Hodge-Tate decomposition

§1. Cartier duality

Let $\mathcal{O} \in \mathrm{CRing}$

M finite flat $/ \mathcal{O}$

$M^* = \mathcal{O}\text{-Mod}(M, \mathcal{O})$, also finite flat

Then $M \xrightarrow[\sim]{ev} M^{**}$ iso, as these are locally free.

Now, let $G = \operatorname{Spec} A$ be a finite flat group scheme $/ \mathcal{O}$.

$$\mathcal{O} \underset{\varepsilon}{\overset{e}{\rightleftarrows}} A \underset{m}{\overset{\mu}{\rightleftarrows}} A \otimes_{\mathcal{O}} A, \qquad A \circlearrowleft \iota$$

$$\mathcal{O} \underset{\varepsilon^*}{\overset{e^*}{\leftrightarrows}} A^* \underset{m^*}{\overset{\mu^*}{\leftrightarrows}} A^* \otimes_{\mathcal{O}} A^*, \qquad A^* \circlearrowleft$$

The Hopf algebra axioms are symmetric.

e.g.

$$\begin{array}{ccc} \mathcal{O} \otimes_{\mathcal{O}} A & \xrightarrow{e \otimes id} & A \otimes_{\mathcal{O}} A \\ & \searrow^{id} & \downarrow m \\ & & A \end{array} \qquad \begin{array}{ccc} \mathcal{O} \otimes_{\mathcal{O}} A & \xleftarrow{\varepsilon \otimes id} & A \otimes_{\mathcal{O}} A \\ & \nwarrow & \uparrow \mu \\ & & A \end{array}$$

Def. Then let $G^* = \operatorname{Spec} A^*$, a finite flat group scheme $/\mathcal{O}$

Facts. - G a finite group $/\mathcal{O}$. Then $\mathcal{O}[G]^* = \mathcal{O}^G$, so $\mu_{n,\mathcal{O}}^* = \underline{\mathbb{Z}/n\mathbb{Z}}_{\mathcal{O}}$

- $(G^*)^* = G^*$

- $G \longmapsto G^*$ is exact

Lemma. Let R be an $\mathcal{O}$-algebra. Then

$$G^*(R) = \operatorname{Hom}_{\mathrm{Sch}/R}(G_R, \mathbb{G}_{m,R})$$

Pf. By base change WLOG take $R = \mathcal{O}$.

Let $f \in G^*(\mathcal{O}) = \mathcal{O}\text{-}\mathrm{Alg}(A^*, \mathcal{O}) \subseteq A^{**} \underset{ev}{\cong} A$

So $f = ev_a$ for a unique $a \in A$. When is this an $\mathcal{O}$-algebra map?

In A^*, $\alpha\beta = \alpha \otimes \beta \circ \mu$

$$\therefore ev_a(\alpha\beta) = (\alpha \otimes \beta)(\mu(a))$$

$$ev_a(\alpha)\, ev_a(\beta) = \alpha(a)\beta(a) = (\alpha \otimes \beta)(a \otimes a)$$

atdcx

So $\mu(a) = a \otimes a$

Similarly, $ev_a(1) = 1 \iff \varepsilon(a) = 1$

Thus $G^*(\mathcal{O}) = \{a \in A \mid \mu(a) = a \otimes a,\ \varepsilon(a) = 1\}$

$= \operatorname{Hom}(\mathcal{O}[t, t^{-1}], A)$

$= \operatorname{Hom}(G, \mathbb{G}_m)$ $\square$

Def. Let $G = \varinjlim_{v} G_v$ be a p-divisible group.

Then let $G^* = \varinjlim_{v} G_v^*$

e.g. $\mu_{p^\infty, \mathcal{O}}^* = \underline{\mathbb{Q}_p/\mathbb{Z}_p}_{\mathcal{O}}$

Fact. $\dim G + \dim G^* = ht(G) = ht(G^*)$

(Here $\dim G = \dim G^0 = \#$ power series variables)

pf idea. Work over the perfect residue field. Let $F: G \to G^{(p)}$ Frobenius. The Verschiebung $G^{(p)} \xrightarrow{V} G$ is defined as $(F_{G^*})^*$

Then $VF = [p]$, $FV = [p]$,

$$\begin{array}{ccccccccc} 0 & \to & \ker(F) & \to & G & \xrightarrow{F} & G^{(p)} & \to & 0 \\ & & \downarrow & & \downarrow [p] & & \downarrow V & & \\ 0 & \to & 0 & \to & G & \xrightarrow{id} & G & \to & 0 \end{array}$$

By Snake, $0 \to \ker(F) \to \ker[p] \to \ker(V) \to 0$

$$p^{\dim G} \qquad p^{ht(G)} \qquad p^{\dim G^*}$$

§ Tangent spaces and logs

Setup. K complete discrete valuation field of char 0

$\mathbb{C}_K = \widehat{K_{alg}}$

$\mathcal{O} = \mathcal{O}_K$

$R = \mathcal{O}_{\mathbb{C}_K}$

G: a p-div grp / $\mathcal{O}_K$

$= \operatorname{colim}_{\underset{v}{\to}} G_v$, $G_v = \operatorname{Spec} A_v$

Def. Let M be an $\mathcal{O}$-module. We let

$$t_G(M) := t_{G^0}(M) = \mathcal{O}\text{-Mod}(I/I^2, M)$$

with $I = \ker(\varepsilon) \subseteq A^0 = \varprojlim_n A_n^0$ the augmentation ideal. This is the tangent space of G with values in M.

Def. In the above setting, $t_G^*(M) = I/I^2 \otimes_{\mathcal{O}} M$, the cotangent space.

Now, recall $1 \longrightarrow G^0(R) \longrightarrow G(R) \longrightarrow G^{\acute{e}t}(R) \longrightarrow 1.$

$$G^{\acute{e}t}(R) \overset{\text{(as } G_v \text{ are finite)}}{=} G^{\acute{e}t}(\mathbb{C}_K) \overset{?}{\cong} \frac{\mathbb{Q}_p}{\mathbb{Z}_p}^{ht(G)}_{\mathcal{O}_K}$$

So $G(R)/G^0(R)$ is torsion

Thus $\forall f \in G(R)$ and $\forall i \gg 0$, $f^{p^i} \in G^0(R)$.
$$G^0(R) = \mathrm{ContAlg}_{\mathcal{O}}(A^0, R)$$

Def. $$\log_G(f)(a) = \lim_{i\to\infty} \frac{f^{p^i}(a)}{p^i}$$

Lemma. - This converges in $\mathbb{C}_K$ for $a \in I$

- This vanishes for $a \in I^2$

- $\log_G(fg) = \log_G(f) + \log_G(g)$

Thus, $\log_G : G(R) \longrightarrow t_G(\mathbb{C}_K)$ $\mathbb{Z}_p$ - linear

Lemma. - $\log_G : F^\lambda G^0(R) \xrightarrow{\sim} \{\tau \in t_G(\mathbb{C}_K) \mid v(\tau(a)) \geq \lambda \ \forall a \in I/I^2\}$, where $F^\lambda G^0(R) = \{f \mid v(f(a)) \geq \lambda \ \forall a \in I\}$

for $\lambda > \frac{v(p)}{p-1}$.

- $1 \longrightarrow G(R)_{tors} \longrightarrow G(R) \xrightarrow{\log} t_G(\mathbb{C}_K) \longrightarrow 0$,

a G_K - equivariant SES.

e.g. $G = \mu_{p^\infty, \mathcal{O}}$. Then $G(R) = 1 + m_R$ where $G(R)_{tors} = \mu_{p^\infty}(R)$.

$$1 \longrightarrow \mu_{p^\infty}(R) \longrightarrow 1 + m_R \xrightarrow{\log} \mathbb{C}_K \longrightarrow 0$$

This is the usual p-adic log!

§ Hodge–Tate

Thm. Let G be a p-divisible group $/\mathcal{O}$.

$$\mathrm{Hom}(T_p(G), \mathcal{O}_{\mathbb{C}_K}) \cong t_{G^*}(\mathcal{O}_{\mathbb{C}_K}) \oplus t_G^*(\mathcal{O}_{\mathbb{C}_K})(-1)$$

Recall $T_p(G) = \varprojlim_n G_n(\mathcal{O}_{\mathbb{C}_K}) = \varprojlim_n G_n(R)$

and $\mathbb{Z}_p(1) = T_p(\mu_{p^\infty})$ the cyclotomic character.

Rmk. The Tate module is dual to H^1_{et}, which has a comparison to complex de Rham cohomology,

$$H^1 \cong H^{0,1} \oplus H^{1,0}$$

(via the $(1,1)$ Kähler form)

We blackbox the following

Thm (Tate–Sen). $H^i(K, \mathbb{C}_K(j)) = \begin{cases} K & i=0,1 \text{ and } j=0 \\ 0 & \text{o/w} \end{cases}$

Onto the proof.

First, $T_p(G^*) = \varprojlim_v G_v^*(R)$

$$= \varprojlim_v \mathrm{GrpSch}_R(G_{v,R}, \mu_{p^v,R}) \quad (\text{as } G_v \text{ is } p^v \text{ torsion})$$

$$= p\text{-Div}_{/R}(G_R, \mu_{p^\infty,R})$$

Taking R-points we get a map

$$T_p(G^*) \cong p\text{-Div}_{/R}(G_R, \mu_{p^\infty}, R)$$
$$\Big\downarrow x$$
$$\mathbb{Z}_p\text{-Mod}(G(R), \mu_{p^\infty}(R)) \leq \mathbb{Z}_p\text{-Mod}(G(R), 1+m_R)$$

inducing
$$G(R) \xrightarrow{\alpha} \mathbb{Z}_p\text{-Mod}(T_p(G^*), 1+m_R)$$
$$g \longmapsto (z \longmapsto x(z)(g))$$

Similarly, taking tangent spaces yields a map

$$T_p(G^*) \longrightarrow \mathcal{O}_K\text{-Mod}(t_G(\mathcal{O}_K), t_{\mu_{p^\infty},\mathcal{O}}(\mathcal{O}_K))$$

whence a map $t_G(\mathcal{O}_K) \xrightarrow{d\alpha} \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mathcal{O}_K)$

In summary, we have the following G_K-invariant diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(R)_{tors} & \longrightarrow & G(R) & \xrightarrow{\log} & t_G(C_K) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle d\alpha} & & \\
1 & \longrightarrow & \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mu_{p^\infty}(R)) & \longrightarrow & \mathbb{Z}_p\text{-Mod}(T_p(G^*), 1+\mathfrak{m}_R) & \xrightarrow{\log} & \mathbb{Z}_p\text{-Mod}(T_p(G^*), C_K) & \longrightarrow & 0
\end{array}$$

Step 1. α_0 is an isomorphism.

pf. $$G(R)_{tors} = \varinjlim_v G_v(O_K)$$

$$= \varinjlim_v \mathrm{GrpSch}_{O_K}(G_{v, O_K}^*, \mu_{p^v, O_K})$$

$$= \varinjlim_v \mathbb{Z}_p\text{-Mod}(G_v^*(O_K), \mu_{p^v}(O_K))$$

$$= \mathbb{Z}_p\text{-Mod}(\varprojlim_v G_v^*(O_K), \varinjlim_v \mu_{p^v}(O_K))$$

$$= \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mu_{p^\infty}(R))$$ □

Step 2. α, $d\alpha$ are injective.

pf. As α_0 is an iso, the snake lemma shows that

$$\ker(\alpha) \xrightarrow[\sim]{\log} \ker(d\alpha)$$

is an iso.

Idea. Reduce to showing injectivity of $d\alpha$ on $\log_G(G(O))$.

Step 3. Take Galois invariants

$$\alpha_K : G(\mathcal{O}_K) \xrightarrow{\sim} \mathbb{Z}_p[G_K]\text{-Mod}(T_p(G^*), 1+\mathfrak{m}_K)$$

$$d\alpha_K : t_G(K) \xrightarrow{\sim} \mathbb{Z}_p[G_K]\text{-Mod}(T_p(G^*), \mathbb{C}_K)$$

Pf.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(R) & \xrightarrow{\alpha} & \mathbb{Z}_p\text{-Mod}(T_p(G^*), 1+\mathfrak{m}_R) & \longrightarrow & \operatorname{cok}(\alpha) & \longrightarrow & 1 \\ & & \downarrow \log_G & & \downarrow \log & & \downarrow \wr \ (\text{snake}) & & \\ 0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mathbb{C}_K) & \longrightarrow & \operatorname{cok}(d\alpha) & \longrightarrow & 0 \end{array}$$

Take Galois fixed parts. Then we show $d\alpha_K$ onto.

$W = \mathbb{Z}_p\text{-Mod}(T_p(G), \mathbb{C}_K)$

$V = \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mathbb{C}_K)$

$0 \to t_G(K) \xrightarrow{d\alpha_K} V^{G_K}$ so $\dim(V^{G_K}) \geq \dim t_G(K) = \dim G$,

OTOH, $\dim(W^{G_K}) \geq \dim G^\vee$

so $\dim(V^{G_K}) + \dim(W^{G_K}) \geq ht(G)$

We claim the reverse inequality holds.

We have a G_K-iso $T_p(G) \cong \mathbb{Z}_p\text{-Mod}(T_p(G^*), \mathbb{Z}_p(1))$,

hence a G_K-pairing $T_p(G) \otimes T_p(G^*) \longrightarrow \mathbb{Z}_p(1)$,

hence a G_K-pairing $V \otimes W \longrightarrow \mathbb{C}_K(-1)$,

Take G_K-invariants $V^{G_K} \otimes W^{G_K} \longrightarrow \mathbb{C}_K(-1)^{G_K} = 0$ by Tate-Sen.

Thus, $V^{G_K} \otimes \mathbb{C}_K \perp W^{G_K} \otimes \mathbb{C}_K$, so the sum of their dimensions is $\leq \dim V = ht(G)$, as $T_p(G)$ is a free $\mathbb{Z}_p$-module of rank $ht(G)$. $\square$

Step 4. Hodge Tate

Pf. As before, $V^{G_K} \cong t_G(K)$
$W^{G_K} \cong t_{G^*}(K)$

Tensoring up to $\mathbb{C}_K$ yields
$V^{G_K} \otimes \mathbb{C}_K \cong t_G(\mathbb{C}_K)$
$W^{G_K} \otimes \mathbb{C}_K \cong t_{G^*}(\mathbb{C}_K)$

which are orthogonal under the above pairing for $\mathbb{C}_K(-1)$.

$$0 \longrightarrow t_{G^*}(\mathbb{C}_K) \xrightarrow{d\alpha} \mathbb{Z}_p\text{-Mod}(T_p(G), \mathbb{C}_K) \longrightarrow \mathbb{C}_K\text{-Mod}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad t_{G}^{*}(\mathbb{C}_K)(-1)$$

$\dim G^*$ $\qquad\qquad ht(G) \qquad\qquad \dim G$

These add up, so it's exact on the right.

Now, we must show this splits. Indeed,

$$Ext^1(t_{G^*}(\mathbb{C}_K)(-1), t_{G^*}(\mathbb{C}_K)) \cong Ext^1(\mathbb{C}_K(-1)^{\dim G}, \mathbb{C}_K^{\dim G^*})$$
$$\cong H^1(K, \mathbb{C}_K(-1))^{(\dim G)(\dim G^*)}$$
$$= 0 \text{ by Tate-Sen} \qquad \square$$

Cor. Let A be an abelian variety $/K$ with good reduction.

Then $H^n_{ét}(A_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j)$

Pf. Let $\mathcal{A}$ be the abelian scheme $/\mathcal{O}$ w/ generic fiber $\mathcal{A}_K = A$.

Claims. i) $\mathcal{A}^*[p^\infty] = \mathcal{A}[p^\infty]^*$, Cartier duality and exactness.

ii) $H^1_{ét}(A_{\bar{K}}, \mathbb{Q}_p) = \mathbb{Z}_p\text{-Mod}(T_p\mathcal{A}[p^\infty], \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

iii) $\mathcal{A}[p^\infty]^\circ$ is the formal completion of $\mathcal{A}$ at its unit e.

iv) $H^0(A, \Omega^1_{A/K}) \cong t_e^*(A)$

v) $H^1(A, \mathcal{O}_A) \cong t_e(A^*)$

$f \mapsto 1+\varepsilon f$ ⇝ deformation of the trivial bundle on A

$0 \to \mathcal{O}_A \to \mathcal{O}^*_{A_{K[\varepsilon]/\varepsilon^2}} \to \mathcal{O}_A^* \to 1$

vi) $H^*_{ét}(A_{\bar{K}}, \mathbb{Q}_p) \cong \Lambda^* H^1_{ét}(A_{\bar{K}}, \mathbb{Q}_p)$

topological: Künneth on $(S^1)^{2d}$

Dolbeault: $dz_1, \dots, dz_d, d\bar{z}_1, \dots, d\bar{z}_d$ are a basis of $H^1(U; \mathbb{C})$ and are Λ-invariant. Use the de Rham iso thm.

Étale: $m: A \times A \to A$ ⇝ $m^*: H^*_{ét}(A; \mathbb{Q}_p) \to H^*_{ét}(A; \mathbb{Q}_p) \otimes H^*(A; \mathbb{Q}_p)$

vii) $H^i(A, \Omega^j_{A/K}) \cong \Lambda^i H^1(A, \mathcal{O}_A) \otimes \Lambda^j H^0(A, \Omega^1_{A/K})$

Then apply Hodge-Tate to $\mathcal{A}[p^\infty]$. □