimension shifting.

Hence, $H^i_{\text{mot}}(\mathbb{F}_q; \mu_m^{\otimes i}) = 0$ for $i \geq 2$.

As $m \equiv 1 \bmod q$, $\mu_m \subseteq \mathbb{F}_q$ so $\mu_m^{\otimes i} \cong \mathbb{Z}/m\mathbb{Z}$ as $\hat{\mathbb{Z}}$-modules.

Thus,

$$H^i(\mathbb{F}_q; \mathbb{Z}/m\mathbb{Z}(i)) = \begin{cases} 0 & i \neq 0,1 \\ \mathbb{Z}/m\mathbb{Z} & \text{o/w} \end{cases}$$

If $\mu_m \notin F$, $H^0(\mathbb{F}_q; \mu_m^{\otimes j}) = (\mu_m^{\otimes j})^{\hat{\mathbb{Z}}}$

and $H^1(\mathbb{F}_q; \mu_m^{\otimes j}) \cong H^0(\mathbb{F}_q; \mu_m^{\otimes -j})^\vee$ as $\mathbb{Z}/m\mathbb{Z}$-modules

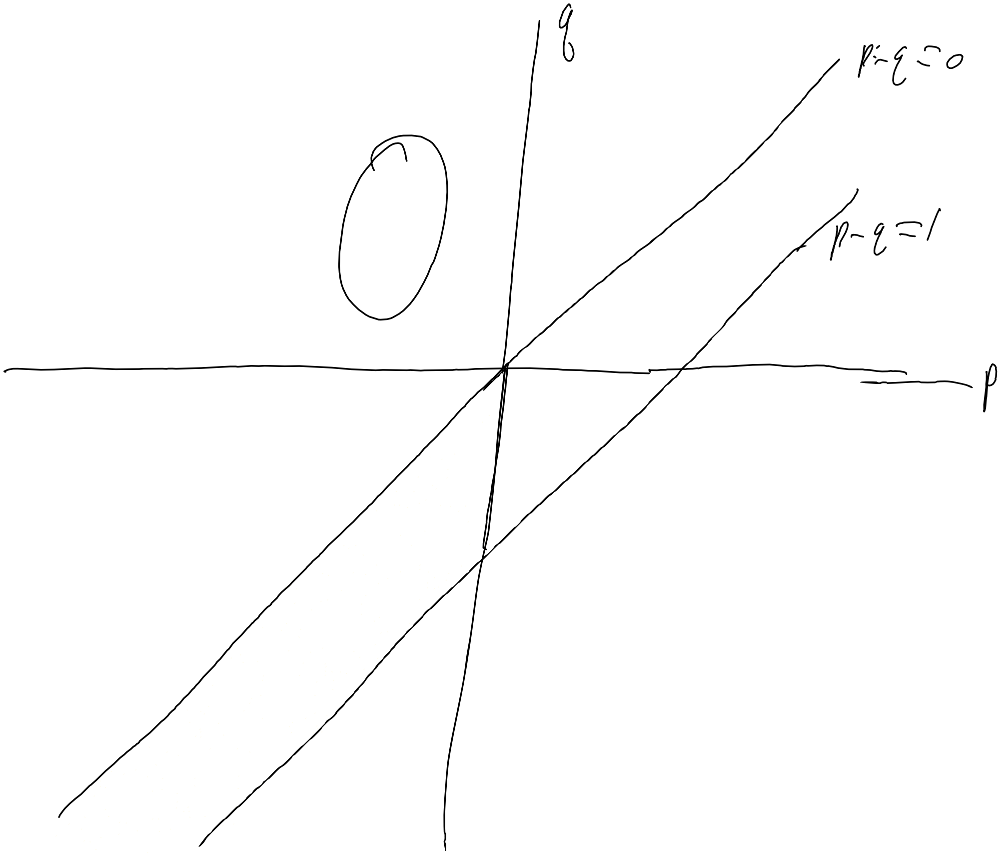
Now, we can compute the K theory of $\mathbb{F}_q$

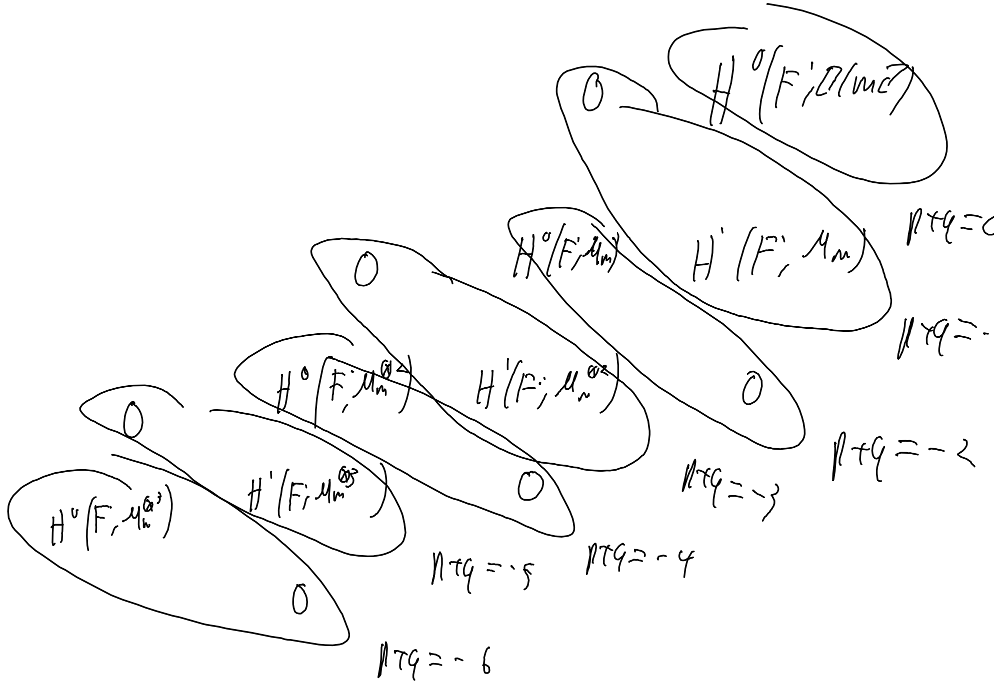
with finite coefficients. For simplicity, let $m \equiv 1 \bmod q$ again.

Recall $E_2^{pq} = H^{p-q}(F; \mathbb{Z}/m\mathbb{Z}(-q))$

$$\Downarrow$$

$$K_{-p-q}(F; \mathbb{Z}/m\mathbb{Z})$$


q
p - q = 0
p - q = 1
p
0



Thus, $p+q=-n$ has exactly $\underline{\text{one}}$ nonzero entry. Hence,

$$K_n(F;\mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} H^0(F;\mu_m^{\otimes(n-1)}) & n \text{ even} \\ H^1(F;\mu_m^{\otimes(n-1)}) & n \text{ odd} \end{cases}$$

For $\mu_m \subseteq F$, these are all $\mathbb{Z}/m\mathbb{Z}$. Else, get

$$K_{2i}(F) \cong \mathbb{Z}/(m, q^i-1)\mathbb{Z}$$

$$K_{2i-1}(F) \cong \mathbb{Z}/(m, q^i-1)\mathbb{Z}$$

Rmk. K-theory has a universal coefficient theorem

$$0 \to K_n(X) \otimes \mathbb{Z}/m\mathbb{Z} \to \underbrace{K_n(X, \mathbb{Z}/m\mathbb{Z})} \to K_{n-1}(X)[m] \to 0$$

$$\pi_n(A; \mathbb{Z}/m\mathbb{Z}) = [\underbrace{P^n(\mathbb{Z}/m\mathbb{Z})}, A]_0$$

attach an n-cell to S^{n-1} via a degree m map

"mod-m Moore space"

Quillen computed

$$K_n(\mathbb{F}_q) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}/q^i - 1 & n = 2i - 1 \end{cases}$$

There are other fields where the ss degenerates at E_2

e.g. let $F = k(C)$ for $k = \bar{k}$, C a curve/k.

Then by Tsen's theorem, $H^i_{et}(F, \mathbb{G}_m) = 0$ for $i \geq 2$

we have

$$H^0(F, \mu_m^{\otimes j}) \cong \mathbb{Z}/m\mathbb{Z}$$

$$H^1(F, \mu_m^{\otimes j}) \cong F^\times/F^{\times m}$$

whence

$$K_n(F; \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m\mathbb{Z} & n \text{ even} \\ F^\times/F^{\times m} & n \text{ odd} \end{cases}$$

Even more trivial is $F = F_{sep}$,

whence $H^i(F, \mu_m^{\otimes i}) = 0$ for $i \geq 1$.

Then

$$K_n(F; \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m\mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

This can be used to compute

$$K_n(C) \cong \begin{cases} (\mathbb{Z}/m\mathbb{Z})^2 & n \geq 0 \text{ even} \\ \mathrm{Pic}(C)[m] & n \geq 0 \text{ odd} \end{cases}$$

by localization,

$$E_1^{p,q} = \bigoplus_{x \in C^{(p)}} K_{-p-q}(k(x)) \Rightarrow K_{-p-q}(X)$$