Outline

- What is a moduli space?
  - conics
  - functors/Yoneda
- What is $\mathcal{M}_g$?
  - no fine moduli scheme
  - GIT
  - sm, q-proj / Spec $\mathbb{Z}$, irred $3g-3$ dim'l
  - $\overline{\mathcal{M}}_{g,n}$
- What is $\mathcal{A}_g$?
  - what is ab var?
  - what is a polarization? dual?
  - Siegel upper half space
  - $\mathcal{A}_{g,d,n}$ has a fine moduli sch / Spec $\mathbb{Z}[1/n]$ which is q-proj.

    $\mathcal{A}_{g/k}$ irred $\forall k$ (Faltings, Chai)

    $\mathcal{A}_g|_{\mathbb{C}}$ general type $g \geq 7$
  - Compactifications?

§1. Moduli spaces

A plane conic is defined by a homogeneous equation

$$\{a_0 x^2 + a_1 xy + \cdots + a_5 z^2 = 0\}$$

where not all a_i are 0.

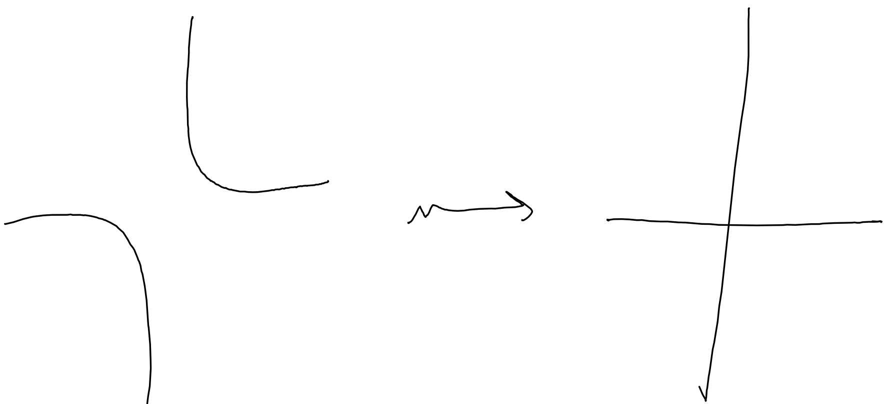
Two equations define the same plane conic iff the equations are off by a nonzero scalar.

Thus

$$\{\text{plane conics}\} \xleftrightarrow{\sim} \mathbb{P}^5$$

This realizes $\mathbb{P}^5$ as the moduli space of plane conics.

Consider $\{xy = tz^2\}_t$



$t=1$ $\rightsquigarrow$ $t=0$

(in $\{z=1\}$)

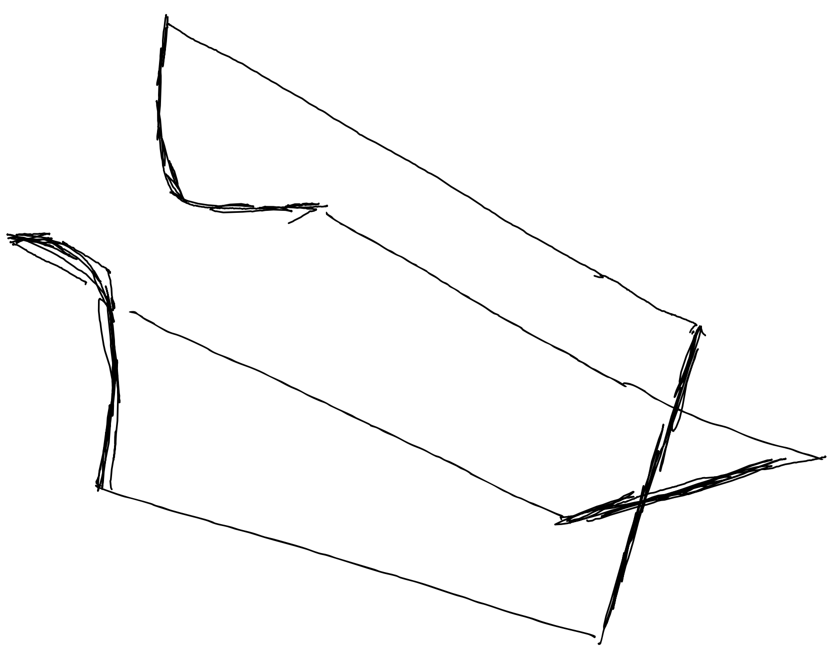
In other words, $\mathbb{A}^1 \longrightarrow \mathbb{P}^5$

$t \longmapsto \{xy = tz^2\}$

More generally, a map $S \longrightarrow \mathbb{P}^5$ corresponds to an S-parametrized family of plane conics. That is,

$$
\begin{tikzcd}
Z \arrow[dr] & \subseteq \mathbb{P}^2 \times S \arrow[d] \\
& S
\end{tikzcd}
$$

such that the fibers are all plane conics



This is natural,

$$\begin{tikzcd} T \arrow[r] \arrow[rr, bend right] & S \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} Z \arrow[r, hook] & \mathbb{P}^3_S \\ Z_T \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\lrcorner"] & \mathbb{P}^3_T \arrow[u] \end{tikzcd}$$

§2. M_g

We first define M_g purely formally.

Def. $\underline{M_g} : Sch^{op} \longrightarrow Set$

$$S \longmapsto \left\{ C \to S \;\middle|\; \begin{array}{l} \text{smth, proper, geom fibers} \\ \text{are genus } g \end{array} \right\} \Big/ \text{isomorphism}$$

Def. M_g is a representative for $\underline{M_g}$.

Thm. M_g does not exist.

Pf. Suppose it did. Then there would be a "universal family" over M_g. That is, under $Sch(-, M_g) \cong \underline{M_g}$, $M_g \xrightarrow{id} M_g$ yields some $\mathcal{U}_g \in \underline{M_g}(M_g)$.

e.g. $\quad id : \mathbb{P}^5 \longrightarrow \mathbb{P}^5$ yields

$$\begin{tikzcd} Z = \{ a_0 x^2 + \dots + a_5 z^2 = 0 \} \arrow[dr] \arrow[r, phantom, "\subseteq"] & \mathbb{P}^2_{x,y,z} \times \mathbb{P}^5_{a_0 \dots a_5} \arrow[d] \\ & \mathbb{P}^5 \end{tikzcd}$$

universal family

Given $S \longrightarrow \mathbb{P}^5$, have

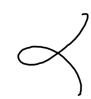
$$\begin{tikzcd} Z_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ S \arrow[r] & \mathbb{P}^5 \end{tikzcd}$$

Let $C \to \operatorname{Spec} \mathbb{C}$ be a curve with a nontrivial automorphism σ

e.g. C hyperelliptic

Consider the family $\mathcal{C} = C \times \mathbb{A}^1 / (x,0) \sim (\sigma(x),1)$

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ S = \mathbb{A}^1/0\sim 1 \end{tikzcd}$$

$=$ 

This is isotrivial but nontrivial.

By assumption, $\mathcal{C} \to S$ is classified by $S \to \mathcal{M}_g$.

That is,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U}_g \arrow[d] \\ S \arrow[r, dashed, "\exists"] & \mathcal{M}_g \end{tikzcd}$$

As $\mathcal{C} \to S$ is iso trivial, every $s \in S$ has fiber $\mathcal{C}_s \cong C$ classified by a single point $\operatorname{Spec} \mathbb{C} \to M_g$.

Thus,

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{U}_g \arrow[dd] \\
& C \arrow[ur] \arrow[dd] & \\
S \arrow[rr, dashed] \arrow[dr] & & M_g \\
& \operatorname{Spec} \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

So $S \to M_g$ is constant so $\mathcal{C} \to S$ is trivial!

Issue: $/\cong$! Resolve via stacks or approximations.

We refer to M_g as a <u>fine</u> moduli scheme for M_g. This does not exist.

Def. Let $\underline{M} : Sch^{op} \longrightarrow Set$.

A coarse moduli scheme of $\underline{M}$ is

a scheme M and a map

$$\underline{M} \longrightarrow Sch(-, M)$$

which is initial among all such maps, and

which is a bijection for any acf input.

Thm. There exists a $\overset{h^2}{\uparrow}$smooth, $\boxed{\text{irreducible}}$ quasiprojective scheme

M_g over Spec $\mathbb{Z}$ which is a coarse moduli

space for $\underline{M_g}$.

- Low genus examples
- dim computation via def. theory (dual numbers)

Thm. $\exists$ a scheme Hilb_n whose S-points are naturally iso to $\{ Z \subseteq \mathbb{P}^n \times S \mid Z \to S \text{ flat}\}$, for S locally Noetherian.

$\mathrm{Hilb}_n = \coprod_P \mathrm{Hilb}_{n,P}$ classifying Z w/ fixed Hilbert polynomial P.

Each $\mathrm{Hilb}_{n,P}$ is projective $/\operatorname{Spec} \mathbb{Z}$

Let C be a curve of genus $g \geq 2$.

Then ω_C is ample and $\omega_C^{\otimes k}$ is very ample for $k \geq 3$.

Then $C \hookrightarrow \mathbb{P}^{(2k-1)(g-1)-1}$ via $\omega_C^{\otimes k}$.

Thm. There is a locally closed subscheme $K_g \subseteq \mathrm{Hilb}_N$ parametrizing curves in $\mathbb{P}^N$ which are

- smooth — open on base
- genus g
- degree $d = 2(g-1)k$ — Hilbert polynomial
- k-pluricanonically embedded — where $\mathcal{O}_{\mathcal{C}}(1) \cong \omega_{\mathcal{C}/\mathcal{H}}^{\otimes k}$

$\begin{matrix}\mathcal{C}\\ \downarrow \\ \mathcal{H}\end{matrix}$

idea: these define maps to Pic, which is separated.

Let C, C' be smooth curve of genus g.

If $C \cong C'$, $H^0(C, \omega_C^{\otimes k}) \cong H^0(C', \omega_{C'}^{\otimes k})$ for all k.

Hence, fixing bases, C, C' embed into $\mathbb{P}^N$ then are projectively equivalent (off by $\mathrm{Aut}(\mathbb{P}^N) = PGL_{N+1}$).

Hence "$M_g = PGL_{N+1} \backslash X_g$"

as a ...

- stack quotient

- $M_g = PGL_{N+1} \backslash X_g$ as a GIT quotient

$\downarrow$

universal property and

$X_g \longrightarrow PGL_{N+1} \backslash X_g$

has orbits as fibers.

§3. $\mathcal{A}_g$.

Def. An abelian scheme$/S$ is a proper smooth group scheme $/S$ w/ geom fibers connected and g-dim'l.

Lemma. A is commutative.

Thm (Abrashkin, Fontaine) $\nexists$ ab schemes over $\operatorname{Spec}\mathbb{Z}$ (everywhere good reduction)

Ab vars $/\mathbb{C}$ are complex tori. $\mathbb{C}^g/\Lambda$

Def. Let A be an ab var $/k$. The dual abelian variety $A^\vee$ parametrizes deg 0 line bundles on A. That is,

$$\mathrm{Sch}(T, A^\vee) \cong \left\{ L \text{ on } A\times T \;\middle|\; \begin{array}{l} L|_{A\times\{t\}} \text{ is a deg } 0 \text{ line bundle} \\ L|_{\{0\}\times T} \text{ is trivial} \end{array} \right\}$$

The universal family $\mathcal{P}$ on $A\times A^\vee$ is the Poincaré bundle.

Fact. $A^\vee$ ab var of same dimension.

If L is a line bundle on A, we have the family

$$\{t_a^* L \otimes L^\vee\}_{a \in A}$$

of deg 0 line bundles on A, where t_a is translation by a.

Thus, this corresponds to a map $A \xrightarrow{\varphi_L} A^\vee$. This is a homomorphism.

Thm. (Milne's notes) φ_L isogeny $\Leftrightarrow$ L ample.

Def. A polarization on A is an ample line bundle.

Its degree is $\deg(\varphi_L)$.

It's principal if $\deg(\varphi_L) = 1$, i.e. φ_L an iso.

e.g. $\mathrm{Jac}(C)$ is principally polarized.

Def. $\underline{\mathcal{A}_g}$ ppav dim g

Thm. $\exists$ fine moduli scheme $\mathcal{A}_g / \operatorname{Spec} \mathbb{Z}$

(can also do $\mathcal{A}_{g,d,n} / \operatorname{Spec} \mathbb{Z}[1/n]$)

$\quad d \downarrow \quad n \downarrow$

degree of polarization, level structure

$\underline{\mathcal{A}_g(\mathbb{C})}$?

$A = \mathbb{C}^g/L$. Note $L = H_1(A;\mathbb{Z})$

$H^1(A;\mathbb{Z}) = \Lambda L^\vee$

so $H^2(A;\mathbb{Z}) = \Lambda^2 L^\vee$

A line bundle determines a class E in

ample iff $\longrightarrow$ H pos. def. Hermitian form

$E = \mathrm{Im}(H)$ integer valued on L

Def. $\boxed{H(u,v) = E(iu,v) + iE(u,v)}$

Take a basis of L and period matrix Π $g \times 2g$.

Write $\Pi = (I \mid \Omega)$. Take $E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (D = I iff principal)

Then Ω symmetric and $\mathrm{Im}(\Omega)$ pos. def.

Let $\mathcal{H}_g = \{ \Omega \in M_{g\times g}(\mathbb{C}) \mid \Omega \text{ sym} + \mathrm{Im}(\Omega) > 0 \}$

$\mathcal{A}_g = Sp_{2g}\backslash\mathcal{H}_g$. $\binom{g}{2}$ dim'l