Automorphic Forms and the GL_1 Case

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Abstract

We present here a brief introduction to automorphic forms and representations. The generalities of this subject are quite vast, and when convenient we will stick to simple cases. For instance, we will mostly ignore technicalities of reductive groups and focus our definitions instead on GL_n. Furthermore, we will discuss a more classical and particularly simple case of this theory – the GL_1 case via Tate's thesis. This forms the basis for the modern Langland's program.

Our primary reference will be [Bump]. Another reference focusing on the GL_n case is [GH]. A succinct exploration of many of the topics covered can also be found in [Knapp]. Furthermore, a more advanced reference covering general reductive groups is [BC]. For Tate's thesis, one can use [Bump]. The original thesis is presented in [CF], along with many other chapters on the prerequisite knowledge. Finally, [RV] is another reference with the goal of presenting Tate's thesis to a reader with minimal background.

0 Tate's Thesis

We will present a short introduction to Tate's thesis. In his Ph.D. thesis, Tate reformulated prior work of Hecke on functional equations for L - functions. He did this using harmonic analysis on the adèles. As such, one must be familiar with harmonic analysis on abelian groups to understand this material. We invite the reader to refer to [Rudin] and [RV] for this theory.

0.1 Adèles

We now begin with a summary of the theory of adèles. See [RV] for more details.

Convention. When we refer to "almost all" elements of some set, we will mean all but finitely many.

Definition 0.1. Let I be some indexing set, G_i be a collection of locally compact Hausdorff groups, and $H_i \subseteq G_i$ be compact open subgroups. We define the restricted topological product to be

$$\prod_{i\in I}{}'(G_i, H_i) = \left\{ a = (a_i)_{i\in I} \in \prod_{i\in I} G_i \;\middle|\; a_i \in H_i \text{ for almost all } i \right\}$$

This is clearly a subgroup of the product, but we topologize it differently. We take a neighborhood basis of the identity element via $\prod_{i\in I} V_i$ for $V_i \subseteq G_i$ open so that $V_i = H_i$ for almost all i.

This is a better topology than that induced from the product topology because it is locally compact. For instance, if we let $\widehat{G}$ denote the Pontryagin dual, we have the result

Proposition 0.1.

$$\prod_{i\in I}{}'(\widehat{G_i}, \widehat{H_i\backslash G_i}) \longrightarrow \widehat{\prod_{i\in I}{}'(G_i, H_i)}$$

via $(\chi_i)_{i\in I} \mapsto \prod_{i\in I} \chi_i$ *is an isomorphism of topological groups.*

In Tate's thesis, he specifically looked at the ring of adèles, introduced by Chevalley.

Definition 0.2. Let F be a global field. We will let V_F denote the set of places of F. The adèle ring of F is

$$\mathbb{A}_F = \prod_{v \in V_F}{}' (F_v, \mathcal{O}_v)$$

with $\mathcal{O}_v$ defined for the finite places, which are almost all of them.

That this is the restricted topological product affords us the following useful results.

Proposition 0.2. *(a) The diagonal embedding $F \longrightarrow \mathbb{A}_F$ via $a \mapsto (a, a, \dots)$ is a ring isomorphism onto its image, which is a discrete subspace of $\mathbb{A}_F$.*

(b) The quotient $F\backslash\mathbb{A}_F$ is compact.

(c) $\widehat{F\backslash\mathbb{A}_F} = F$.

(d) Each $\widehat{F_v} = F_v$, so $\widehat{\mathbb{A}_F} = \mathbb{A}_F$. Let's note here that the isomorphism $\widehat{F_v} = F_v$ arises by some fixed choice of additive character $\psi_v : F_v \longrightarrow \mathbb{C}$ so that the Fourier transform is written as $\widehat{\theta}(y) = \int_F \theta(x)\psi_v(xy)\ dx$

Compare this to classical harmonic analysis.

Proposition 0.3. *(a) The inclusion $\mathbb{Z} \longrightarrow \mathbb{R}$ has discrete image.*

(b) The quotient $S^1 = \mathbb{Z}\backslash\mathbb{R}$ is compact.

(c) $\widehat{S^1} = \mathbb{Z}$.

(d) $\widehat{\mathbb{R}} = \mathbb{R}$. The associated additive character witnessing this fact is $\psi_\infty(t) = e^{2\pi i t}$.

It was exploiting this similarity that allowed Tate to utilize classical techniques of harmonic analysis, such as Poisson summation, to study the adèles.

We also must consider the multiplicative group of $\mathbb{A}_F$, which is known as the idèle group.

Definition 0.3. The group of idèles is

$$\mathbb{A}_F^* = \prod_{v \in V_k}{}' (F_v^*, \mathcal{O}_v^*)$$

We remark that this is not the subspace topology from the inclusion $\mathbb{A}_F^* \longrightarrow \mathbb{A}_F$. Rather, it is the inclusion $\mathbb{A}_F^* \longrightarrow \mathbb{A}_F \times \mathbb{A}_F$ via $a \mapsto (a, a^{-1})$. Furthermore, to connect this with the coming GL_n case of automorphic forms, we note that $GL_1(R) = R^*$. Then we have

$$GL_1(\mathbb{A}_F) = \prod_{v \in V_F}{}' (GL_1(F_v), GL_1(\mathcal{O}_v^*))$$

and the topology on GL_1 via the embedding $a \mapsto (a, a^{-1})$ is the usual way to put an algebraic group structure on the general linear group.

0.2 L - functions and functional equations

The restricted direct product structure on the idèles affords a fruitful domain to develop local – global connections. From this, the structure of Tate's thesis appears. To develop a functional equation globally, he will develop them in the simpler local settings and multiply them all together to get a global result via the idèles.

Definition 0.4. Let F be a global field. A Hecke character on F is an element of $\widehat{F^*\backslash\mathbb{A}_F^*}$, i.e. a continuous group homomorphism $\mathbb{A}_F^* \longrightarrow S^1$ which is trivial on F^*.

Remark. The group $F^*\backslash\mathbb{A}_F^*$ is known as the idèle class group – an essential object of class field theory.

Furthermore, Hecke characters arise classically over the rational numbers as Dirichlet characters.

Proposition 0.4. *Let χ be a Hecke character on $\mathbb{Q}$. Then $\chi = \chi_1|\cdot|^{it}$ for some real number t and some finite order Hecke character χ_1. Furthermore, finite order Hecke characters on $\mathbb{Q}$ correspond to primitive Dirichlet characters.*

With this set up, we can begin to define L - functions.

Definition 0.5. Let χ be a Hecke character on a global field F. Furthermore, let S be a finite set of places of F containing any infinite places, as well as any places v for which χ_v is ramified. Here, a character χ_v is unramified if it is trivial for all elements of norm 1. For the nonarchimedean places, the norm 1 part corresponds to the inertia group by class field theory, hence the term unramified.

For $v \notin S$ we define the local L - factor.

$$L_v(s,\chi) = (1 - \chi_v(\pi_v)q_v^{-s})^{-1}$$

where s is a complex number, π_v is a uniformizer of F_v and q_v is the cardinality of the residue field of F_v.

And we define the restricted L - function, for $\chi = \chi_1|\cdot|^\lambda$, as

$$L_S(s,\chi) = \prod_{v\notin S} L_v(s+\lambda,\chi_1)$$

Remark. The space of quasicharacters, i.e. maps $\mathbb{A}_F^* \longrightarrow \mathbb{C}^*$, can be given the structure of a Riemann surface via the decomposition of any such quasicharacter χ as $\chi = \chi_1|\cdot|^s$ for χ_1 of finite order and $s \in \mathbb{C}$. The two parameters (s,χ) can therefore be thought of as a single quasicharacter $\chi|\cdot|^s$, so as to view L_S as having a single parameter. This explains why the λ moves to the left parameter above.

Example. These are a generalization of Dirichlet characters. For instance, consider $F = \mathbb{Q}$, χ the trivial character, and $S = V_{\mathbb{Q}} - \{\infty\}$. Then $L_S(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$, which is known by the Euler product formula as the Riemann – Zeta function $\zeta(s)$. More generally, the trivial character recovers the Dedekind zeta function.

We now begin an exploration of Tate's method to prove a functional equation for these L - functions. First off, we extend the classical notion of a Schwartz function to the adèles.

Definition 0.6. Let F be a global field and v be a place. For v an infinite place, we let $\mathcal{S}(F_v)$ be the usual Schwartz functions $F_v \longrightarrow \mathbb{C}$. For v a finite place, we let $\mathcal{S}(F_v)$ be the locally constant functions $F_v \longrightarrow \mathbb{C}$. Finally, we let $\mathcal{S}(\mathbb{A}_F)$ be the set of sums of functions of the form $\phi(x) = \prod_{v\in V_F} \phi_v(x_v)$ for $\phi_v \in \mathcal{S}(F_v)$.

Let χ be a Hecke character, $s \in \mathbb{C}$, $\phi \in \mathcal{S}(A)$. We consider the zeta integral

$$\zeta(s, \chi, \phi) = \int_{\mathbb{A}_F^*} \phi(x)\chi(x)|x|^s \, d^*x$$

where d^*x is a self dual Haar measure on $\mathbb{A}_F^*$. Also, by the above remark, we can think of (χ, s) as describing a single quasicharacter $\chi|\cdot|^s$. We describe now how to analyze ζ and its relation to L_S.

1. The zeta integral is linear in ϕ, so without loss of generality say $\phi(x) = \prod_v \phi_v(x_v)$. Then by properties of idèlic integration, we can factor this global zeta integral into local zeta integrals

$$\zeta(s, \chi, \phi) = \prod_v \zeta_v(s, \chi_v, \phi_v)$$

where

$$\zeta_v(s, \chi_v, \phi_v) = \int_{F_v^*} \phi_v(x)\chi_v(x)|x|_v^s \, d^*x$$

2. The local zeta integrals converge for $Re(s) > 0$ and, away from some finite set S containing the infinite places, we have $\zeta_v(s, \chi_v, \phi_v) = (1 - \chi(\pi_v)q_v^{-s})^{-1}$. That is, the local zeta integrals are, except for some finite set S, the same as the local L - factors.

3. The local zeta integrals have a meromorphic continuation to all $s \in \mathbb{C}$. Furthermore, we have the local functional equation

$$\zeta_v(1 - s, \chi_v^{-1}, \widehat{\phi_v}) = \gamma_v(s, \chi_v, \psi_v)\zeta_v(s, \chi_v, \phi_v)$$

where here ψ_v is the fixed additive character of F_v witnessing its self duality, as above.

4. Using the above facts about convergence of the local zeta integrals, one shows that $\zeta(s, \chi, \phi)$ has a meromorphic continuation to all $s \in \mathbb{C}$. Furthermore, this is holomorphic unless χ is trivial on the norm 1 idèles. In that case, χ is of the form $x \mapsto |x|^\lambda$ for $\lambda \in i\mathbb{R}$, where there are potential poles at $1 - \lambda$ and λ for F a number field and potential poles at $1 - \lambda + 2\pi in/\log(q)$ and $-\lambda + 2\pi in/\log(q)$ for function fields over $\mathbb{F}_q$ and $n \in \mathbb{Z}$.

   By similar techniques to the local functional equation, we have the global functional equation

$$\zeta(s, \chi, \phi) = \zeta(1 - s, \chi^{-1}, \widehat{\phi})$$

5. By comparing now L_S and ζ, we can form a functional equation for L_S, where S is a finite set of places containing the infinite places.

$$L_S(s, \chi) = \left(\prod_{v \in S} \zeta_v(s, \chi_v, \phi_v) \right) L_S(1 - s, \chi^{-1})$$

   with poles as above.

6. We can define local L - factors for the places we have missed (noting that $|x|_v$ is the square of

the complex norm for v complex).

$$L_v(s,\chi) = \begin{cases} 1 & v \text{ finite and ramified} \\ \pi^{-(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2) & v \text{ real and } \chi_v(x) = \left(\frac{x}{|x|_v}\right)^{\varepsilon} \text{ where } \varepsilon = 0,1 \\ 2(2\pi)^{s+\alpha+|k|/2}\Gamma(s+\alpha+|k|/2) & v \text{ complex and } \chi_v(x) = |x|_v^{\alpha}\left(\frac{x}{\sqrt{|x|_v}}\right)^{k} \end{cases}$$

Upon this, we define

$$\varepsilon_v(s,\chi_v,\psi_v) = \frac{\gamma_v(s,\chi_v,\psi_v)L_v(s,\chi_v)}{L_v(1-s,\chi_v^{-1})}$$

and this is equal to 1 for almost all v. Their product is

$$\varepsilon(s,\chi) = \prod_v \varepsilon(s,\chi,\psi_v)$$

with the ψ_v dependence dropped. Finally, we have our desired L - function $L(s,\chi) = \prod_v L_v(s,\chi)$ with a meromorphic continuation to $s \in \mathbb{C}$ and potential poles at $s = 0,1$, satisfying the functional equation

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\chi^{-1})$$

1 Automorphic Forms

We will now present the definition of automorphic forms on GL_n. One can proceed with much of the same theory for GL_n replaced by any reductive group, but we will not seek such generality.

Definition 1.1. Let F be a global field and $n \geq 1$. We let

$$GL_n(\mathbb{A}_F) = \prod_{v \in V_F} (GL_n(F_v), GL_n(\mathcal{O}_v))$$

which admits a closed embedding into $\mathbb{A}_F^{n^2+1}$ via the map $A \mapsto (A, 1/\det(A))$. If we think of GL_n as an affine algebraic group via this embedding into the affine space A^{n^2+1}, then $GL_n(\mathbb{A}_F)$ is the group of $\mathbb{A}_F$ points of GL_n.

Similarly to the GL_1 theory presented above, this is a locally compact group with a diagonally embedded subgroup $GL_n(F) \longrightarrow GL_n(\mathbb{A}_F)$ which has discrete image. We no longer have cocompactness, and must instead accept the following weaker result

Lemma 1.1. *$Z(GL_n(\mathbb{A}_F))GL_n(F)\backslash GL_n(\mathbb{A}_F)$ has finite volume in the Haar measure. Here $Z(GL_n(\mathbb{A}_F))$ denotes the center of this group, which is known to be $GL_1(\mathbb{A}_F)$ embedded as diagonal matrices.*

We now have to fix a maximal compact subgroup of $GL_n(\mathbb{A}_F)$. Take this to be $K = \prod_v K_v$ where

$$K_v = \begin{cases} O(n) & v \text{ real} \\ U(n) & v \text{ complex} \\ GL_n(\mathcal{O}_v) & v \text{ finite} \end{cases}$$

Let ω be a quasicharacter of $GL_1(\mathbb{A}_F)$ as defined in section 0. That is, $\omega : GL_1(\mathbb{A}_F) \longrightarrow \mathbb{C}^*$. We can then think of this as a central quasicharacter of GL_n.

Definition 1.2. An automorphic form for $GL_n(\mathbb{A}_F)$ with central quasicharacter ω is a function $f : GL_n(\mathbb{A}_F) \longrightarrow \mathbb{C}$ satisfying the following:

(i) f is smooth.

(ii) f is K finite.

(iii) f is $\mathcal{Z}$ finite.

(iv) f has moderate growth.

(v) $f(zg) = \omega(z)f(g)$ for $z \in GL_1(\mathbb{A}_F)$ the center, $g \in GL_n(\mathbb{A}_F)$.

(vi) $f(\gamma g) = f(g)$ for $\gamma \in GL_n(F)$ and $g \in GL_n(\mathbb{A}_F)$. That is, f is defined on the quotient $GL_n(F)\backslash GL_n(\mathbb{A}_F)$.

We must now define each of these terms (i) - (iv).

(i) If F is a function field, smoothness is the same as being locally constant. If F is a number field, f is smooth if it is locally of the form $g^\infty g^{fin} \mapsto \phi(g^\infty)$ for some function $\phi : GL_n(\mathbb{A}_F)^\infty \longrightarrow \mathbb{C}$, which is smooth in the sense of calculus. Here, the superscript ∞ mean the product over the infinite places and the superscript fin means the restricted product over the finite places. Compare this to the definition of Schwartz functions from section 0.

(ii) There is a right action of K on functions $GL_n(\mathbb{A}_F) \longrightarrow \mathbb{C}$ via $(fk)(g) = f(kg)$. We say that f is K finite if the span of the orbit fK is a finite dimensional $\mathbb{C}$ vector space.

(iii) Take the below definition per infinite place v of F. We define an action of the Lie algebra $\mathfrak{gl}_n(F_v)$ on the space of K finite maps $f : GL_n(\mathbb{A}_F) \longrightarrow \mathbb{C}$ via

$$(Xf)(g) = \frac{d}{dt} f(g \exp(tX))\bigg|_{t=0}$$

for $g \in GL_n(\mathbb{A}_F)$ and $X \in \mathfrak{gl}_n(F_v)$.

This in turns defines an action of the universal enveloping algebra $U(\mathfrak{gl}_n(F_v))$, and hence of its center $\mathcal{Z}$. We then say that f is $\mathcal{Z}$ finite if it lives in a finite dimensional vector space fixed by the action of $\mathcal{Z}$.

(iv) For each place v, we have a norm $|\cdot|_v$ on $\mathbb{A}_F^{n^2+1}$ via $|a|_v = \max_{1 \le i \le n^2+1} |a_i|_v$. Let $||a|| = \prod_v |a|_v$. For $g \in GL_n(\mathbb{A}_F)$, let $||g||$ be defined via the inclusion $GL_n(\mathbb{A}_F) \longrightarrow \mathbb{A}_F^{n^2+1}$. We say that f has moderate growth if there are constants C and N so that $|f(g)| \le C||g||^n$ for all g.

The space of all automorphic forms with central quasicharacter ω is denoted $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F), \omega)$.

Example. Let's return to the GL_1 case and see how it fits into this greater context. A central quasicharacter for GL_1 is a quasicharacter $\omega : GL_1(\mathbb{A}_F) \longrightarrow \mathbb{C}^*$, which as discussed above must decompose to be of the form $\omega = \omega_1|\cdot|^s$ for some finite order character ω_1 and $s \in \mathbb{C}$. Let $f \in \mathcal{A}(GL_1(F)\backslash GL_1(\mathbb{A}_F), \omega)$. By condition (v), we have that $f(g) = f(1)\omega(g)$. As such, any automorphic form on $GL_1(\mathbb{A}_F)$ is of the form $c\omega$ for $c \in \mathbb{C}$ and $\omega : GL_1(\mathbb{F})\backslash GL_1(\mathbb{A}_F) \longrightarrow \mathbb{C}^*$. Furthermore, as f must be constant on $GL_1(F)$ it is a continuous map from a compact group $GL_1(F)\backslash GL_n(\mathbb{A}_F)$ and hence maps into S^1. So really, ω is not just a quasicharacter but a Hecke character. These are, up to scalars, the objects studied in Tate's thesis.

Example. We also provide a brief explanation of how the GL_2 case of automorphic forms generalize the classical Maass forms, which themselves are closely related to the classical modular forms. More details can be found in [Bump, 3.2, 3.6]. Indeed, a Maass form is a function $\mathbb{H} \longrightarrow \mathbb{C}$ satisfying a transformation property with respect to the action of $G = GL_2^+(\mathbb{R})$ and some discrete subgroup Γ, which is an eigenform of the weight k Laplacian Δ_k, and which satisfies certain growth conditions. The transformation properties for G and Γ are related to equation (v) and (vi) above. Viewing the Lie algebra in terms of differential operators, being an eigenform of Δ_k relates to $\mathcal{Z}$ finiteness.

This suggests a connection to automorphic forms. And indeed, a Maass form $\mathbb{H} \longrightarrow \mathbb{C}$ can be lifted to a function on $\Gamma\backslash G$ which is an eigenfunction of the Laplacian Δ and satisfies related transformation properties. This then lifts to the adèles via the fact that $\Gamma_0(N)\backslash SL_2(\mathbb{R})$ is a right quotient of $Z(GL_2(\mathbb{A}_F))GL_2(F)\backslash GL_2(\mathbb{A}_F)$. See [Bump, 3.3.1] for more on this.

Definition 1.3. An automorphic form f is called cuspidal if

$$\int_{M_{r\times s}(F)\backslash M_{r\times s}(\mathbb{A}_F)} f\left(\begin{pmatrix} I_r & X \\ 0 & I_s \end{pmatrix} g\right) dX = 0$$

for $g \in GL_n(\mathbb{A}_F)$ and any $r + s = n$ with $r, s < n$.

This generalizes the classical notion of cuspidality for Maass and modular forms. The space of cuspidal automorphic forms with central quasicharacter ω is called $\mathcal{A}^0(GL_n(F)\backslash GL_(\mathbb{A}_F), \omega)$.

2 Automorphic Representations

We are interested in representations of $GL_n(\mathbb{A}_F)$. A natural candidate would seem to be the automorphic forms $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F), \omega)$. Indeed, there is a right action of $GL_n(\mathbb{A}_F)$ on the space of functions $GL_n(\mathbb{F}) \longrightarrow \mathbb{C}$ via $(fg)(a) = f(ga)$ for $f : GL_n(\mathbb{A}_F) \longrightarrow \mathbb{C}$, $a, g \in GL_n(\mathbb{F})$. Many properties of automorphic forms are preserved under this action, but there is one notable omission: K finiteness needn't be preserved at the infinite places. For instance, this [MO] answer gives an example of $SO(2)$ finiteness which is not preserved under right translation by $SL_2(\mathbb{R})$.

This is not an issue on the finite places, that is to say that $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F), \omega)$ is a (right) representation of $GL_n(\mathbb{A}_F)^{fin}$. For function fields, this is a completely satisfactory answer, but for number fields we care deeply about the infinite places. For instance, the place at ∞ over $\mathbb{Q}$ recovers for us the auxillary Γ and π terms in the functional equation for the Riemann zeta function. We seek therefore some sort of action at ∞ on $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F), \omega)$ to define an automorphic representation.

Definition 2.1. Let $\mathfrak{g}_\infty = \prod_{v|\infty} \mathfrak{gl}_n(F_v)$ and $K_\infty = \prod_{v|\infty} K_v$. A $(\mathfrak{g}_\infty, K_\infty)$ module is a vector space V with representations $\pi_{\mathfrak{g}_\infty}$ and π_{K_∞} of $\mathfrak{g}_\infty$ and of K_∞ respectively, which are subject to the following conditions:

(i) The π_{K_∞} action yields a decomposition of V into a direct sum of finite dimensional irreducible subspaces.

(ii) $\pi_{\mathfrak{g}_\infty}(X)v = \frac{d}{dt}(\pi_{K_\infty}(\exp(tX)v))|_{t=0}$ for $X \in \mathfrak{g}_\infty$ and $v \in V$.

(iii) $\pi_{\mathfrak{g}_\infty}(X)\pi_{K_\infty}(k) = \pi_{K_\infty}(k)\pi_{\mathfrak{g}_\infty}(k^{-1}Xk)$ for $k \in K_\infty$ and $X \in \mathfrak{g}_\infty$.

We have such a structure on $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F),\omega)$. Here, K_∞ acts by right translation as above and $\mathfrak{g}_\infty$ acts by differential operators on the infinite places, recalling that automorphic forms are assumed to be smooth. Furthermore, the $GL_n(\mathbb{A}_F)^{fin}$ action and the $(\mathfrak{g}_\infty, K_\infty)$ action commute with each other.

Definition 2.2. An automorphic representation of $GL_n(\mathbb{A}_F)$ is a subquotient of $\mathcal{A}(GL_n(F)\backslash GL_n(\mathbb{A}_F),\omega)$ which is irreducible.

A cuspidal representation is the same, but with $\mathcal{A}$ replaced by $\mathcal{A}^0$.

Motivated by the local – global nature of adèlic results, we would like to determine the extent to which automorphic representations, a global object, are decomposed locally. To do so, we need the notion of a restricted tensor product to align with our restricted products.

Definition 2.3. Let V_i be modules indexed by I. For almost all $i \in I$, let there be some fixed nonzero vector v_i^0 in V_i. The finite subsets S of I for which v_i^0 exists for all $i \notin S$ form a directed system under inclusion. Given an inclusion $S \subseteq S'$ we can form the map $\bigotimes_{i\in S} V_i \longrightarrow \bigotimes_{i\in S'} V_i$ by tensoring by v_i^0 in the components $i \in S' - S$. We then define

$$\bigotimes_{i\in I}{}' V_i = \varinjlim_{S} \bigotimes_{i\in S} V_i$$

So the elements of $\bigotimes_{i\in I}{}'V_i$ are sums of pure tensors indexed by I, where almost all components are v_i^0.

For instance, say we have a restricted direct product $G = \prod_{i\in I}{}'(G_i, H_i)$ and V_i representations of G_i. Suppose there are nonzero vectors v_i^0 in V_i for almost all i so that v_i^0 are fixed by H_i. Then the restricted tensor product $\bigotimes_{i\in I}{}'V_i$ has the structure of a representation of G.

We would like to decompose automorphic representations into restricted tensor products, but we first need to insist on some regularity of the representation.

Definition 2.4. Let V be a vector space with commuting actions from $GL_n(\mathbb{A}_F)^{fin}$ and $(\mathfrak{g}_\infty, K_\infty)$. We say V is admissible if in its direct sum decomposition via K, no isomorphism type has infinite multiplicity and if the orbit vK has finite dimensional span for all v.

The definition of admissibility applies for any $GL_n(F_v)$ representation for v finite or any $(\mathfrak{g}_\infty, K_v)$ module for v infinite.

We now state a structure theorem for irreducible admissible automorphic representations.

Theorem 2.1 (The tensor product theorem)**.** *Let (V,π) be an irreducible admissible representation of $GL_n(\mathbb{A}_F)$. Then for each infinite v, there is an admissible $(\mathfrak{g}_\infty, K_v)$ module (V_v, π_v). Additionally, for each finite v there are irreducible admissible representations (V_v,π_v) of $GL_n(F_v)$. Furthermore, for almost all v there is some nonzero vector $\xi_v^0 \in V_v$ which is fixed by K_v so that (V,π) is the restricted tensor product of the (V_v,π_v).*

3 L - Functions Attached to Automorphic Representations

We now return to studying L - functions. In Tate's thesis, we began with Hecke characters and developed a place-by-place description of an associated L - function, whose global functional equations arose by multiplying the local functional equations. We will turn now to associating L - functions to automorphic representations of $GL_2(\mathbb{A}_\mathbb{F})$, and describe very briefly how the resulting functional equations arises similarly to in Tate's thesis. Thus, we place Tate's thesis in the general context of the study of automorphic representations as the GL_1 case.

We begin with local L - factors for $\mathbb{F}$ a local nonarchimedean field, as we did with GL_1. Let $\chi_1, \chi_2 : \mathbb{F}^* \longrightarrow \mathbb{C}^*$ quasicharacters and consider the space of smooth functions $f : GL_n(\mathbb{F}) \longrightarrow \mathbb{C}$ subject to the condition

$$f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g\right) = \left|\frac{y_1}{y_2}\right|^{1/2} \chi_1(y_1)\chi_2(y_2)f(g)$$

$GL_n(\mathbb{F})$ acts on this by right translation, and we call the resulting representation $\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$. For χ_i unramified, this is irreducible. Any representation of $GL_n(\mathbb{F})$ arising automorphically with a nonzero vector fixed by $GL_n(\mathcal{O}_\mathbb{F})$ is of this form.

Definition 3.1. An admissible representation of $GL_n(\mathbb{F})$ is called spherical if it admits a nonzero vector fixed by $GL_n(\mathcal{O}_\mathbb{F})$.

Now we can move on to defining local L - factors for unramifed nonarchimedean local fields, and subsequently piecing them together to get a partial L - function.

Definition 3.2. If χ_i are unramified, we let $\alpha_i = \chi_i(\varpi)$ for ϖ a uniformizer of $\mathbb{F}$. These are called the Satake parameters of $\pi = \pi(\chi_1, \chi_2)$.

We can now define the local L - factor on $\mathbb{F}$ to be $L(s, \pi) = (1 - \alpha_1 q^{-s})^{-1}(1 - \alpha_2 q^{-s})^{-1}$ for q the cardinality of the residue field of $\mathbb{F}$. Note the similarity with the GL_1 local L - factors.

Definition 3.3. Let (V, π) be an automorphic cuspidal representation of $GL_n(\mathbb{A}_F)$, and suppose the central quasicharacter ω is actually unitary (compare this to Hecke characters, which are also supposed to be unitary). Write, by the tensor product theorem, $\pi = \bigotimes' \pi_v$. Let S be a finite set of places for which π_v is spherical outside S. In particular, S contains the infinite places. Then we define the partial L - function as

$$L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi_v)$$

where L_v is the local L - factor just defined.

Now, to avoid technical complexities, we will again return to vagueness in our presentation, but will give citations in [Bump] for where the precise statements are found. One sees that the following process is in direct analogy with the GL_1 case in section 0.

1. We define a global zeta integral $Z(s, \phi)$ for $\phi \in V$ ([Bump, Eq. 3.5.29]), and local zeta integrals $Z_v(s, W_v)$ ([Bump, Eq. 3.5.32]). Here W_v is a "Whittaker model", taking a similar role to ψ_v from the GL_1 case. The beginning of [Bump, 5.3] covers Whittaker models. These are related via a product formula $Z(s, \phi) = \prod_v Z_v(s, W_v)$.

2. For almost all v, $Z_v(s, W_v) = L_v(s, \pi_v)$. See [Bump, Prop. 3.5.3]

3. The local zeta integrals have meromorphic continuations and functional equations of the form

$$Z_v\left(1 - s, \pi_v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_v\right) = \gamma_v(s, \pi_v, \psi_v) Z_v(s, W_v)$$

   See [Bump, Prop. 3.5.4].

4. We form a global functional equation for the partial L - function via the local functional equations

$$L_S(s,\pi) = \left(\prod_{v\in S} \gamma_v(s,\pi_v,\psi_v)\right) L_S(1-s,\widehat{\pi})$$

[Bump, Thm. 3.5.6].

5. Extend the definition of the local L factors to the remaining finitely many places, and form similar ε factors as in GL_1 ([Bump, Eq. 3.5.49]) to yield a functional equation for L itself.

4 Conclusion

We have seen that the generality of automorphic forms, which encapsulates the GL_1 theory of Hecke characters, affords us a generalization of Tate's method to prove functional equations for L functions associated to such characters. The theory of automorphic forms extends to all GL_n, as well as more general reductive groups. For instance, there is still a notion of L functions attached to cuspidal automorphic representations of reductive groups. Langland's conjectured that these L - functions have functional equations, and in fact conjectured the vaster functoriality conjecture. This is far beyond the scope of this report, so we invite the reader to read [Bump, 3.9]. One can also see [Knapp] and Borel's article "Automorphic L - functions" in [BC, III].

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