The First and Second Inequalities in Class Field Theory

Author. Jas Singh

Abstract

We present in this report an account of the first and second inequalities from class field theory. Our focus will primarily be the modern idèle theoretic approach via group cohomology. We will also discuss consequences of these inequalities, such as properties of the reciprocity map and the Hasse norm theorem.

Our primary references will be [Milne] and [CF] – especially Tate's article in chapter VII.

0 Notation and Conventions

K will denote a global field and L will be some finite extension. If v is a place of K, we will often write L_v where we mean L_w for some choice of w over v. Similarly, G_v will denote the decomposition group for any choice of w. We will only use this notation when the choice of w is irrelevant. $N_{L/K}$ denotes the norm, and we will write this simply as N when the fields are understood.

$\mathbb{I}_K$ will denote the idèles of K and $\mathbb{C}_K = \mathbb{I}_K/K^*$ will be the idèle class group.

1 First Inequality

Convention. In this section, L/K will be abelian.

1.1 Statement and proof

The proofs given here can be found in more detail in [Milne, VII §4] and [CF, VII §8]. We begin with the idèlic statement of the first inequality.

Theorem 1.1 (The first inequality)**.** *Let L/K be cyclic. Then we have the bound*

$$[\mathbb{I}_K : K^* N_{L/K}\mathbb{I}_L] \geq [L : K]$$

The key to proving this inequality is to compute the Herbrand quotient of the idèle class group.

Definition 1.1. For a cyclic group G and a G module A, the Herbrand quotient $h(A) = h(G, A)$ is the quotient $\frac{|H^2(G,A)|}{|H^1(G,A)|}$, whenever this expression makes sense.

Lemma 1.2. *Let L/K be cyclic with Galois group G. Then $H^2(G, \mathbb{C}_L) \cong \mathbb{I}_K/K^* N\mathbb{I}_L$.*

Proof. As G is cyclic the Tate cohomology groups are periodic, and we have $H^2(G, \mathbb{C}_L) = \widehat{H}^0(G, \mathbb{C}_L)$. By definition, the latter is $\mathbb{C}_L^G / N\mathbb{C}_L = \mathbb{C}_K / N\mathbb{C}_L$. The fact that $\mathbb{C}_L^G = \mathbb{C}_K$ is justified by the short exact sequence

$$0 \longrightarrow L^* \longrightarrow \mathbb{I}_L \longrightarrow \mathbb{C}_L \longrightarrow 0$$

and Hilbert's theorem 90, which says that $H^1(G, L^*) = 0$.

Now, we seek to show $\mathbb{C}_K / N\mathbb{C}_L \cong \mathbb{I}_K / K^* N\mathbb{I}_L$. Indeed, by definition of the idèle class group, the left hand side here is

$$\mathbb{C}_K / N\mathbb{C}_L = \frac{\mathbb{I}_K / K^*}{N(\mathbb{I}_L / L^*)}$$

This is isomorphic to $\mathbb{I}_K / K^* N\mathbb{I}_L$. For instance, one can use the Yoneda lemma and observe that maps out of $\frac{\mathbb{I}_K / K^*}{N(\mathbb{I}_L / L^*)}$ are precisely maps out of $\mathbb{I}_K$ which vanish on the subgroup $K^* N\mathbb{I}_L$. $\square$

As such, we have shown that the numerator of the Herbrand quotient $h(\mathbb{C}_L)$ is exactly the term we wish to control in the first inequality. The computation of $h(\mathbb{C}_L)$ as follows.

Proposition 1.1. $h(\mathbb{C}_L) = [L : K]$ *in the above setting.*

From this, the first inequality follows shortly.

Proof of the first inequality. By Lemma 1.2, the Herbrand quotient can be computed as

$$h(\mathbb{C}_L) = \frac{|\mathbb{I}_K / K^* N\mathbb{I}_L|}{|H^1(G, \mathbb{C}_L)|}$$

Proposition 1.1 ensures that this quantity equals $[L : K]$, so the numerator must be at least $[L : K]$. $\square$

We will now explain how to compute $h(C_L) = [L : K]$. First, a few facts about the Herbrand quotient. These can be found in most texts about group cohomology, such as [CF, IV §8].

(i) The Herbrand quotient is multiplicative on short exact sequences. That is, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of G modules, then $h(B) = h(A)h(C)$.

(ii) If A is finite then $h(A) = 1$.

Now we return to computing the Herbrand quotient of $\mathbb{C}_L$.

Proof of Proposition 1.1. Let A be a finite set of primes in L generating its ideal class group. Let S be some finite set of primes of K containing the infinite places, the ramified places, and A. Now let T be the primes in L lying over S.

Define $\mathbb{I}_T = \prod_{w \in T} L_w \times \prod_{w \notin T} \mathcal{O}_w^*$. Let I_L be the ideal group of L. Then $\mathbb{I}_T$ is the kernel of the map $\mathbb{I}_L \longrightarrow I_L \longrightarrow I_L/\langle T \rangle$. Hence, $\mathbb{I}_L/\mathbb{I}_T \cong I_L/\langle T \rangle$. But we chose T so that it contains a set of generators for the ideal class group, and as such, $\langle T \rangle L^* = I_L$. Thus, by the isomorphism $\mathbb{I}_L/\mathbb{I}_T \cong I_L/\langle T \rangle$, we have that $\mathbb{I}_L = \mathbb{I}_T L^*$.

We now compute $\mathbb{C}_L$ in terms of $\mathbb{I}_T$. Indeed,

$$\begin{aligned} \mathbb{C}_L &= \frac{\mathbb{I}_L}{L^*} \\ &= \frac{\mathbb{I}_T L^*}{L^*} \\ &= \frac{\mathbb{I}_T}{L^* \cap \mathbb{I}_T} \end{aligned}$$

Furthermore, by definition of $\mathbb{I}_T$ that the elements of $L^* \cap \mathbb{I}_T$ are those elements a of L so that $w(a) = 0$ for all $w \notin T$. We refer to this group as $U(T)$ – the group of T units. We have therefore written $\mathbb{C}_L = \mathbb{I}_T/U(T)$, so by multiplicativity of Herbrand quotients, we have $h(\mathbb{C}_L) = h(\mathbb{I}_T)/h(U(T))$. As such, we have reduced to computing these two remaining Herbrand quotients.

Herbrand quotient of $\mathbb{I}_T$.

We have by definition of $\mathbb{I}_T$ that

$$\begin{aligned} \mathbb{I}_T &= \prod_{w \in T} L_w^* \times \prod_{w \notin T} \mathcal{O}_w^* \\ &= \prod_{v \in S} \prod_{w \mid v} L_w \times \prod_{v \notin S} \prod_{w \mid v} \mathcal{O}_w^* \end{aligned}$$

Hence, we see that

$$h(\mathbb{I}_T) = h\left(\prod_{v \in S} \prod_{w \mid v} L_w^* \right) h\left(\prod_{v \notin S} \prod_{w \mid v} \mathcal{O}_w^* \right)$$

We consider the first term here. By multiplicativity, we have $h\Big(\prod_{v \in S} \prod_{w \mid v} L_w^*\Big) = \prod_{v \in S} h\Big(\prod_{w \mid v} L_w^*\Big)$. By Hilbert's theorem 90, we have $h\Big(\prod_{w \mid v} L_w^*\Big) = |H^2(G, \prod_{w \mid v} L_w^*)|$. We use Shapiro's lemma to calculate $H^2(G, \prod_{w \mid v} L_w^*) = H^2(G_v, L_v^*)$ via restriction and corestriction. Furthermore, this cohomology group is isomorphic to $\frac{1}{[L_v:K_v]}\mathbb{Z}/\mathbb{Z}$ via the invariant map. One can see this in [CF, VI]. We will let $n_v = [L_v : K_v]$, so that we have shown $h\Big(\prod_{v \in S} \prod_{w \mid v} L_w^*\Big) = \prod_{v \in S} n_v$.

Now we consider the second term $h\Big(\prod_{v \notin S} \prod_{w \mid v} \mathcal{O}_w^*\Big)$. Note that cohomology commutes with products in the coefficients. One can see this immediately via cochains, or more abstractly by computing an isomorphism on H^0 and using uniqueness of derived functors. As such, we are left to consider $h(\mathcal{O}_w^*)$, which we claim to be 1.

Lemma 1.3. *There is an open subgroup $U \subseteq \mathcal{O}_w^*$ with trivial cohomology, i.e. $H^i(G,U) = 0$ for all i.*

Proof. See [CF, VI §1.4]. $\square$

This subgroup U is open in $\mathcal{O}_w^*$ and is hence finite index. The Herbrand quotient is therefore trivial on the quotient $\mathcal{O}_w^*/U$ so by multiplicativity, we conclude $h(\mathcal{O}_w^*) = h(U) = 1$.

Thus, we have computed $h(\mathbb{I}_T) = \prod_{v\in S} n_v$.

Herbrand quotient of $U(T)$.

The strategy is to take inspiration from Dirichlet's unit theorem and compare $U(T)$ to a codimension 1 lattice in some vector space. Indeed, consider $V = \mathrm{Hom}(T, \mathbb{R})$ and $\lambda : U(T) \longrightarrow V$ via $a \mapsto (w \mapsto \log |a|_w)$. V has a G action via the usual action on T and the trivial action on $\mathbb{R}$. λ is a morphism of G modules, and following the proof of Dirichlet's unit theorem in this case leads us to conclude that $M^0 = \mathrm{im}(\lambda)$ is a lattice in $V^0 = \{f \in V : \sum_{w\in T} f(w) = 0\}$. Furthermore, the kernel of λ will be the roots of unity in L, and is hence finite. As such, $h(U(T)) = h(M^0)$.

We consider the constant function $g : T \longrightarrow \mathbb{R}$ via $g(w) = 1$. Then $V = V^0 \oplus \mathbb{R}g$ and $M = M^0 \oplus \mathbb{Z}g$ is a lattice in V. Furthermore, $\mathbb{Z}g$ is a trivial G module and the above direct sum is as G modules. We compute

$$\begin{aligned} h(M) &= h(M^0)h(\mathbb{Z}) \\ &= h(U(T))[L : K] \end{aligned}$$

so we are left to compute $h(M)$. This is done by showing that all lattices in V have the same Herbrand quotient, and then finding a more convenient lattice to compute with.

Lemma 1.4. *Let A and B be G stable lattices in a finite dimensional $\mathbb{R}[G]$ module V. Then $h(A) = h(B)$.*

Proof. First off, as these are G stable lattices in V we have that $A\otimes\mathbb{R} \cong B\otimes\mathbb{R} \cong V$ as $\mathbb{R}[G]$ modules. Such an isomorphism arises from some G invariant matrix with nonvanishing determinant. A linear algebraic argument shows that those conditions can descend to $\mathbb{Q}$, and as such we have the isomorphism $A \otimes \mathbb{Q} \cong B \otimes \mathbb{Q}$.

From this, we can express A as a submodule of $\frac{1}{N}B$, where N arises by taking bases and clearing denominators. Upon multiplying by N, we can therefore view A as a submodule of B. Both are free abelian groups of the same rank, so their quotient is finite. Hence $h(A) = h(B)$.

See [CF, IV §8] for more details. □

Our next lattice will be $\mathrm{Hom}(T, \mathbb{Z})$. By decomposition into orbits, this becomes $\bigoplus_{v \in S} \mathrm{Hom}(G/G_v, \mathbb{Z})$. By Shapiro's lemma, we have

$$h(G, \mathrm{Hom}(G/G_v, \mathbb{Z})) = h(G_v, \mathbb{Z})$$

which we can compute to be n_v. Hence, we have compute $h(\mathrm{Hom}(T, \mathbb{Z})) = \prod_{v \in S} n_v$.

By the lemma, this therefore computes $h(M) = \prod_{v \in S} n_v$. So we may compute

$$\begin{aligned} h(U(T)) &= \frac{h(M)}{[L : K]} \\ &= \frac{\prod_{v \in S} n_v}{[L : K]} \end{aligned}$$

and finally, we conclude

$$\begin{aligned} h(\mathbb{C}_L) &= \frac{h(\mathbb{I}_T)}{h(U(T))} \\ &= \frac{\prod_{v \in S} n_v}{\frac{\prod_{v \in S} n_v}{[L:K]}} \\ &= [L : K] \end{aligned}$$

□

1.2 Corollaries

Our main goal will be to show surjectivity of the reciprocity map, i.e. that Frobenius elements generate the Galois group.

Proposition 1.2. *Let L/K be abelian with Galois group G. Let T be a finite set of primes in L containing those which are ramified over K. Then the Frobenius elements $(\mathfrak{P}, L/K)$ for $\mathfrak{P} \notin T$ generate the Galois group G.*

The proof will use the following two lemmas, which arise from the first inequality.

Lemma 1.5. *Suppose there is a subgroup D of $\mathbb{I}_K$ so that*

(i) $D \subseteq N\mathbb{I}_L$

(ii) K^*D is dense in $\mathbb{I}_K$

Then $L = K$.

Proof. Take some $L/E/K$ with E/K cyclic. Then we have

$$\begin{aligned} D &\subseteq N_{L/K}\mathbb{I}_L \\ &= N_{E/K}N_{L/E}\mathbb{I}_L \\ &\subseteq N_{E/K}\mathbb{I}_E \end{aligned}$$

so as K^*D is dense in $\mathbb{I}_K$, we have that $K^*N_{E/K}\mathbb{I}_E$ is dense in $\mathbb{I}_K$. The K^* action on $\mathbb{I}_K$ is continuous and $N_{E/K}\mathbb{I}_E$ is open in $\mathbb{I}_K$, so $K^*N_{E/K}$ is an open dense subgroup of $\mathbb{I}_K$. Open subgroups are also closed, so $K^*N_{E/K}\mathbb{I}_E$ is a closed dense subgroup of $\mathbb{I}_K$, so they are equal.

Now, by the first inequality, we get the upper bound $[E : K] \leq [\mathbb{I}_K : K^*N_{E/K}\mathbb{I}_E] = 1$. As such, $E = K$. So there are no intermediate cyclic extensions in L/K. As this extension is abelian (or even weaker, solvable) this shows that $L = K$. □

Lemma 1.6. *Suppose there are only finitely many primes in K which are not completely split in L. Then $L = K$.*

Proof. We want to show $L = K$ so by the above lemma, we seek an appropriate subgroup D. Take S to be a finite set of primes containing the infinite primes as well as all the primes not completely split in L. Our candidate for D will be

$$D = \{a \in \mathbb{I}_K : a_v = 1 \text{ for } v \in S\}$$

We check the two conditions needed for D.

(i) We show that $D \subseteq N\mathbb{I}_L$. Certainly 1 is a norm, so we need only show that a_v for $v \notin S$ is a norm when $a \in D$. Indeed, for $v \notin S$, v is completely split in L so L_v/K_v is trivial. Hence, $N_{L_v/K_v} = \mathrm{id}$ and a_v is a norm. We have shown that all elements of D are norms from L.

(ii) We want that K^*D is dense in $\mathbb{I}_K$. This is an application of weak approximation.

So by the lemma, $L = K$. □

Now, we can return to the proof of surjectivity of the reciprocity map.

Proof of Proposition 1.2. Let T be a finite set of primes in L containing those ramified from K. Let H be the subgroup of G generated by the Frobenius elements $(\mathfrak{P}, L/K)$ for $\mathfrak{P} \notin T$. To show $H = G$ is equivalent to showing $L^H = K$. To do this, we have our above lemma. Namely, we will show that only finitely many primes are not completely split in L^H/K.

Indeed, take some $\mathfrak{P} \notin T$. Then $(\mathfrak{P} \cap L^H, L^H/K) = (\mathfrak{P}, L/K)|_{L^H}$. By definition of H, $(\mathfrak{P}, L/K) \in H$, which is therefore id on L^H. As such, $(\mathfrak{P} \cap L^H, L^H/K) = \mathrm{id}$. It follows then that any primes in K which do not lie under T are completely split in L^H. This is a finite set of primes in K, so by the lemma we have that $L^H = K$. Hence, the reciprocity map is surjective. □

2 Second Inequality

We no longer assume L/K is abelian, but we will assume that K is a number field. The proof presented will also work for K a function field so that $[L : K]$ is prime to the characteristic of K, but for brevity we omit this.

2.1 Statement and proof

Theorem 2.1 (The second inequality). *Let L/K be finite Galois. Then*

(i) $|\widehat{H}^0(G, \mathbb{C}_L)|$ *and* $\widehat{H}^2(G, \mathbb{C}_L)$ *divide* $[L : K]$.

(ii) $\widehat{H}^1(G, \mathbb{C}_L) = 0$

The proof of this uses the first inequality. It will be quite technical at some points, so we will focus mostly on summarizing. Chapter VII of [CF] and [Milne, VII §5] will contain the missing details.

Proof. **Reductions.**

Prime cyclic case. We reduce first to L/K a cyclic extension of prime degree. Recall first Serre's "Ugly Lemma" (which is misstated in [CF]).

Lemma 2.2 (Ugly lemma). *Let G be finite and A a G module. Let $i \geq 0$. Suppose that for all $H \leq G$ that $\widehat{H}^j(H, A) = 0$ for $0 < j < i$ and that $|\widehat{H}^i(H/K, A)|$ divides $[H : K]$ for all $K \leq H$ prime index. Then $|\widehat{H}^i(G, A)|$ divides $|G|$*

If we know the second inequality in the prime cyclic case, the Ugly lemma lets us pass to the general case.

Furthermore, in the cyclic case, we have access to the first inequality. And in fact, we have access to the computation $h(\mathbb{C}_L) = [L : K]$, which means that $|\widehat{H}^2(G, \mathbb{C}_L)| = [L : K]|\widehat{H}^1(G, \mathbb{C}_L)|$. Furthermore, we know that $\widehat{H}^0(G, \mathbb{C}_L) \cong \widehat{H}^2(G, \mathbb{C}_L)$ in the cyclic case. As such, the second inequality in the cyclic case reduces to the statement that $\widehat{H}^0(G, \mathbb{C}_L)$ has order dividing $[L : K]$. Let's recall that $\widehat{H}^0(G, \mathbb{C}_L) = \mathbb{C}_K/N\mathbb{C}_L$.

Containing the roots of unity. Now that we have reduced to L/K cyclic of prime order p, we reduce further to assume K contains p^{th} roots of unity. Let $K' = K(\zeta_p)$ and $L' = LK'$.

Then $[L' : K'] = [L : K]$. Consider the diagram

$$\begin{array}{ccccccc}
\mathbb{C}_L & \longrightarrow & \mathbb{C}_K & \longrightarrow & \mathbb{C}_K/N\mathbb{C}_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{C}_{L'} & \longrightarrow & \mathbb{C}_{K'} & \longrightarrow & \mathbb{C}_{K'}/N\mathbb{C}_{L'} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{C}_L & \longrightarrow & \mathbb{C}_K & \longrightarrow & \mathbb{C}_K/N\mathbb{C}_L & \longrightarrow & 0
\end{array}$$

where the first vertical maps are the conorms and the second are the norms. One can chase this to conclude $N_{K'/K} : \mathbb{C}_{K'}/N\mathbb{C}_{L'} \longrightarrow \mathbb{C}_K/N\mathbb{C}_L$ is onto. Hence, if $[\mathbb{C}_{K'} : N\mathbb{C}_{L'}]$ divides $[L' : K'] = [L : K]$ then so will $[\mathbb{C}_K : N\mathbb{C}_L]$, which is what we need for the second inequality.

So we have reduced to proving $[\mathbb{C}_K : N\mathbb{C}_L] \mid [L : K]$ for L/K cyclic of prime order p with K containing p^{th} roots of unity. As such, we are free to use Kummer theory to understand this extension. It turns out that it's just as easy to prove this for L/K abelian with exponent p, i.e. $G(L/K) \cong (\mathbb{Z}/p)^r$. Our goal then is to show that $[\mathbb{C}_K : N\mathbb{C}_L] \mid [L : K] = p^r$.

First off, Kummer theory tells us that L is of the form $K(a_1^{1/p}, \ldots, a_r^{1/p})$ for $a_i \in K$. Take a finite set S of primes of K satisfying the following properties.

(a) S contains the infinite places.

(b) S contains divisors of p.

(c) All a_i are in $U(S)$ as defined above.

(d) S contains generators of the ideal class group of K.

Let $s = |S|$.

Take now $M = K(U(S)^{1/p})$. By Kummer theory this corresponds to $U(S)K^{*p}/K^{*p}$, which can be computed as $(\mathbb{Z}/p)^s$. So we have $M/L/K$ with $[L : K] = p^r$ and $[M : K] = p^s$. We then let $[M : L] = p^t$ where $t = s - r$.

Now we seek to understand t.

Lemma 2.3. *There is a finite set of primes T of K which is disjoint from S so that the $(\mathfrak{p}, M/L)$ for $\mathfrak{p} \in T$ are an $\mathbb{F}_p$ basis of $G(M/L)$. In particular, $t = |T|$.*

Proof. The set T is constructed as the primes lying under some $w_1, \ldots, w_t$ of L whose Frobenius elements form a basis for the Galois group. $\square$

We are trying to compute $[\mathbb{C}_K : N\mathbb{C}_L] = [\mathbb{I}_K/K^*N\mathbb{I}_L]$. To do so, we shrink the denominator slightly as follows.

Lemma 2.4. *Consider*

$$E = \prod_{v\in S} K_v^{*p} \times \prod_{v\in T} K_v^* \times \prod_{v\notin S\cup T} \mathcal{O}_v^*$$

Then $E \subseteq N\mathbb{I}_L$.

Proof. We work place by place.

For the $v \in S$, we use the local reciprocity isomorphism $K_v^*/NL_v^* \cong G(L_v/K_v)$ to conclude that the domain has exponent divisible by p.

For $v \in T$, one can check that $L_v = K_v$.

For $v \notin S \cup T$, the norm map on units is surjective because the extension L_v/K_v is unramified. □

As such, we have the divisibility

$$[\mathbb{C}_K : N\mathbb{C}_L] = [\mathbb{I}_K : K^*N\mathbb{I}_L] \mid [\mathbb{I}_K : K^*E]$$

so we are left to show that the latter group is divisible by p^r. We know that S contains a set of generators of the class group of K, so

$$\mathbb{I}_K = K^*\mathbb{I}_S = K^*\mathbb{I}_{S\cup T}$$

so we compute

$$\begin{aligned}[\mathbb{I}_K : K^*E] &= [K^*\mathbb{I}_{S\cup T} : K^*E] \\ &= \frac{[\mathbb{I}_{S\cup T} : E]}{[U(S\cup T) : K^*\cap E]}\end{aligned}$$

Now, we must compute the orders of the numerator and denominator here.

Lemma 2.5. $[\mathbb{I}_{S\cup T} : E] = p^{2s}$

Proof. Both $\mathbb{I}_{S\cup T}$ and E are defined as a place-by-place product, so the quotient is readily computed as

$$\mathbb{I}_{S\cup T}/E = \prod_{v\in S} K_v^*/K_v^{*p}$$

One can furthermore compute $[K_v^* : K_v^{*p}] = \frac{p^2}{|p|_v}$ □

Lemma 2.6. $[U(S\cup T) : K^*\cap E] = p^{s+t}$

Proof. We have the following two facts

(i) $U(S \cup T)^p \subseteq K^* \cap E$, which holds by definition of E.

(ii) $[U(S \cup T) : U(S \cup T)^p] = p^{s+t}$ which follows from a computation using Dirichlet's unit theorem.

So the problem is reduced to showing the reverse containment $K^* \cap E \subseteq U(S \cup T)^p$.

One can apply a counting argument to show that $U(S) \longrightarrow \prod_{v \in T} \mathcal{O}_v^* / \mathcal{O}_v^{*p}$ is surjective. With this surjectivity, we can show that if $a \in K^*$ satisfies

(a) a is a unit outside $S \cup T$, i.e. $a \in U(S \cup T)$

(b) a is a p^{th} power locally in S, i.e. $a \in K_v^{*p}$ for $v \in S$

then a is a global p^{th} power, i.e. $a \in K^{*p}$. This is shown by proving $K(a^{1/p}) = K$ via Lemma 1.5. $\square$

With the above two lemmas, we have computed

$$\begin{aligned}[\mathbb{I}_K : K^* E] &= \frac{[\mathbb{I}_{S \cup T} : E]}{[U(S \cup T) : K^* \cap E]} \\ &= \frac{p^{2s}}{s+t} \\ &= p^{s-t} \\ &= p^r\end{aligned}$$

so we conclude

$$[\mathbb{C}_K : N\mathbb{C}_L] \mid [\mathbb{I}_K : K^* E] = p^r$$

which proves the second inequality in this case. By our above reduction, we conclude the second inequality for all L/K finite Galois. $\square$

2.2 Corollaries

Corollary 2.6.1. *The reciprocity map $\mathbb{C}_K / N\mathbb{C}_L \longrightarrow G(L/K)$ is an isomorphism for L/K abelian.*

Proof. Surjectivity was shown as a corollary to the first inequality above, though we have suppressed the translation between the ideal theoretic statement there and the idèle theoretic statement here. By the second inequality, $|\mathbb{C}_K / N\mathbb{C}_L| = |\widehat{H}^0(G, \mathbb{C}_L)| \leq [L : K]$, so this map must be an isomorphism. $\square$

Corollary 2.6.2. *We have an exact sequence*

$$0 \longrightarrow Br(L/K) \longrightarrow \bigoplus_v Br(L_v/K_v)$$

where $Br(L/K) = H^2(G(L/K), L^*)$ is the Brauer group.

Proof. Take the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow L^* \longrightarrow \mathbb{I}_L \longrightarrow \mathbb{C}_L \longrightarrow 0$$

and use the fact that $H^2(G, \mathbb{I}_L) = \bigoplus_v H^2(G_v, L_v^*)$. We then get an exact sequence

$$H^1(G, \mathbb{C}_L) \longrightarrow Br(L/K) \longrightarrow \bigoplus_v Br(L_v/K_v)$$

and $H^1(G, \mathbb{C}_L) = 0$ by the second inequality. □

Corollary 2.6.3 (Hasse norm theorem)**.** *Let L/K be cyclic. Then $a \in K^*$ is a norm from L if and only if a_v is a norm from L_v for all v.*

Proof. As L/K is cyclic, $H^2 = \widehat{H}^0$, so the exact sequence of Brauer groups above becomes

$$0 \longrightarrow K^*/NL^* \longrightarrow \bigoplus_v K_v^*/NL_v^*$$

Injectivity of this map is exactly what we wanted to show. □

References

[CF] J. W. S. Cassels and A. Fröhlich, eds. *Algebraic number theory: proceedings of an instructional conference organized by the London Mathematical Society (a NATO advanced study institute) with the support of the International Mathematical Union*. 2nd ed. OCLC: ocn573377924. London: London Mathematical Society, 2010. ISBN: 9780950273426.

[Milne] J.S. Milne. *Class Field Theory (v4.03)*. Available at www.jmilne.org/math/. 2020.