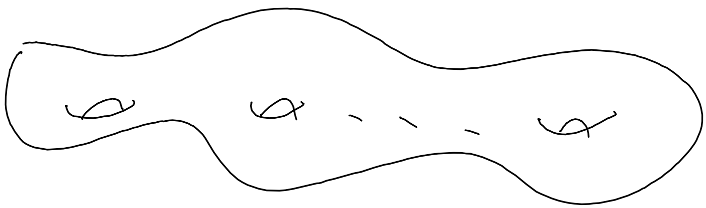
odge

lruclure

§1. Review of Hodge Theory on Kähler Manifolds



$2g$ holes

g holomorphic differentials

Let X^n be a complex manifold.

In local holomorphic coordinates $z_1, \ldots, z_n$ on X, complex differential forms are of the form

$$\sum_{I,J} f_{I,J}(z)\, dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$$

Let $\Omega^{p,q}(X)$ be the differential forms

with p many dz's

and q many $d\bar{z}$'s

Thm. • $\Omega^n(X) = \bigoplus_{p+q=n} \Omega^{p,q}(X)$

• $d = \partial + \bar{\partial}$ where $\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$

$\bar{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$

Rmk. $\ker\left(\Omega^{0,0}(X) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X)\right)$ are the holomorphic functions $X \longrightarrow \mathbb{C}$, by the Cauchy-Riemann equations.

Now, let X be Hermitian, so ∂ and $\bar{\partial}$ admit adjoints and Laplacians.

Def. $\mathcal{H}^n_\partial(X) = \{\omega \in \Omega^n(X) \mid \Delta_\partial(\omega) = 0\}$, similarly for $\bar{\partial}$

$\mathcal{H}^{p,q}_\partial(X) = \mathcal{H}^n_\partial(X) \cap \Omega^{p,q}(X)$

Thm. $\mathcal{H}^k_{\bar\partial}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial}(X)$ for X Hermitian.

Thm. $\mathcal{H}^{p,q}_{\bar\partial}(X) \xrightarrow{\sim} H^{p,q}(X)$

for X <u>compact</u> Hermitian.

Now, let X be a smooth projective complex variety, so that it admits a polarization

$$\omega = c_1(L) \in H^{1,1}(X) \cap H^2(X;\mathbb{Q})$$

where $L = \iota^* \mathcal{O}(1)$ for $\iota: X \hookrightarrow \mathbb{P}^N$.

Thm. For such X, ∂ and $\bar\partial$-harmonicity are equivalent

- $H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$

- $\overline{H^{p,q}(X)} = H^{q,p}(X)$

This is the "bidegree decomposition".

§2. Hodge Structures

Let $R \subseteq \mathbb{R}$ a subring.

Def 1. An R-Hodge structure consists of the following data.

- A free R-module $V = V_R$ of finite rank
- A decomposition

$$V_{\mathbb{C}} = V \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

for finite dimensional complex vector spaces $V^{p,q}$.

Such that

$$\overline{V^{p,q}} = V^{q,p}$$

Def. The weight k part of an R-Hodge structure is the real vector space $V^{(k)} = \bigoplus_{p+q=k} V^{p,q}$.

e.g. • $H^*(X;\mathbb{Z})$ is an integral Hodge structure for X a smooth projective variety / $\mathbb{C}$.

• $\mathbb{Z}(1)$ is the weight -2 integral Hodge structure on $2\pi i\,\mathbb{Z}$

$$\mathbb{Z}(1)\otimes\mathbb{C} = \mathbb{Z}(1)^{-1,-1} = \mathbb{C}$$

• $\mathbb{Z}(m)$ is the weight $-2m$ integral Hodge structure on $(2\pi i)^m\mathbb{Z}$

$$\mathbb{Z}(m)\otimes\mathbb{C} = \mathbb{Z}(1)^{-m,-m} = \mathbb{C}$$

Hodge structures may also be viewed as representations of an algebraic group.

Recall that a $\mathbb{Z}$-grading on a complex vector space V is equivalent to an action of $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$, as the character group of $\mathbb{G}_m$ is $\mathbb{Z}$.

Let $f: T \longrightarrow S$ be a map of schemes. We may form

$$\begin{aligned} Sh(T_{zar}) &\xrightarrow{f_*} Sh(S_{zar}) \\ F &\longmapsto (Y \mapsto F(Y_T)) \end{aligned}$$

Def. If $X \longrightarrow T$ has $f_* X$ representable, we denote its representing object as $\mathrm{Res}_{T/S}(X)$, the "Weil restriction of scalars".

That is, $\mathrm{Res}_{T/S} X(Z) = X(Z_T)$.

e.g., $f : \mathrm{Spec}\, \mathbb{C} \longrightarrow \mathrm{Spec}\, \mathbb{R}$

$X = \mathbb{G}_m / \mathbb{C}$

$= \mathrm{Spec}\,(\mathbb{C}[z_1, z_2]/(z_1 z_2 - 1))$

Let $z_j = x_j + i y_j$

$z_1 z_2 - 1 = (x_1 x_2 - y_1 y_2 - 1) + i(x_1 y_2 + x_2 y_1)$

Then $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is given by

$$\mathrm{Spec}\, \frac{\mathbb{R}[x_1, y_1, x_2, y_2]}{(x_1 x_2 - y_1 y_2 - 1,\ x_1 y_2 + x_2 y_1)}$$

$\mathbb{S}$ is the "Deligne torus".

$\mathbb{S}(\mathbb{R}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$

$\mathbb{S}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}^{\times 2}$

The inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ induces

$$\begin{array}{ccc} \mathbb{S}(\mathbb{R}) & \longrightarrow & \mathbb{S}(\mathbb{C}) \\ \| & & \| \\ \mathbb{C}^\times & \longrightarrow & \mathbb{C}^{\times 2} \\ z & \longmapsto & (z, \bar{z}) \end{array}$$

$\mathbb{S}_\mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$ w/ $G(\mathbb{C}/\mathbb{R})$ acting by swapping coordinates.

Prop. (Def. 2) Let V be a real vector space. Defining a Hodge structure on V is the same as defining a representation $\mathbb{S} \to GL(V)$ over $\mathbb{R}$

Pf. Let $\mathbb{S} \xrightarrow{\rho} GL\, V$.

Then let $V^{p,q} = \{ v \in V_\mathbb{C} \mid \rho(z) v = z^{-p} \bar{z}^{-q} v \}$ $\square$

(See Jacob's notes for details)

Def. $w: \mathbb{G}_m \longrightarrow \mathbb{S}$ $/\mathbb{R}$ is defined

on $\mathbb{R}$ points as $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ inclusion

on $\mathbb{C}$ points as $\mathbb{C}^\times \xhookrightarrow{\Delta} \mathbb{C}^{\times 2}$

(Rmk. Milne has the opposite sign convention to Peters-Steenbrink.)

Then we have

$$V^{(k)} = \{ v \in V \mid h(w(a))\, v = a^{-n} v \mid a \in \mathbb{S}(\mathbb{R}) \}$$

Rmk. An $\mathbb{R}$-Hodge structure on $V = V_\mathbb{R}$ arises from a real representation $\mathbb{S} \xrightarrow{\rho} GL(V_\mathbb{R})$ so that $\rho \circ w$ is defined$/\mathbb{R}$.

e.g. $\mathbb{Z}(m)$ is the integral representation

$$\mathbb{S} \longrightarrow \mathbb{G}_m$$
$$z \longmapsto (z\bar{z})^m$$

Thus, we may define morphism, $\otimes$'s, etc. of Hodge structures via their representations

e.g. $\mathbb{Z}(m) = \mathbb{Z}(1)^{\otimes m}$

Def. $V(m) = V \otimes \mathbb{Z}(m)$

§3, Polarizations

Let (V, ρ) be a weight k Hodge structure

Def. $C = \rho(i)$ is the Weil operator,

so $C|_{V^{p,q}} = i^{p-q}$

Def. A polarization of an $\mathbb{R}$-Hodge structure V of weight k is a map of Hodge structures

$$V \otimes V \xrightarrow{\psi} \mathbb{R}(-k)$$

s.t.

- $\psi(v, w) = (-1)^k \psi(w, v)$
- $\psi_C(u, v) := Q(u, v) := (2\pi i)^k \psi(Cu, v)$

is symmetric and positive definite on $V_{\mathbb{R}}$

Rmk. As $\mathbb{R}(-k)_{\mathbb{C}}$ is supported in only one bidegree, this says $V^{p,q} \perp V^{p',q'}$ if $(p,q) \neq (p',q') \neq (k,k)$

Rmk. Orthogonal complements show the category of polarized $\mathbb{R}$-Hodge structures is semisimple e.g. (V, ρ) an $\mathbb{R}$-Hodge structure of weight 1

w/ $V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$

Let $J = \rho(i)$.

A polarization is thus an alternating form

$$V_{\mathbb{R}} \otimes V_{\mathbb{R}} \xrightarrow{\psi_{\mathbb{R}}} \mathbb{R}(1)$$

s.t.
- $\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v)$
- $\frac{1}{2\pi i} \psi_{\mathbb{R}}(u, Jv) > 0$ for $u \neq 0$

Such V arises naturally as $H_1(A, \mathbb{Z})$ for an abelian variety $A / \mathbb{C}$.

e.g. The Hodge-Riemann bilinear relations yields a polarization on the primitive cohomology of a smooth complex projective variety.

§ Variations of Hodge Structures

Consider $\pi: X \to S$ smooth projective

w/ X, S quasiprojective complex manifolds.

We view this as a holomorphic family $X_s = \pi^{-1}(s)$

paramotrized by $s \in S$.

As discussed, $H^i(X_s; \mathbb{Z})$ admits an integral Hodge structure, so we ask what it means for (polarized) Hodge structures to vary in a family?

First,

Def. ? An $\mathbb{R}$-Hodge structure of weight k on $V_\mathbb{R}$ is given by a finite decreasing filtration $F^\bullet V_\mathbb{C}$

s.t. $\forall\, p+q = k+1$, $F^p V_\mathbb{C} \cap \overline{F^q V_\mathbb{C}} = 0$

$F^p V_\mathbb{C} \oplus \overline{F^q V_\mathbb{C}} = V_\mathbb{C}$

Rmk. $V^{p,q} = F^p V \cap \overline{F^q V}$

$F^p V = \bigoplus_{i \geq p} V^{i, n-i}$

Rmk. X a complex manifold,

$$\Omega^\bullet = [\Omega^0 \to \Omega^1 \to \dots]$$

Let $F^p\Omega^\bullet = [\dots \to 0 \to \Omega^p \to \Omega^{p+1} \to \dots]$

$\underline{\mathbb{C}} \to \Omega^\bullet$ is an acyclic resolution, hence a quasi-isomorphism.

$$\rightsquigarrow \underset{\substack{\parallel \\ E_1^{pq}}}{H^q(X, \Omega^p)} \Rightarrow H^{p+q}(X; \mathbb{C})$$

Hodge theory says that for X compact Kähler, this degenerates.

Def. A variation of Hodge structures of weight k over a complex manifold S is

- a local system $V_\mathbb{Z}$ of abelian groups on S
- A finite decreasing filtration $\mathcal{F}^p$ of
$$\mathcal{V} = V_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_S$$
by holomorphic subbundles

s.t.

- $\mathcal{F}^p_s$, a filtration on $V_{\mathbb{Z},s} \otimes_\mathbb{Z} \mathbb{C}$, yields a weight k Hodge structure
- $\mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S^1$ have $\ker(\nabla) = V_\mathbb{C}$.
$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S$$

The latter condition is "Griffiths transversality"

e.g. $A \to S$ abelian variety $/ S$ a smooth variety $/ \mathbb{C}$. Then $R^1 f_* \mathbb{Z}$ is a (polarized) variation of Hodge structures $/S$ of type $\{(-1,0), (0,-1)\}$

Thm (Griffiths, '68). $X \xrightarrow{f} S$ smooth proper morphism between smooth quasiprojective varieties. Then $R^q f_* \mathbb{Z}$ has the natural structure of a variation of Hodge structures on S via the relative de Rham complex.

$$\left(F^p R^q f_* \Omega^\bullet_{X/S} = \operatorname{im}\left(R^q f_* \tau^{\geq p} \Omega^\bullet_{X/S} \to R^q f_* \Omega^\bullet_{X/S} \right) \right)$$

Note. In such a context, f is a topological fibration so all fibers have the same cohomology, but not canonically. If S is simply connected, any local system is constant so we can consider variations of Hodge structures on a single V.

Def. Let V be a finite free $\mathbb{R}$-module. A variation of Hodge structures of weight k on V over S is a choice of Hodge structures $V_s^{p,q}$ for $s \in S$ s.t.

- $$\begin{aligned} S &\longrightarrow Gr(V_{\mathbb{C}}) \\ s &\longmapsto V_s^{p,q} \end{aligned}$$ is continuous $\forall p,q$

- $$\begin{aligned} S &\xrightarrow{\alpha} Flag(d, V_{\mathbb{C}}) \\ s &\longmapsto F_s^{\bullet} V_{\mathbb{C}} \end{aligned}$$ is holomorphic

  where $d(p) = \dim F_s^p V$

  and $Flag(d, V_{\mathbb{C}}) = \{(\cdots \subseteq W_p \subseteq \cdots \subseteq V_{\mathbb{C}}) \mid \dim W_p = d(p)\}$

- $d\alpha : T_s S \longrightarrow T_{F_s^\bullet}(Flag(d, V_{\mathbb{C}})) \subseteq \bigoplus_p Hom\left(F_s^p, \frac{V(\mathbb{C})}{F_s^p}\right)$

  has image in $\bigoplus_p Hom(F_s^p, F_s^{p-1}/F_s^p)$

  "Griffiths transversality"

e.g. - $V_{\mathbb{Z}} = \mathbb{Z}^2$, $k=1$ (De Jong AWS'02)

Then $V_{\mathbb{C},s} = V_s^{1,0} \oplus V_s^{0,1}$

Let $V_s^{1,0} = \left\langle \begin{pmatrix} 1 \\ f(s) \end{pmatrix} \right\rangle$

so $V_s^{0,1} = \left\langle \begin{pmatrix} 1 \\ \bar{f}(s) \end{pmatrix} \right\rangle$

for $f : S \longrightarrow \mathbb{P}^1$ holomorphic

- Now suppose we wanted this to be <u>polarized</u>.

$$\psi : V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \longrightarrow \mathbb{Z}(-1)$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) \longmapsto \frac{1}{2\pi i}(ad - bc)$$

Need $0 < \frac{1}{2\pi i} \psi\left(i \begin{pmatrix} 1 \\ f(s) \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{f}(s) \end{pmatrix} \right)$

$$= i\bar{f}(s) - if(s)$$

$$= 2\,\mathrm{Im}(f(s))$$

So $f : S \longrightarrow \mathbb{H}^*$

Rmk. $\mathbb{H}^*$, $\mathbb{P}^1$ carry (polarized) variations of Hodge structures over them which pull back to these.

Def. A Hermitian symmetric space is a Hermitian manifold S s.t.

- $\mathrm{Aut}(S) \curvearrowright S$ transitively
- $\forall p \in S$ $\exists s_p \in \mathrm{Aut}(S)$ an involution s.t. in a nbhd of p, p is the only fixed point of s_p.

e.g. $\mathbb{H}$ w/ $s_i = \frac{-1}{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\mathbb{CP}^1$ w/ reflection

abelian varieties w/ $z \mapsto -z$

(Ref. Milne Shimura Varieties + Moduli)

Fix $V_{\mathbb{Z}}$, $F_0^\bullet$ a Hodge filtration of weight k on V, $\psi_0 : V \otimes V \longrightarrow \mathbb{Z}(m)$ a polarization.

Let $D = D(V, F_0, \psi_0) = \left\{ \begin{array}{l} \text{Hodge filtrations} \\ F^\bullet \text{ on } V \text{ of} \\ \text{weight } k \end{array} \middle| \begin{array}{l} \dim F^p = \dim F_0^p \ \forall p \\ \psi_0 \text{ polarizes } F^\bullet \end{array} \right\}$

Thm. Let $d(p) = \dim F_0^p$. The natural inclusion

$$D \hookrightarrow \mathrm{Flag}(d, V_{\mathbb{C}})$$

realizes D as a locally closed submanifold.

This is the period domain (due to Griffiths) of (V, F_0, ψ_0).

Thm. $(\mathbb{V}, F, \psi)$ a polarized variation of Hodge structures on S w/ constant local system $\mathbb{V} = V$. Fix $0 \in S$.

$$\varphi : S \longrightarrow D(V, F_0, \psi_0)$$
$$s \longmapsto F_s$$

is holomorphic.

In this case, (V, F, ψ) is pulled back from a universal polarized variation of Hodge structures on $D = D(V, F_0, \psi_0)$.

Thm. If the universal polarized variation of Hodge structures on D satisfies Griffiths transversality, then D is a Hermitian symmetric domain.

Def. In the above setting of (V, F_0) suppose we also have $\{\psi_i\}_{i \in I}$ a family of tensors on (V, F_0) w/ $0 \in I$ s.t. ψ_0 is a polarization. We let $D(V, F_0, \psi)$ be a component of $D(V, F_0, \psi_0)$ where all ψ_i are Hodge tensors,

(i.e. $V^{\otimes 2r} \longrightarrow \mathbb{Z}(-mr)$ is a map of Hodge structures)

This is a so-called period sub domain, and $D(V, F_0, \psi) \subseteq D(V, F_0, \psi_0)$

Thm. Every Hermitian symmetric domain is a period subdomain.