Reduction of Drinfeld Modules and Tate Uniformization

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§1. Classical Tate uniformization

§2. Reduction of Drinfeld modules

§3. Tate uniformization of Drinfeld modules

§1.

Ref. Silverman AAEC, ch. 5

First, work / $\mathbb{C}$.

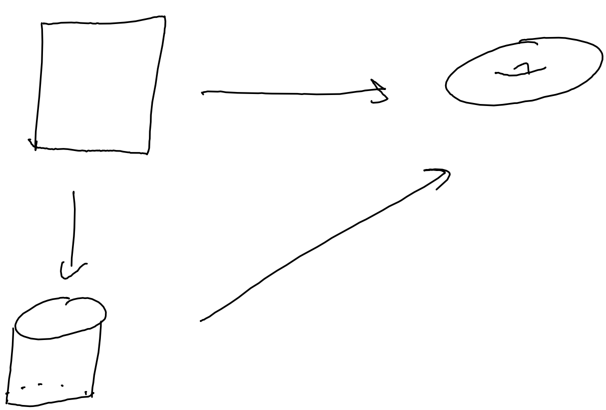
Let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathbb{H}$.

$$\mathbb{C}/\Lambda_\tau \xrightarrow[{[\wp : \wp' : 1]}]{\sim} E_\tau(\mathbb{C}), \quad E_\tau = \{y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\}$$

w/ $\wp = \wp(z;\tau)$

Let $u = e^{2\pi i z}$, $q = e^{2\pi i \tau}$. As $\tau \in \mathbb{H}$, $|q| < 1$.

$$\begin{tikzcd}
z \arrow[d, mapsto] & \mathbb{C}/\Lambda_\tau \arrow[r, "\sim", "{[\wp : \wp' : 1]}"'] \arrow[d, "\sim"'] & E_\tau(\mathbb{C}) \\
u & \mathbb{C}^\times/q^{\mathbb{Z}} \arrow[ur, "\sim", "(?)"'] &
\end{tikzcd}$$



Recall the q-expansions

$$\frac{1}{(2\pi i)^4}\, g_2(\tau) = \frac{1}{12}\left(1 + 240\, S_3(q)\right)$$

$$\frac{1}{(2\pi i)^6}\, g_3(\tau) = \frac{1}{216}\left(-1 + 504\, S_5(q)\right)$$

$$\left(S_k(q) = \sum \sigma_k(n)\, q^n \qquad \sigma_k(n) = \sum_{d \mid n} d^k \right)$$

We may similarly determine $\wp(u;q)$ and $\wp'(u;q)$

$$\frac{1}{(2\pi i)^2}\, \wp(u;q) = \underbrace{\sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2}} + \frac{1}{12} - 2S_1(q)$$

model for double poles w/ multiplicative q-periodicity

$$\frac{1}{(2\pi i)^3}\, \wp'(u;q) = \sum_{n\in\mathbb{Z}} \frac{q^n u\,(1+q^n u)}{(1-q^n u)^3}$$

As well as

$$\Delta(q) = (2\pi i)^{12}\, q \prod_{n\geq 1} (1-q^n)^{24}$$

$$j(q) \in \frac{1}{q} + q\,\mathbb{Z}[[q]]$$

We change variables

$$\frac{1}{(2\pi i)^2}\, x = x' + \frac{1}{12}$$

$$\frac{1}{(2\pi i)^3}\, y = 2y' + x'$$

To rewrite the Weierstrass equation as

$$y'^2 + x'y' = x'^3 + a_4(q)\, x' + a_6(q)$$

$$a_4(q) = -5 s_3(q), \qquad a_6(q) = \frac{-5 s_3(q) + 7 s_5(q)}{12}$$

In summary, let

$$E_q = \{ y^2 + xy = x^3 + a_4(q)\, x + a_6(q) \}$$

w/ $a_4(q), a_6(q) \in \mathbb{Z}[[q]]$ as above.

Then E_q is an elliptic curve w/ an isomorphism of complex Lie groups

$$\mathbb{C}^\times / q^{\mathbb{Z}} \xrightarrow[{[x(u;q),\, y(u;q);\, 1]}]{\sim} E_q(\mathbb{C}), \text{ with } \begin{aligned} X(u;q) &= \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2 s_1(q) \\ Y(u;q) &= \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q) \end{aligned}$$

w/ $\Delta(q) \in \mathbb{Z}[[q]]$, $j(q) \in \frac{1}{q} + \mathbb{Z}[[q]]$.

Thm. All e.c./$\mathbb{C}$ are of this form for some $|q|<1$.

Now, let K be a complete nonarchimedean field

Note that if $\Lambda \subseteq K$ a discrete subgroup, then for any $\lambda \in \Lambda$, $p^n \lambda \in \Lambda$. But $\lim_{n\to\infty} p^n \lambda = 0$, so 0 is an accumulation point of Λ. Hence, $\Lambda = 0$.

But $K^\times$ has many lattices.

$$\left(\begin{aligned} \text{Recall} \quad \mathbb{Q}_p^\times &= \langle p \rangle \times \mathbb{Z}_p^\times \\ &= \langle p \rangle \times \mu_{p-1}(\mathbb{Z}_p) \times \mathbb{Z}_p \end{aligned} \right)$$

such as ... $q^{\mathbb{Z}}$ for $|q|<1$.

Thm. Define $a_4, a_6, \Delta, j, X, Y$ formally in q. Specializing to $q \in K^\times$ w/ $|q|<1$ yields convergent series and a Tate elliptic curve E_q admitting an G_K-equivariant isomorphism

$$K_{sep}^\times / q^{\mathbb{Z}} \xrightarrow[{[X, Y, 1]}]{\sim} E_q(K_{sep})$$

so that $L^\times / q^{\mathbb{Z}} \xrightarrow{\sim} E_q(L)$

for any L/K separable.

Pf idea. These all reduce to power series identities, which hold formally in q as they hold over $\mathbb{C}$. Surjectivity is hard.

Rmk. $|j(q)| = \left|\frac{1}{q} + O(1)\right| = \frac{1}{|q|} > 1.$

Hence, $|j(q)| > 1$ is necessary for an elliptic curve $E_{/K}$ to be uniformized by the Tate curve.

Thm. Given any $E_{/K}$ w/ $|j| > 1$, $\exists!\ q \in K^\times$ w/ $|q| < 1$ so that

$$E \cong_{/K_{sep}} E_q$$

by specializing the power series.

Furthermore, let $\gamma(E/K) = -c_4/c_6 \in K^\times/K^{\times 2}$, which is independent of the choice of Weierstrass equation. Then TFAE.

- $E \cong_{/K} E_q$
- $\gamma(E/K) = 1$
- E has split multiplicative reduction.

Rmk. For fixed $j \neq 0, 1728$, γ determines K-isomorphism. We have, in that case, $\mathrm{Twist}(E/K) = H^1(K, \mathrm{Aut}(E)) = H^1(K, \mu_2) = K^\times/K^{\times 2}$.

Let's at least observe that the Tate curve has split multiplicative reduction. Indeed, $a_4(q)$ and $a_6(q)$ are both in $q\mathbb{Z}[[q]]$.

Hence, $\overline{E_q} = \{y^2 + xy = x^3\}$.

§2

Refs: Papikian, Poonen

Notation: $F = \mathbb{F}_q(C)$ some curve $C/\mathbb{F}_q$.

∞ a place of F

$A = H^0(C - \{\infty\}, \mathcal{O})$

(K, v) a dvf, $\mathcal{O}_K$ its ring of integers, k its residue field.

suppose we have $A \longrightarrow \mathcal{O}_K$.

Def. Let $\varphi : A \longrightarrow K\{\tau\}$ be a Drinfeld module.

we say

- φ has integral coefficients if φ lands in $\mathcal{O}_K\{\tau\}$
- φ has stable reduction if it is isomorphic to a Drinfeld module w/ integral coefficients ψ so that $A \xrightarrow{\bar{\psi}} k\{\tau\}$ is a Drinfeld module of positive rank
- φ has good reduction if it has stable reduction and the rank does not drop.

Recall

$$C/\Lambda = \begin{cases} \mathbb{G}_a(\mathbb{C}) & \mathrm{rk}\ \Lambda = 0 \\ \mathbb{G}_m(\mathbb{C}) & \mathrm{rk}\ \Lambda = 1 \\ E(\mathbb{C}) & \mathrm{rk}\ \Lambda = 2 \end{cases}$$

Heretofore, let $C = \mathbb{P}^1_{\mathbb{F}_q}$

$A = \mathbb{F}_q[T]$

e.g. Let $\varphi_T = T + \frac{1}{\pi}\tau + \tau^2$. Any isomorphic Drinfeld module is of the form

$$u^{-1}\varphi_T u = T + \frac{u^{q-1}}{\pi}\tau + u^{q^2-1}\tau^2$$

for some $u \in K^\times$.

Then integrality requires $v(u) \geq 1$, which forces $\mathrm{rk}(\overline{u^{-1}\varphi_T u}) \leq 1$. If $q > 2$, the reduction is just T, so not a Drinfeld module.

More generally, insisting $u^{-1}(T + c_1\tau + c_2\tau^2)u = T + u^{q-1}c_1\tau + u^{q^2-1}c_2\tau$ is integral uniquely specifies $v(u) \in \mathbb{Q}$. So we may need to pass to a ramified extension.

Now suppose $\varphi_T = T + g_1 \tau + \dots + g_r \tau^r$.

Then

$$u^{-1}\varphi_T u = T + \sum_{i=1}^{r} c^{q^i-1} g_i \tau^i$$

$$\frac{v(c^{q^i-1} g_i)}{q^i - 1} = v(c) + \frac{v(g_i)}{q^i-1}$$

Def. Let $e(\varphi) = \min_{1\le i\le r} \frac{v(g_i)}{q^i-1}$

$$r'(\varphi) = \max\{1 \le i \le r \mid e(\varphi) = \frac{v(g_i)}{q^i-1}\}$$

Lemma. Let $\varphi : A \longrightarrow K\{\tau\}$ a Drinfeld module, L/K finite.

i) φ is stable$/L$ $\iff$ $e(\varphi) \in v(L)$.

ii) If φ is stable$/L$, $r'(\varphi)$ is the rank of any reduction.

Pf. idea. $e(c^{-1}\varphi c) = v(c) + e(\varphi)$

stable means $\exists c$ s.t. $e(c^{-1}\varphi c) = 0$

e.g. The Carlitz module $\varphi_T = T + \tau$ has good reduction.

Cor. All Drinfeld modules have potentially stable reduction.

Lemma. Suppose $a \in A$ has positive degree and $v(a) \geq 0$. If the slope of the first line segment of the Newton polygon of $\frac{\varphi_a(x)}{x}$ is an integer, φ has stable reduction.

§3

Papikian

As before, complete nonarchimedean fields have no lattices.

Before, we went to $K^\times$ using an exponential.
Here, we just change the A-action via a Drinfeld module.

Def. Let $\varphi: A \longrightarrow K\{\tau\}$ a Drinfeld module.
A φ-lattice is a free A-submodule $\Lambda \subseteq {}^{\varphi}K^{sep}$ of finite rank which is G_K-invariant and is discrete in $\mathbb{C}_K$, i.e. $\Lambda \cap B$ is finite for all balls B.

e.g. $\varphi: A \longrightarrow \mathcal{O}_K\{\tau\}$ with good reduction.
Let $\omega \in K^\times$. Define $\Lambda = \{ e_a(\omega) \mid a \in A \}$ (c.f. $q^{\mathbb{Z}}$).
If ω is not φ-torsion, this is free of rank 1.
Also, if $|\omega| \leq 1$ then $\Lambda \subseteq \mathcal{O}_K = \bar{B}_1(0)$, so this fails to be discrete. Hence, this is a φ-lattice $\Leftrightarrow |\omega| > 1$.

Let Λ be a φ-lattice.

We thus have an entire exponential

$$e_\Lambda(x) = x \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{x}{\lambda}\right)$$

$\varphi_a^{-1}\Lambda$ is not a lattice, as it contains torsion, but it is discrete as Λ has finite index in $\varphi_a^{-1}\Lambda$. We have

$$0 \longrightarrow \varphi[a] \longrightarrow \varphi_a^{-1}\Lambda/\Lambda \longrightarrow \Lambda/\varphi_a\Lambda \longrightarrow 0$$

(By snake lemma on

$$\begin{tikzcd}
 & \Lambda \arrow[r, hook] \arrow[d, "\varphi_a"'] & \varphi_a^{-1}\Lambda \arrow[r] \arrow[d, "\varphi_a"] & \varphi_a^{-1}\Lambda/\Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "\mathrm{id}"'] & \Lambda \arrow[r] & 0 &
\end{tikzcd}$$

)

The zeroes of e_Λ are Λ, so the zeroes of $e_\Lambda \circ \varphi_a$ are $\varphi_a^{-1}\Lambda$.

Exponentials are uniquely determined, up to scalars, by their zero set. Hence,

$$a e_{\Lambda'} = e_\Lambda \circ \varphi_a$$

As $\Lambda \subseteq \varphi_a^{-1}\Lambda$ is finite index, let $z_1, \dots, z_m$ be coset representatives. Then let $P_a(x) = x \prod_{\substack{i=1 \\ z_i \notin \Lambda}}^{m} \left(1 - \frac{x}{e_\Lambda(z_i)}\right)$, a polynomial.

It follows that

$$e_\Lambda(\varphi_a(x)) = a P_a(e_\Lambda(x))$$

Let $\psi_a(x) = a P_a(x)$, so that

$$e_\Lambda \circ \varphi_a = \psi_a \circ e_\Lambda$$

Prop. $a \longmapsto \psi_a$ is a Drinfeld module of rank $rk(\varphi) + rk(\Lambda)$.

Pf. Being a Drinfeld module is the same proof as in the analytic theory showing lattices correspond to Drinfeld modules.

$rk(\psi) = \deg_\tau \psi_T = \log_q([\varphi_T^{-1}\Lambda/\Lambda])$ by def'n of P_a

$= \log_q(|\varphi[T]|) + \log_q([\Lambda : T\Lambda])$ by the SES

$= rk(\varphi) + rk(\Lambda)$ $\square$

We have

$$0 \longrightarrow \Lambda \longrightarrow {}^{\varphi}\mathbb{C}_K \xrightarrow{e_\Lambda} {}^{\psi}\mathbb{C}_K \longrightarrow 0$$

Def. If ψ has good reduction, we say (φ, Λ) is the Tate uniformization of ψ.

Thm. $$\left\{ (\varphi,\Lambda) \;\middle|\; \begin{array}{l} \varphi: A \to \mathcal{O}_K\{\tau\} \text{ good reduction rk } r \\ \Lambda \text{ a rk } d \ \varphi\text{-lattice} \end{array} \right\} \Big/ \cong \longrightarrow \left\{ \begin{array}{l} \text{rk } r+d \text{ integral Drinfeld} \\ \text{modules with rk } r \text{ reduction} \end{array} \right\} \Big/ \cong$$

$$(\varphi,\Lambda) \longmapsto \psi$$

is a bijection satisfying

- $\bar{\varphi} = \bar{\psi}$
- $\mathrm{Hom}(\psi_1, \psi_2) \xrightarrow{\sim} \{ u \in \mathrm{Hom}(\varphi_1, \varphi_2) \mid u\Lambda_1 \subseteq \Lambda_2 \}$

$$w \longmapsto e_{\Lambda_2}^{-1}\, w\, e_{\Lambda_1}$$

We must reverse this process to $\psi \longmapsto (\varphi, \Lambda)$.

We need to solve

$$e_\Lambda \circ \varphi_T = \psi_T \circ e_\Lambda$$

for both Λ and φ.

Idea: We know $\bar{\varphi}_T = \bar{\psi}_T$. We can construct Λ, φ_T by iterative lifting.

Lemma. B an $\mathbb{F}_q$-algebra, $d \in \mathbb{Z}^{\geq 0}$.

Let $f = \sum_{i=0}^{n} f_i z^i \in B\{z\}$ s.t.

- $f_d \in B^\times$
- $f_{d+1}, \dots, f_n \in \mathrm{nil}(B)$

$\exists!$ $u = \sum u_j z^j \in B\{z\}^\times$ s.t.

- $u_0 = 1$
- $u_j \in \mathrm{nil}(B)$ $j \geq 1$
- $g = u^{-1} f u$ has degree d w/ $\mathrm{lead}(g) \in B^\times$.

Lifting lemma. Let $f \in \mathcal{O}_K\{z\}$, $d = \deg(\bar{f}) \geq 0$.

$\exists!$ $u \in \mathcal{O}_K\{\{z\}\}$ s.t.

- $u = 1 + \sum_{i \geq 1} \alpha_i z^i$, $|\alpha_i| < 1$, $\alpha_i \to 0$
- $g = u^{-1} f u \in \mathcal{O}_K\{z\}$ with $\deg(g) = \deg(\bar{g}) = d$.
- $u(x)$ entire

Pf. For all $k \geq 1$, the prior lemma implies the existence of a unique unit $u_k \in (\mathcal{O}_K/\mathfrak{m}^k)\{z\}^\times$ w/ constant term 1 s.t. that $u_k^{-1} \bar{f} u_k$ is a degree d polynomial with $\mathrm{lead}(u_k^{-1} \bar{f} u_k) \in (\mathcal{O}_K/\mathfrak{m}^k)^\times$. By uniqueness, $u_k \equiv u_{k-1} \bmod \mathfrak{m}^{k-1}$. Then $u = \lim_{k \to \infty} u_k$ works $\square$

Now, suppose we have a Drinfeld module

ψ of rank $r+d$ with reduction $\bar{\psi}$ of rank r,

By the lifting lemma, $\exists!\ e = 1 + \sum_{i \geq 1} a_i \tau^i$ w/ $a_i \in \mathfrak{m}$

s.t.

- $\varphi_T := e^{-1} \psi_T e \in \mathcal{O}_K\{\tau\}$ has deg r
- $\bar{\varphi}_T = \bar{\psi}_T$
- $e(x)$ is entire

Application: Néron-Ogg-Shafarevich

Thm. Let $E_{/K}$ be an elliptic curve over a local field K. For $\ell \neq p = \text{char}(k)$

i) E has good reduction

ii) $E[m]$ is unramified $\forall (m,p)=1$.

iii) $\exists$ infinitely many m coprime to p s.t. $E[m]$ is unramified

iv) $T_\ell E$ is unramified.

Pf sketch. Suppose E has bad reduction w/ i), ii). Additive reduction is ruled out as $\tilde{E}_{smth}(\bar{k})$ would have no torsion. So suppose E has multiplicative reduction.

$\exists$ finite extension k'/k s.t. $\tilde{E}_{k', smth} \cong \mathbb{G}_{m,k'}$. Then let K'/K be the corresponding unramified extension. So WLOG, E has split multiplicative reduction. Hence, E is K-isomorphic to its Tate uniformization.

$$\left(K_{sep}^\times / q^{\mathbb{Z}}\right)[m] = q^{\frac{1}{m}\mathbb{Z}} / q^{\mathbb{Z}}$$

$$= \mu_m \langle q^{1/m} \rangle / q^{\mathbb{Z}}$$

For m prime to $\mathrm{char}(k)$, $K(\mu_m)/K$ is unramified, so WLOG $\mu_m \subseteq K$.

Now, $K(q^{1/m}) = K\left(\left(K_{sep}^\times / q^{\mathbb{Z}}\right)[m]\right) / K$ is ramified, so inertia is nontrivial. Thus,

$E[m] \cong \left(K_{sep}^\times / q^{\mathbb{Z}}\right)[m]$ is ramified. $\square$

Thm. Let φ be a Drinfeld module over a local field $K/\mathbb{F}_\mathfrak{p}$ for $\mathfrak{p} \subseteq A$ prime. For $\mathfrak{l} \subseteq A$ prime distinct from $\mathfrak{p}$ TFAE.

i) φ has good reduction

ii) $\varphi[a]$ is unramified $\forall\, \mathfrak{p} \nmid a$

iii) $\exists\, \infty$ly many a coprime to $\mathfrak{p}$ s.t. $\varphi[a]$ is unramified

iv) $T_\mathfrak{l}\varphi = \varprojlim \varphi[\mathfrak{l}^n]$ is unramified

Pf sketch. The hard part again is proving good reduction from the Galois representation.

We show (iii) $\Rightarrow$ (i).

Suppose indeed $\varphi[a]$ is unramified for ∞ly many $\mathfrak{p} \nmid a$.

Step 1. φ has stable reduction.

Pf. WLOG φ has integral coefficients.

Let $a \in A$ have positive degree s.t. $\mathfrak{p} \nmid a$ and $\varphi[a]$ is unramified.

$K(\varphi[a])$ is unramified, so the roots of $\frac{\varphi_a(x)}{x}$ have integral valuation. Hence, the Newton polygon has integral slopes, so we have stable reduction.

Step 2. Let $r' = rk(\bar{\varphi})$. Suppose $r' < r$. Then let (e, Λ) be the Tate uniformization

$((\text{cf. } E \rightsquigarrow (\mathbb{G}_m, q^{\mathbb{Z}})))$

Pass to a finite extension to ensure $\Lambda \subseteq K$.

If $\exists\, \lambda \in \Lambda - \{0\}$ s.t. $|\lambda| \le 1$ then $\underbrace{|\varphi_a(\lambda)| \le 1}$ $\forall a \in A$,

(∞ set as torsion free)

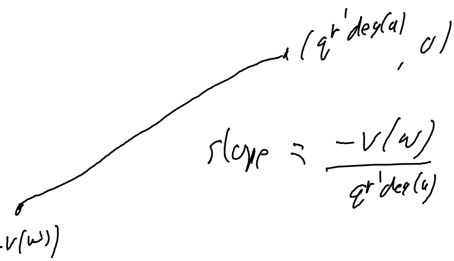
contradicting discreteness.

Hence, $|\lambda| > 1$ for all $\lambda \neq 0$.

Let $w \in \Lambda - \{0\}$ of minimal norm, i.e. maximal valuation.

Let $a \in A$ s.t. $\mathfrak{p} \nmid a$ and $\deg(a) > 0$.

Consider $\varphi_a(x) - w$. The nonconstant coefficients are in $\mathcal{O}_K$ and its leading coefficient is a unit, so the Newton polygon is



Let $\varphi_a(z) = w$. As $e_1 \circ \varphi_a = \psi_a \circ e_1$, we have

$\psi_a(e_1(z)) = e_1\, \varphi_a(z) = e_1(w) = 0$, as $w \in \Lambda$.

That is, $e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right) \in \psi[a]$

$$v\left(\frac{z}{\lambda}\right) = \frac{v(w)}{q^{r' \deg(a)}} - v(\lambda)$$

$$\geq \frac{v(w)}{q^{r' \deg(a)}} - v(w)$$

$$> 0 \quad \text{as} \quad r' \deg(a) > 0.$$

Hence, $v\left(1 - \frac{z}{\lambda}\right) = 0$ for $\lambda \neq 0$,

so $v(e_\Lambda(z)) = v(z) = \frac{v(w)}{q^{r' \deg(a)}}$

Thus, $\psi[a]$ contains element of valuation $\frac{v(w)}{q^{r' \deg(a)}}$

For all but finitely many choices of $\deg(a)$, this is not in $\mathbb{Z}$, implying $K(\psi[a])$ is ramified for all but finitely many $\mathfrak{p} \nmid a$. $\times$ $\square$